Lecture-4

Anjana Mandal
(8972339491)

Monotone sequences

Till now we have seen how to work with convergent sequences. All the methods require that we already know (or at least suspect) the value of the limit, and we verify that our suspicion is correct.

All these methods do not tell us how to determine (quickly) if a sequence does converge.

Here we shall establish results that can be used to show a sequence is convergent even though the value of the limit is not known. We shall find a condition on a bounded sequence which ensures the convergence of ~~the~~ sequence.

Def^n:- A sequence ~~of~~ $\{x_n\}$ of real numbers is called monotonically increasing if $x_n \leq x_{n+1}$ $\forall n$, and $\{x_n\}$ is called a monotonically decreasing if $x_n \geq x_{n+1}$ $\forall n$.

If a sequence is monotonically increasing or decreasing, we call that ~~an~~ a monotone sequence or monotonic sequence.

~~Examples:~~

If $x_n < x_{n+1}$ $\forall n$ → ~~call it~~ Strictly increasing monotone sequence

$x_n > x_{n+1}$ → Strictly monotone decreasing sequence.

Scanned with CamScanner

Examples

① $\{1, 2, 3, 4, \ldots, n, \ldots\}$ → strictly increasing

② $\{a, a^2, a^3, \ldots, a^n, \ldots\}, a > 1$ → " "

③ $\{1, 2, 2, 3, 3, 3, \ldots\}$ → increasing sequence

④ $\{1, 1/2, 1/3, \ldots, 1/n, \ldots\}$

⑤ $\{1, 1/2, 1/2^2, \ldots, 1/2^{n-1}, \ldots\}$

⑥ $\{b, b^2, b^3, \ldots, b^n, \ldots\}, 0 < b < 1$

} → Strictly decreasing

⑦ $\{1, -1, 1, \ldots, (-1)^{n+1}, \ldots\}$

$\{-1, 2, -3, \ldots, (-1)^n n, \ldots\}$

} not monotonic sequence

Theorem:- Let $\{x_n\}$ be monotonically increasing and bounded above, then $\{x_n\}$ converges. $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$.

Proof:- LUB axiom plays a very important role in this theorem.

[**LUB axiom:-** Every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound (or a supremum)]

Since $\{x_n\}$ is bounded above, it has the least upper bound, say x.

Scanned with CamScanner

We have to show that for every $\varepsilon > 0$, $\exists\, n_0$ s.t. $|x_n - x| < \varepsilon$ $\forall n \geq n_0$.

So, let us take $\varepsilon > 0$.

If x is the least upper bound, then it means that

$\exists\, n_0 \in \mathbb{N}$ s.t. $x_{n_0} > x - \varepsilon$ [x is lub so, $x - \varepsilon$ is ~~an~~ not an upper bound]

let $n \geq n_0$.

Then

$$x - \varepsilon < x_{n_0} \leq x_n \leq x < x + \varepsilon$$ [$\because x$ is the lub]

So, $\Rightarrow x - \varepsilon < x_n < x + \varepsilon$ [As $n_0 \leq n$ $\Rightarrow x_{n_0} \leq x_n$ as $\{x_n\}$ is an increasing sequence]

$\Rightarrow |x_n - x| < \varepsilon$

$\Rightarrow \lim_{n \to \infty} x_n = x$

<u>Theorem</u>:- Let $\{x_n\}$ be monotonically decreasing and bounded below. Then $\{x_n\}$ converges and converges to it's glb.

$\Rightarrow$ Proof is similar.

Either you can initiate the proof in the same way or the other possibility is that if $\{x_n\}$ is monotonically increasing then $\{-x_n\}$ is monotonically decreasing. Then use the theorem ~~to~~ for increasing sequences and get the required thing.

Scanned with CamScanner

Monotone Convergence Theorem:-

A monotone sequence of real numbers is convergent if and only if it is bounded.

⇒ proving this theorem, means proving the ~~above~~ ~~two~~ last two theorems.

The Monotone convergence theorem establishes the existence of the limit of a bounded monotone sequence.

~~It also give us a way of calculating the limit of the sequence provided we can evaluate the supremum~~

Because sequence is a function from the set of all natural numbers and the set of all natural numbers satisfy principle of induction.

So, for example, I can give the value of x_1, and then I can tell how to compute x_2 from x_1, x_3 from x_2 etc.

This is what is called defining a sequence inductively, that is give the value of x_1 and then say how x_{n+1} is computed from x_n, then that defines all the numbers of the sequence. This is one possibility.

And I can also give the value of x_1 & x_2 and then say how to compute

x_{n+2} from x_n and x_{n+1}.
That will also define the whole sequence.
So, this is called defining a sequence inductively.
But once a sequence is defined inductively then every property of that sequence has to be cheched by induction.
And one of the ways of showing that such sequences are convergent and divergent is that you show that the sequence is monotonically increasing and bounded above or the other way decreasing & bounded below.

① Defined $\{x_n\}$ by

$$x_n = 1$$
$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \; ; n = 1, 2, 3, \ldots$$

$\Rightarrow$ [Whenever a sequence is defined in this way to show sequence is convergent divergent, you just to see whether it is increasing & bounded above or decreasing and bdd below.]

