Ring - Theory

Ring :- let R be a non empty set. $(R,+,\cdot)$ is said to be a ring if

(i) $(R,+)$ is commutative group :-

(i) $\forall a \in R, \forall b \in R \Rightarrow a+b \in R$

(ii) $a+(b+c) = (a+b)+c, \forall a,b,c \in R$

(iii) $e \in R$ s.t. $a+e = e+a = a \ \forall a \in R$

(iv) for each $a \in R$ $\exists$ $b \in R$ s.t. $a+b = b+a = e$

(v) $a+b = b+a, \forall a,b \in R$

B) $(R,\cdot)$ is semi-group :-

(i) $\forall a \in R, \forall b \in R \Rightarrow a \cdot b \in R$

(ii) $a \cdot (b \cdot c) = (a \cdot b) \cdot c, \forall a,b,c \in R$

c) left and Right distribution law :-

(i) $a \cdot (b+c) = a \cdot b + a \cdot c \ \forall a,b,c \in R$

(ii) $(a+b) \cdot c = a \cdot c + b \cdot c \ \forall a,b,c \in R$

Q: Show that $(Z', +, \cdot)$ is Ring.

Soln :- (A) $(Z',+)$ is commutative group

$(Z',+)$ is cyclic group then $(Z',+)$ is commutative.

B) $(Z',\cdot)$ is semigroup

(i) let $a,b \in Z'$ s.t. $a \cdot b \in Z'$

(ii) $a \cdot (b \cdot c) = abc = (a \cdot b) \cdot c$

$\Rightarrow a \cdot (b \cdot c) = (a \cdot b) \cdot c, \ \forall a,b,c \in Z'$

c) left and Right distribution

(i) $a \cdot (b+c) = a \cdot b + a \cdot c$

(ii) $(a+b) \cdot c = a \cdot c + b \cdot c \ \forall a,b,c \in Z'$

then $(\mathbb{Z},+,\cdot)$ is Ring.

Similary $(\mathbb{Q},+,\cdot)$, $(\mathbb{R},+,\cdot)$, $(\mathbb{C},+,\cdot)$, $(\mathbb{Z}_n,+,\cdot)$ are Ring.

Ques:- Show that $(\mathbb{Z}\times\mathbb{Z},+,\cdot)$ is ring.

Soln:- $\mathbb{Z}\times\mathbb{Z} = \{(a,b) / a\in\mathbb{Z}, b\in\mathbb{Z}\}$

(A) $(\mathbb{Z}\times\mathbb{Z},+)$ is commutative group

(I) let $x=(a,b)\in\mathbb{Z}\times\mathbb{Z}$ and $y=(c,d)\in\mathbb{Z}\times\mathbb{Z}$

s.t. $x+y = (a,b)+(c,d)$

$= (a+c, b+d) \in \mathbb{Z}\times\mathbb{Z}$

$\Rightarrow x+y \in \mathbb{Z}\times\mathbb{Z}$

$\Rightarrow x+y \in \mathbb{Z}\times\mathbb{Z}$, $\forall\ x,y \in \mathbb{Z}\times\mathbb{Z}$

[$a\in\mathbb{Z}$, $b\in\mathbb{Z}$ and $(\mathbb{Z},+)$ is group then $a+c\in\mathbb{Z}$

(II) $x+(y+z) = (x+y)+z = x+y+z$

$\forall\ x,y,z \in \mathbb{Z}\times\mathbb{Z}$

(III) $e=(0,0)\in\mathbb{Z}\times\mathbb{Z}$ s.t.

$x+e = (a,b)+(0,0) = (a+0, b+0) = (a,b) = x$, $\forall\ x\in\mathbb{Z}\times\mathbb{Z}$

(IV) let $x=(a,b)\in\mathbb{Z}\times\mathbb{Z}$, $a\in\mathbb{Z}$, $b\in\mathbb{Z}$

$a\in\mathbb{Z}$, and $\mathbb{Z}$ is a group then $-a\in\mathbb{Z}$

Similarly $-b\in\mathbb{Z}$ then $-x=(-a,-b)\in\mathbb{Z}\times\mathbb{Z}$

~~s.t. $-x=(-a,-b)\in\mathbb{Z}\times\mathbb{Z}$~~

s.t. $x+(-x) = (-x)+x = (0,0)$

(V) let $x=(a,b)\in\mathbb{Z}\times\mathbb{Z}$ and $y=(c,d)\in\mathbb{Z}\times\mathbb{Z}$

s.t. $x+y = (a,b)+(c,d)$

$\Rightarrow x+y = (a+c, b+d)$

$= (c+a, d+b)$

$= (c,d)+(a,b)$

$= y+x$

$\Rightarrow x+y = y+x$, $\forall\ x,y\in\mathbb{Z}\times\mathbb{Z}$

then $(\mathbb{Z}\times\mathbb{Z},+)$ is commutative group.

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(B) $(\mathbb{Z}\times\mathbb{Z}, \cdot)$ is Semigroup

[$\mathbb{Z}$ means Semi direct product]

(i) let $x=(