Here $x_1 = 1$, $x_2 = \frac{1}{2}(1+2) = \frac{3}{2}$

$x_2 > x_1$; true for $n=1$

$$x_{n+1} - x_n = -\frac{1}{2}x_n + \frac{1}{x_n} = \frac{1-x_n^2}{2x_n} = \frac{(1-x_n)(1+x_n)}{2x_n} \geq 0 \; [\because x_n \geq 1]$$

$\Rightarrow \{x_n\}$ is monotonically increasing

~~What about the~~

[Or, $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \geq x_n$, as $\frac{1}{x_n} \geq \frac{1}{2}x_n$]

Scanned with CamScanner

Claim 1 :- $1 \le x_n \le 2 \ \forall n$.

Tr $x_1 = 1$, $x_n = 3/2$

So, true for $n = 1, 2$.

Assume true for some $n \in \mathbb{N}$ and then we try for $n+1$.

$$1 \le x_n \le 2$$

$$\Rightarrow \frac{1}{2} \le \frac{1}{x_n} \le 1$$

$$\Rightarrow x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

$$\le \frac{1}{2}(2+2) = 2$$

So, $x_{n+1} \le 2$.

$$x_{n+1} \ge \frac{1}{2}(1+1) = 1$$

$$\Rightarrow 1 \le x_{n+1} \le 2$$

So, by induction it holds for all n.

Claim 2 :- ~~$x_n \le x_{n+1} \ \forall n$~~, $x_n \ge x_{n+1} \ \forall n$

$x_1 = 1$, $x_2 = 3/2$, $x_1 \le x_2$

~~So, true for $n = 1$~~

~~To show $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \ge x_n$~~

~~$\left[\Rightarrow \frac{1}{x_n} \ge \frac{1}{2} x_n \right.$~~

~~$\left. \Rightarrow x_n^2 \le 2 \right]$~~

Assume that $x_n \le x_{n+1}$ and ~~show that $x_{n+1} \le x_{n+2}$~~

For $n = 2$

$$x_3 = \frac{1}{2}\left(x_2 + \frac{2}{x_2}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{4}{3}\right)$$

$$= \frac{1}{2} \times \frac{17}{6} = 1.417 < x_2$$

Scanned with CamScanner

So for $n=2$
$x_3 < x_2$
True for $n=2$.
Let it is true for some n.
$x_n \geq x_{n+1}$
$\Rightarrow x_n \geq \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$
$\Rightarrow \frac{1}{2}x_n \geq \frac{1}{x_n} \Rightarrow x_n^2 \geq 2$

we wish to show

$x_{n+1} \geq x_{n+1}$ $\qquad 1 \leq x_n \leq 2$
$\Rightarrow x_{n+1}^2 \geq 2$ $\qquad \Rightarrow \frac{1}{2} \leq \frac{1}{x_n} \leq 1$

$$x_{n+1}^2 = \frac{1}{4}(x_n + 2/x_n)^2 = \frac{1}{4}\left(x_n^2 + \frac{4}{x_n^2} + 4\right)$$
$$\geq \frac{1}{4} \times \left(2 + \frac{4}{1/4} + 4\right)$$
$$= \frac{1}{4}(2+16+4) = \frac{22}{4} \geq 2$$

So, for any $n \in \mathbb{N}$, $n \geq 2$

$\cancel{x_n \geq}$ $x_{n+1} \leq x_n$
$\Rightarrow \{x_n\}$ is a monotonically decreasing sequence.

So, we have the sequence $\{x_n\}$ is monotonically decreasing and bounded below.
So, $\{x_n\}$ is a convergent sequence.

Let, $\lim_{n\to\infty} x_n = l$. $\qquad$ as $1 \leq x_n \leq 2$
$\Rightarrow \lim_{n\to\infty} x_{n+1} = l$. $\qquad \Rightarrow 1 \leq \lim x_n \leq 2$
$\Rightarrow 1 \leq l \leq 2$

Scanned with CamScanner

So, $l = \frac{1}{2}\left(l + \frac{2}{l}\right)$

$\Rightarrow 2l^2 = l^2 + 2$

$\Rightarrow l^2 = 2 \Rightarrow l = \pm\sqrt{2}$

But as $1 \le l \le 2$

$\Rightarrow l \ne -\sqrt{2}$

So, $\lim_{n\to\infty} x_n = \sqrt{2}$.

(2) ~~Let $x_1 > 0$~~

Let $x_1 > 0$ be arbitrary and define

$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right), n \in \mathbb{N}$.

~~Claim 1:- $x_n^2 \ge a \ \forall n \ge 2$.~~

~~S_n satis~~

claim 1:- $x_n^2 \ge a, \ \forall n \ge 2$.

$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$

$\Rightarrow x_n^2 - 2x_{n+1}x_n + a = 0$

This equation has a real root as $\{x_n\}$ is a sequence of real numbers.

So, The discriminant $4x_{n+1}^2 - 4a \ge 0$

$\Rightarrow x_{n+1}^2 \ge a \ \forall n \ge 1$

claim 2:- $x_{n+1} \le x_n \ \forall n \ge 2$

$x_n - x_{n+1} = x_n - \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$

$= \frac{1}{2}\left(\frac{x_n^2 - a}{x_n}\right) \ge 0 \ \forall n \ge 2$

Scanned with CamScanner

$\Rightarrow x_{n+1} \leq x_n \ \forall n \geq 2$.

As $\{x_n\}$ is monotonically decreasing and bounded below, by Monotone Convergence Theorem ~~lim~~ $\{x_n\}$ is convergent, that is $\lim x_n$ exists.

Let $\lim x_n = l$

$\Rightarrow \lim x_{n+1} = l$.

So, $l = \frac{1}{2}(l + \frac{a}{l})$

$\Rightarrow l^2 = a \Rightarrow l = \pm\sqrt{a}$

$\Rightarrow l = \sqrt{a}$ [$\because 0 < l \leq a$]

(3) $x_n = 1$, $x_{n+1} = \frac{1}{4}(2x_n + 3)$, for $n \geq 1$.

$\Rightarrow x_1 = 1$, $x_2 = \frac{1}{4}(2+3) = \frac{5}{4}$

$\Rightarrow x_1 < x_2 < 2$.

<u>Claim 1</u>:- $x_n < 2 \ \forall n$.

For $n = 1, 2$, $x_n < 2$.

Let it is true for some $k \in \mathbb{N}$, then

$x_{k+1} = \frac{1}{4}(2x_k + 3) < \frac{1}{4}(4+3) = \frac{7}{4} < 2$

$\Rightarrow x_{k+1} < 2$

$\Rightarrow x_n < 2 \ \forall n \in \mathbb{N}$.

<u>Claim 2</u>:- $x_n < x_{n+1} \ \forall n \in \mathbb{N}$

~~Let~~ ~~Let~~ True for $n = 1$

let for some $k \in \mathbb{N}$

$x_k < x_{k+1}$

Now, $x_{k+1} = \frac{1}{4}(2x_k + 3) < \frac{1}{4}(2x_{k+1} + 3)$

$= x_{k+2}$

Therefore $x_n < x_{n+1}$ $\forall n \in \mathbb{N}$.

We ~~sh~~ have the sequence $\{x_n\}$ is monotonically increasing and bounded above by 2.

So, by Monotone convergence theorem $\lim x_n$ exists, say l.

$\lim x_n = l$.

$\Rightarrow \lim x_{n+1} = l$

$\Rightarrow l = \frac{1}{2}(2l + 3)$

$\Rightarrow l = 3/2$.

(4) $x_1 = 1$, $x_{n+1} = \sqrt{2x_n}$ for $n \in \mathbb{N}$.

(5) $x_1 = 8$, $x_{n+1} = \frac{1}{2}x_n + 2$ for $n \in \mathbb{N}$.

(6) $x_1 > 1$, $x_{n+1} = 2 - 1/x_n$ for $n \in \mathbb{N}$.

(7) $x_1 \geq 2$, $x_{n+1} = 1 + \sqrt{x_n - 1}$ for $n \in \mathbb{N}$

(8) $x_1 = 1$, $x_{n+1} = \sqrt{2 + x_n}$ for $n \in \mathbb{N}$.

~~Solve (4),~~

Find the limits for 4, 5, 6, 7, 8.

Scanned with CamScanner

(9) $x_n = (1+1/n)^n \ \forall n$.

~~$x_{n+1} - x_n = (1+1/n)^n \, 1/n > 0$~~

~~$\Rightarrow x_{n+1} > x_n$~~

~~$\Rightarrow \{x_n\}$ is~~ a monotonic increasing sequence.

Applying A.M. $\geq$ G.M. on $(n+1)$ terms $1+\frac{1}{n}, 1+\frac{1}{n}, \ldots, 1+\frac{1}{n}, 1$ gives you $\Rightarrow x_{n+1} > x_n$

$$(1+1/n)^n = 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \cdots + \frac{1}{n^n}$$

$$= 1 + 1 + 1 \cdot (1-1/n)\frac{1}{2!} + 1(1-1/n)(1-2/n)\frac{1}{3!} + \cdots + \frac{1}{n^n}$$

$\Rightarrow \quad x_n \geq 2$. [$\because$ each terms are positive and less than 1]

Also, $x_n = 1 + 1 + (1-1/n)\frac{1}{2!} + (1-1/n)(1-2/n)\frac{1}{3!} + \cdots + \frac{1}{n^n}$

$$< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}$$

$$< 3$$

[$\because 3! > 2^2$, $4! > 2^3$, $n! > 2^{n-1}$ for $n \geq 3$]

$\Rightarrow 2 \leq x_n < 3 \ \forall n$.

So, $\{x_n\}$ is monotonically increasing and bounded above by 3.

So, it must be a convergent sequence.

$\Rightarrow \lim_{n\to\infty} (1+1/n)^n = l, \quad 2 \leq l < 3$.

~~Here~~ We denote this limit l by

e where $2 < e < 3$.

(9) $x_1 = 3,\ x_{n+1} = \frac{1}{4 - x_n}$.

<u>Claim 1:-</u> $x_n > 0\ \forall n$.

$x_1 = 3,\ x_2 = 1$

$x_1, x_2 > 0$

Let $x_{n+1} > 0$ we want to show $x_{n+2} > 0$

$\Rightarrow \frac{1}{4 - x_n} > 0$

$\Rightarrow$ ~~x_{n+1}~~

<u>Claim 2:-</u> $0 < x_{n+1} \le x_n$.

$x_1 = 3,\ x_2 = 1$

So, $0 < 1 \le 3 < 4$,

Suppose that $0 < x_{n+1} \le x_n$.

$\Rightarrow -x_n \le -x_{n+1} < 0$

$\Rightarrow 4 - x_n \le 4 - x_{n+1} < 4$.

$\Rightarrow 0 < \frac{1}{4} < \frac{1}{4 - x_{n+1}} \le \frac{1}{4 - x_n}$

$\Rightarrow 0 < x_{n+2} \le x_{n+1}$.

So, $\{x_n\}$ is a decreasing sequence & bounded below by 0. Therefore $\{x_n\}$ is a convergent sequence.

Let l be the limit

$\Rightarrow l = \frac{1}{4 - l} \Rightarrow 4l - l^2 = 1 \Rightarrow l^2 - 4l - 1 = 0$

$\Rightarrow l = \frac{4 \pm \sqrt{16+4}}{2}$

$= 1 \pm \sqrt{5}$

(10) $x_1 = 2$, $x_{n+1} = 2 - \frac{1}{x_n}$, $n \in \mathbb{N}$

$\Rightarrow$ Decreasing & bdd below by 0

(11) $x_1 = \sqrt{2}$, $x_{n+1} = \sqrt{2x_n}$, $n \in \mathbb{N}$

$\Rightarrow$ Increasing & bdd above by 2.

(12) $x_1 = 1$, $x_{n+1} = \frac{4+3x_n}{3+2x_n}$, $n \in \mathbb{N}$.

$\Rightarrow x_1 = 1$, $x_2 = \frac{4+3}{3+2} = \frac{7}{5} > x_1$

$x_2 > x_1$

$$x_{n+1} - x_n = \frac{x_n - x_{n-1}}{(3+2x_n)(3+2x_{n-1})}$$

By induction $x_{n+1} - x_n > 0$

$\Rightarrow \{x_n\}$ is an increasing sequence.

$$x_{n+1} = 1 + \frac{1+x_n}{3+2x_n} \leq 2$$

So, $\{x_n\}$ is increasing & bdd above.

Limit is $\sqrt{2}$.

(13) Let $0 < b_1 < a_1$ and define

$a_{n+1} = \frac{a_n + b_n}{2}$, $b_{n+1} = \sqrt{a_n b_n}$ $\forall n \in \mathbb{N}$.

Show that both $\{a_n\}$ & $\{b_n\}$ converge.

$\Rightarrow$ As ~~G.M. ≤ G.M.~~ G.M. $\leq$ A.M.

$\Rightarrow$ Applying on a_{n-1}, b_{n-1}

Scanned with CamScanner

$$\sqrt{a_{n-1}b_{n-1}} \le \frac{a_{n-1}+b_{n-1}}{2}$$

$\Rightarrow b_n \le a_n$.

Also $0 < b_n \le a_n$

Now, $0 \le a_{n+1} = \frac{a_n+b_n}{2} \le \frac{a_n+a_n}{2} = a_n$.

$\Rightarrow 0 \le a_{n+1} \le a_n$

$\Rightarrow \{a_n\}$ is a monotonically sequence bdd below by 0.

Therefore $\{a_n\}$ converges.

Also, $b_{n+1} = \sqrt{a_n b_n} \ge \sqrt{b_n b_n} = b_n$.

so, $b_n \le b_{n+1} \quad \forall n$.

Also, $b_n \le a_n \le a_1$

$\Rightarrow b_n \le a_1$

$\Rightarrow \{b_n\}$ is an increasing sequence bdd above by a_1

$\Rightarrow \{b_n\}$ is a convergent sequence.

(19) Let $\{x_n\}$ be a sequence in $(0,1)$. suppose $4x_n(1-x_{n+1}) > 1 \quad \forall n \in \mathbb{N}$. Show that the sequence is monotonic and find the limit.

Scanned with CamScanner

Applying AM-GM inequality on x_n & $(1-x_{n+1})$ we get,

$$\frac{x_n + (1-x_{n+1})}{2} \geq \sqrt{x_n(1-x_{n+1})} > 1/2$$

$$\left[4x_n(1-x_{n+1}) > 1 \Rightarrow \sqrt{x_n(1-x_{n+1})} > 1/2 \right]$$

$$\Rightarrow \frac{x_n + 1 - x_{n+1}}{2} > 1/2$$

$$\Rightarrow x_n - x_{n+1} > 0$$

$$\Rightarrow x_n > x_{n+1}$$

Also, $0 < x_n < 1 \ \forall n$.

$\Rightarrow \{x_n\}$ is a monotonically decreasing & bounded below.

$\Rightarrow \{x_n\}$ is a convergent sequence.

Let $\lim_{n\to\infty} x_n = x_0$

[$\because$ ~~If~~ as $x_n < y_n \ \forall n$ $\Rightarrow \lim x_n \leq \lim y_n$]

Then, $4x_0(1-x_0) \geq 1$.

$$\Rightarrow 4x_0 - 4x_0^2 - 1 \geq 0$$

$$\Rightarrow x_0^2 - x_0 + 1/4 \leq 0$$

$$\Rightarrow x_0^2 - 2 \cdot 1/2 \cdot x_0 + 1/4 \leq 0$$

$$\Rightarrow (x_0 - 1/2)^2 \leq 0$$

$\cancel{\Rightarrow x_0 = 1/2 \ [\because 0 < x_n < 1}$

$$\Rightarrow (x_0 - 1/2)^2 = 0 \quad [\because (x_0 - 1/2)^2 \not< 0 \text{ as } (x_0 - 1/2)^2 > 0]$$

$$\Rightarrow x_0 = 1/2$$

Scanned with CamScanner

(15) Show that the sequence $\{a_n\}$, where

$$a_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!}$$ is convergent.

$$\Rightarrow a_{n+1} - a_n = \frac{1}{(n+1)!} > 0$$

$$\Rightarrow a_n < a_{n+1} \quad \forall n$$

So, $\{a_n\}$ is m.i.

Also, $n! = n.(n-1)(n-2)\cdots 3.2.1$

$$> 2^{n-1} \quad \forall n \geq 2$$

$$\therefore \frac{1}{n!} \leq \frac{1}{2^{n-1}} \quad \forall n \geq 2$$

$$a_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

$$< 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$$

$$< 1 \cdot \frac{1-(1/2)^n}{1-1/2} < 2 \cdot \left(1 - \frac{1}{2^n}\right)$$

$$\Rightarrow a_n < 2 \quad \forall n$$

So, $\{a_n\}$ is bdd above.

Therefore $\{a_n\}$ is convergent.

(16) Let $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$

Show that $\{x_n\}$ is convergent.

Scanned with CamScanner

$x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} < n \cdot \frac{1}{n} = 1$.

$\Rightarrow$ so, $\{x_n\}$ is bounded above.

$x_{n+1} = \frac{1}{n+1+1} + \frac{1}{n+1+2} + \cdots + \frac{1}{n+1+n}$.

$x_{n+1} - x_n = \frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1}$

$= \frac{1}{2n+1} - \frac{1-2}{(2n+2)(n+1)}$

$= \frac{1}{2n+1} - \frac{1}{2n+2}$

$= \frac{1}{(2n+1)(2n+2)} > 0$.

$\therefore \{x_n\}$ is monotonically increasing.

Therefore $\{x_n\}$ is convergent.

(17) $u_1 = \sqrt{3}$, $u_{n+1} = \sqrt{3u_n}$ $\forall n \geq 1$

(18) $u_1 > 0$, $u_{n+1} = \sqrt{6+u_n}$, $n \geq 1$.

(i) show that $\{u_n\}$ is m.i if $0 < u_1 < 3$

(i) $u_2 = \sqrt{6+u_1}$

$u_2^2 - u_1^2 = 6 + u_1 - u_1^2$

$= (3-u_1)(2+u_1) > 0$

$\therefore u_2^2 > u_1^2$ [~~since~~ $\because 0 < u_1 < 3$]

i.e., $u_2 > u_1$ [$\because u_2 > 0, u_1 > 0$]

also, $u_2 = \sqrt{6+u_1} > \sqrt{6+3} = 3$

$\therefore u_k < u_{k+1}$ and $u_k < 3$

$u_{k+1} < \sqrt{6+u_k} < \sqrt{6+3} = 3$

Scanned with CamScanner

$\therefore u_{k+2} = \sqrt{6+u_{k+1}}$

$u_{k+2}^2 - u_{k+1}^2 = 6+u_{k+1} - u_{k+1}^2$

$= (3-u_{k+1})(2+u_{k+1}) > 0$

$[\because 0 < u_{k+1} < 3]$

$\therefore u_{k+2}^2 > u_{k+1}^2$

$\therefore u_{k+2} > u_{k+1}$ $[u_{k+1} > 0,\ u_{k+2} > 0]$

$\therefore u_n < u_{n+1}$ and $u_n < 3$

$\therefore \{u_n\}$ is monotonically increasing and bounded above. ~~$\therefore$~~

So, $\{u_n\}$ is convergent.

Suppose $\lim u_n = l$.

Now, $u_{n+1} = \sqrt{6+u_n}$.

$\therefore u_{n+1}^2 = 6+u_n$.

$\therefore \lim_{n\to\infty} u_{n+1}^2 = \lim_{n\to\infty} (6+u_n)$

i.e., $l^2 = 6+l$.

or, ~~$l^2=3$~~ $l^2 - l - 6 = 0$

or, $(l-3)(l+2) = 0$

$\Rightarrow l = 3, -2$.

Since $u_1 > 0$, and $\{u_n\}$ is monotonically increasing. $\therefore \lim_{n\to\infty} u_n \neq -2$

So, $\lim u_n = 3$.

Scanned with CamScanner

2nd Step:- (ii) Show that $\{u_n\}$ is m.d. if $u_1 > 3$.

(i) Again

$u_2 = \sqrt{6+u_1}$

$\therefore u_2^2 - u_1^2 = 6 + u_1 - u_1^2 = -(u_1^2 - 6 - u_1)$

$= \cancel{-[(3+u_1)(2}$ $= -[u_1^2 - 3u_1 + 2u_1 - 6]$

$= -[u_1(u_1-3) + 2(u_1-3)]$

$= -(u_1+2)(u_1-3) < 0$

If $u_1 - 3 \cancel{>} 0$

$\Rightarrow u_1 \cancel{>} 3$

$\therefore u_2^2 < u_1^2$

$\Rightarrow u_2 < u_1$.

Also, $u_2 = \sqrt{6+u_1} > \sqrt{6+3} = 3$

$\therefore u_k > u_{k+1}$ and $u_k > 3$.

$u_{k+1} = \sqrt{6+u_k} > \sqrt{6+3} = 3$

$u_{k+2} = \sqrt{6+u_{k+1}}$

$\therefore u_{k+2}^2 - u_{k+1}^2 = 6 + u_{k+1}^2 - u_{k+1}^2$

$= (3 - u_{k+1})(2 + u_{k+1}) < 0$

i.e., $u_{k+2}^2 - u_{k+1}^2 < 0$ $[\because u_{k+1} < 3]$

i.e., $u_{k+2} < u_{k+1}$ $[\because u_{k+1} > 0, u_{k+2} > 0$

$\therefore u_n > u_{n+1}$ $\forall n$ & $u_n > 3$ $\forall n$.

$\therefore \{u_n\}$ is monotonically decreasing & bounded below.

So, $\{u_n\}$ is convergent.

Scanned with CamScanner

Let $\lim u_n = l$.

$\Rightarrow l^2 = 6 + l$

$\Rightarrow l = 3, -2$

~~Since $u_n > 3$, $\forall n$~~ ~~Si~~

~~$\Rightarrow \lim_{n\to\infty}$~~

Since $u_n > 3$ $\forall n$, $\lim u_n \neq -2$

$\Rightarrow \lim u_n = 3$.

(13) A sequence $\{u_n\}$ is defined by $u_1 > 0$ and $u_{n+1} = \frac{3(1+u_n)}{5+u_n}$ for $n \geq 1$. Prove that

(i) the sequence $\{u_n\}$ is a decreasing sequence if $u_1 > 1$

(ii) the sequence $\{u_n\}$ is an increasing sequence if $0 < u_1 < 1$.

$\Rightarrow$ (i) $u_1 > 0$, $u_{n+1} = \frac{3(1+u_n)}{5+u_n}$ $(n \geq 1)$

suppose $u_1 > 1$

$\Rightarrow 2u_1 > 2$

$\Rightarrow 3u_1 > 2 + u_1$

$3 + 3u_1 > 5 + u_1$

$\Rightarrow \frac{3(1+u_1)}{5+u_1} > 1$

$\Rightarrow u_2 > 1$.

$u_1 > 1$ to show $u_2 > 1$

$u_2 > 1$ means

$3(1+u_1) > 5 + u_1$

$\Rightarrow 2u_1 > 2$

$\Rightarrow u_1 > 1$

Scanned with CamScanner

similarly if $u_k > 1$

then $u_{k+1} > 1$.

so, ~~if~~ if $u_1 > 1$ then $u_n > 1 \ \forall n$.

$\Rightarrow \{u_n\}$ is bounded below.

- Next step to show $u_{n+1} < u_n \ \forall n$.

$u_1 > 1$

$\Rightarrow u_1 - 1 > 0$

$\Rightarrow (u_1+3)(u_1-1) > 0$

$\Rightarrow u_1^2 + 2u_1 - 3 > 0$

$\Rightarrow u_1^2 > 3 - 2u_1$

or, $u_1^2 + 5u_1 > 3 + 3u_1$

$\Rightarrow \dfrac{3(1+u_1)}{u_1+5} < u_1$

$\Rightarrow u_2 < u_1$

~~As~~ Back calculation

$u_2 < u_1$

$\Rightarrow \dfrac{3(1+u_1)}{u_1+5} < u_1$

$\Rightarrow 3 + 3u_1 < u_1^2 + 5u_1$

$\Rightarrow u_1^2 > 3 - 2u_1$

$\Rightarrow u_1^2 + 2u_1 - 3 > 0$

$\Rightarrow (u_1+3)(u_1-1) > 0$.

$\Rightarrow (u_1 - 1) > 0$

$\Rightarrow u_1 > 1$,

So, true for $n = 1$

Let it is for some $k \in \mathbb{N}$, i.e, ~~$u_{k+1} < u_{k-1}$~~

$u_{k+1} < u_k$

To show ~~$u_{k+2} < u_{k+1}$~~

$u_{k+2} < u_{k+1}$.

$u_{k+1} > 1$.

similarly you can show

$u_{k+2} < u_{k+1}$.

$u_{k+2} < u_{k+1}$

$\Rightarrow \dfrac{3(1+u_{k+1})}{5+u_{k+1}} < u_{k+1}$

$\Rightarrow 3 + 3u_{k+1} < 5u_{k+1} + u_{k+1}^2$

$\Rightarrow u_{k+1}^2 + 2u_{k+1} - 3 > 0$

$\Rightarrow (u_{k+1}+3)(u_{k+1}-1) > 0$

So, $u_{n+1} < u_n \ \forall n$.

$\Rightarrow \{u_n\}$ is monotonically decreasing & bounded below. Therefore $\{u_n\}$ converges, if $u_1 > 1$.

~~let lim~~ Let $\lim u_n = l$.

Scanned with CamScanner

So, $l = \dfrac{3(1+l)}{5+l}$

$\Rightarrow 5l + l^2 = 3 + 3l$

$\Rightarrow l^2 + 2l - 3 = 0$

$\Rightarrow (l+3)(l-1) = 0 \Rightarrow l = 1, -3$.

Since $u_n > 0 \ \forall n$

$\Rightarrow \lim u_n \neq -3$

So, $\lim u_n = 1$.

Solve (ii) in a similar way.

(20) Prove that the sequence $\{u_n\}$ defined by

(i) $0 < u_1 < u_2$ and $u_{n+2} = \dfrac{2u_{n+1} + u_n}{3}$ for $n \geq 1$, converges to $\dfrac{u_1 + 3u_2}{4}$.

$$\Rightarrow \quad u_3 - u_2 = \frac{2u_2 + u_1}{3} - u_2$$

$$= \frac{u_1 - u_2}{3}$$

$$= (-1/3)(u_2 - u_1)$$

$$u_4 - u_3 = \frac{2u_3 + u_2}{3} - u_3$$

$$= -1/3\,(u_3 - u_2)$$

$$= (-1/3)^2 (u_2 - u_1)$$

$$u_5 - u_4 = (-1/3)^3 (u_2 - u_1)$$

$$\vdots$$

$$u_n - u_{n-1} = (-1/3)^{n-2} (u_2 - u_1)$$

Adding

$$u_n - u_2 = \left\{\left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^3 + \cdots + \left(-\frac{1}{3}\right)^{n-2}\right\}(u_2 - u_1).$$

$$= \left(-\frac{1}{3}\right) \frac{1-\left(-\frac{1}{3}\right)^{n-2}}{1-\left(-\frac{1}{3}\right)} (u_2 - u_1)$$

$$= -\frac{1}{3} \cdot \frac{3}{4} \left\{1 - \left(-\frac{1}{3}\right)^{n-2}\right\} (u_2 - u_1)$$

$$\to -\frac{1}{4}(u_2 - u_1) \text{ as } n \to \infty.$$

$$\therefore \lim_{n\to\infty} (u_n - u_2) = -\frac{1}{4}(u_2 - u_1)$$

$$\therefore \lim_{n\to\infty} u_n = -\frac{1}{4}(u_2 - u_1) + u_2$$

$$= \frac{u_1 + 3u_2}{4}$$

(ii) $0 < u_1 < u_2$ and $u_{n+2} = \dfrac{u_{n+1} + 2u_n}{3}$ for $n \geq 1$, converges to $\dfrac{2u_1 + 3u_2}{5}$.

$$u_3 - u_2 = \frac{u_2 + 2u_1}{3} - u_2 = \frac{2u_1 - 2u_2}{3} = -\frac{2}{3}(u_2 - u_1)$$

$$u_4 - u_3 = \frac{u_3 + 2u_2}{3} - u_3 = \frac{2u_2 - 2u_3}{3} = \left(-\frac{2}{3}\right)^2 (u_2 - u_1)$$

$$u_5 - u_4 = \frac{u_4 + 2u_3}{3} - u_4 = \frac{2u_3 - 2u_4}{3} = \left(-\frac{2}{3}\right)^3 (u_2 - u_1)$$

$$\vdots$$

$$= \left(-\frac{2}{3}\right)^{n-2} (u_2 - u_1)$$

$$u_n - u_{n-1} =$$

Adding

$$u_n - u_2 = \left\{\left(-\frac{2}{3}\right) + \left(-\frac{2}{3}\right)^2 + \cdots + \left(-\frac{2}{3}\right)^{n-2}\right\}(u_2 - u_1)$$

$$= \left\{\left(-\frac{2}{3}\right) \cdot \frac{1-\left(-\frac{2}{3}\right)^{n-2}}{1-\left(-\frac{2}{3}\right)}\right\}(u_2 - u_1)$$

$$\to \left(-\frac{2}{3}\right) \frac{3}{5}(u_2 - u_1) \text{ as } n \to \infty$$

$$\to -\frac{2}{5}(u_2 - u_1) \text{ as } n \to \infty$$

Scanned with CamScanner

$\therefore \lim_{n\to\infty}(u_n - u_2) = -2/5\,(u_2 - u_1)$

$\Rightarrow \lim u_n = \dfrac{2u_1 + 3u_2}{5}$

iii) $0 < u_1 < u_2$ and $u_{n+2} = \sqrt{u_{n+1}u_n}$ for $n \geq 1$, converges to $\sqrt[3]{u_1 u_1^2}$.

$u_3 = \sqrt{u_2 u_1}$ $\quad \dfrac{u_3}{u_2} = \sqrt{\dfrac{u_1}{u_2}} = \left(\dfrac{u_2}{u_1}\right)^{-1/2}$

$u_4 = \sqrt{u_3 u_2}$, $\quad \dfrac{u_4}{u_3} = \sqrt{\dfrac{u_2}{u_3}} = \left(\dfrac{u_2}{u_1}\right)^{(-1/2)^2}$

$\vdots$

$\dfrac{u_n}{u_{n-1}} = \left(\dfrac{u_2}{u_1}\right)^{(-1/2)^{n-2}}$

Multiplying $\dfrac{u_n}{u_2} = \left(\dfrac{u_2}{u_1}\right)^{(-1/2)+(-1/2)^2+\cdots+(-1/2)^{n-2}}$

$= \left(\dfrac{u_2}{u_1}\right)^{(-1/2)\cdot\frac{1-(-1/2)^{n-2}}{1-(-1/2)}}$

$\longrightarrow \left(\dfrac{u_2}{u_1}\right)^{-1/2\cdot 2/3}$ as $n\to\infty$

$\longrightarrow \left(\dfrac{u_2}{u_1}\right)^{-1/3}$

$\therefore \lim_{n\to\infty}\dfrac{u_n}{u_2} = \left(\dfrac{u_2}{u_1}\right)^{1/3}$

$\Rightarrow \lim_{n\to\infty} u_n = u_2 \cdot \left(\dfrac{u_1}{u_2}\right)^{1/3}$

$= \sqrt[3]{u_1 u_2^2}$

Scanned with CamScanner

(iv) $0 < u_1 < u_2$ and $\frac{2}{u_{n+2}} = \frac{1}{u_{n+1}} + \frac{1}{u_n}$ for $n \geq 1$, converges to $3/(1/u_1 + 2/u_2)$

$$\Rightarrow \frac{2}{u_3} = \frac{1}{u_2} + \frac{1}{u_1}$$

$$\Rightarrow \frac{2}{u_3} - \frac{2}{u_2} = \frac{1}{u_2} + \frac{1}{u_1} - \frac{2}{u_2}$$

$$\Rightarrow 2\left(\frac{1}{u_3} - \frac{1}{u_2}\right) = \left(\frac{1}{u_2} - \frac{1}{u_1}\right)$$

$$\Rightarrow \frac{1}{u_2} - \frac{1}{u_3} = (-1/2)\left(\frac{1}{u_1} - \frac{1}{u_2}\right)$$

$$\Rightarrow \frac{1}{u_3} - \frac{1}{u_2} = (-1/2)\left(\frac{1}{u_2} - \frac{1}{u_1}\right)$$

$$\cancel{\frac{2}{u_4} = \frac{1}{u_3} + \frac{1}{u_2}}$$

$$\cancel{= \frac{1}{u_2} +}$$

$$\frac{2}{u_4} = \frac{1}{u_3} + \frac{1}{u_2}$$

$$\frac{2}{u_4} - \frac{2}{u_3} = \frac{1}{u_2} - \frac{1}{u_3}$$

$$\Rightarrow 2\left(\frac{1}{u_4} - \frac{1}{u_3}\right) = (+1/2)\left(\frac{1}{u_2} - \frac{1}{u_1}\right)$$

$$\Rightarrow \left(\frac{1}{u_4} - \frac{1}{u_3}\right) = (-1/2)^2\left(\frac{1}{u_2} - \frac{1}{u_1}\right)$$

$$\vdots$$

$$\left(\frac{1}{u_n} - \frac{1}{u_{n-1}}\right) = (-1/2)^{n-2}\left(\frac{1}{u_2} - \frac{1}{u_1}\right)$$

Adding

$$\frac{1}{u_n} - \frac{1}{u_2} = \left\{(-1/2) + (-1/2)^2 + \cdots + (-1/2)^{n-2}\right\}\left(\frac{1}{u_2} - \frac{1}{u_1}\right)$$

$$= (-1/2)\frac{1 - (-\frac{1}{2})^{n-2}}{1 - (-1/2)}\left(\frac{1}{u_2} - 1/u_1\right)$$

Scanned with CamScanner

$$\lim\left(\frac{1}{u_n}-\frac{1}{u_2}\right)=\left(-\tfrac{1}{2}\right)\left(\tfrac{2}{3}\right)\left(\frac{1}{u_2}-\frac{1}{u_1}\right)$$

$$=-\frac{1}{3}\left(\frac{1}{u_2}-\frac{1}{u_1}\right)$$

$$\therefore \lim\frac{1}{u_n}=\frac{1}{u_2}-\frac{1}{3u_2}+\frac{1}{3u_1}$$

$$=\frac{2}{3u_2}+\frac{1}{3u_1}$$

$$\therefore \lim u_n=\frac{3}{(1/u_1+2/u_2)}$$

(21) If $s_1>0$ and $s_{n+1}=\frac{1}{2}\left(s_n+\frac{4}{s_n}\right)$ for $n\geq 1$, Prove that the sequence is monotone decreasing bounded below and $\lim s_n=2$.

(22) Prove that the sequences $\{x_n\}$ & defined by

$$\cancel{x_{n+1}=\frac{1}{2}(x_n+y_n),\ \frac{2}{y_{n+1}}}$$

$$x_{n+1}=\sqrt{x_n y_n} \text{ and}$$

$$\frac{2}{y_{n+1}}=\frac{1}{x_n}+\frac{1}{y_n} \text{ for } n\geq 1,\ x_1>0, y_1$$

converges to a common limit l, where $l^2=x_1y_1$.

$\Rightarrow$ <u>Case I:</u> Let $x_1\neq y_1$

then $x_2=\sqrt{x_1y_1}$

and $\frac{2}{y_2}=\frac{1}{x_1}+\frac{1}{y_1}$

Scanned with CamScanner

Since $\sqrt{x_1 y_1} < \dfrac{2}{\frac{1}{x_1} + \frac{1}{y_1}}$

$\therefore x_2 < y_2$

$x_3 = \sqrt{x_2 y_2} > \sqrt{x_2 \cdot x_2} = x_2$

and $\dfrac{2}{y_3} = \dfrac{1}{x_2} + \dfrac{1}{y_2} > \dfrac{1}{y_2} + \dfrac{1}{y_2} = \dfrac{2}{y_2}$

$\therefore y_3 < y_2$

$x_3 = \sqrt{x_2 y_2} < \dfrac{2}{\frac{1}{x_2} + \frac{1}{y_2}} = y_3$

So, we have

$x_2 < x_3 < y_3 < y_2$

Continuing this proceeding,

we have, ~~$x_2 < x_3 < \cdots < y_{n+1} < y_{n+1}$~~

~~So, $\{x_n\}$ is m.i. and bounded above~~

$x_2 < x_3 < \cdots < y_3 < y_2 \quad \forall n.$

Therefore the sequence $\{x_n\}$ is m.i. and bounded above and the sequence $\{y_n\}$ is m.d. and bounded below.

Hence both the sequence converges.

Suppose $\lim x_n = l$ & $\lim y_n = m$

Since $x_{n+1} = \sqrt{x_n y_n}$

$\therefore x_{n+1}^2 = x_n y_n.$

$\Rightarrow l^2 = lm$

$\Rightarrow l(l-m) = 0$

$\because l \neq 0, \; l = m.$

Scanned with CamScanner

Therefore the sequence $\{x_n\}$ & $\{y_n\}$ converge to common limit.

<u>Case II</u>:- Let $x_1 = y_1$.

In this case $x_n = y_n = x$, $\forall n \in \mathbb{N}$.

Therefore $\{x_n\}$ & $\{y_n\}$ both converge to the same limit x,

(23) Prove that the sequences $\{x_n\}$ and $\{y_n\}$ defined by

$x_{n+1} = \frac{1}{2}(x_n + y_n)$, $\frac{2}{y_{n+1}} = \frac{1}{x_n} + \frac{1}{y_n}$ for $n \geq 1$,

$x_1 > 0$, $y_1 > 0$, converge to a common limit l where $l^2 = x_1 y_1$.

Scanned with CamScanner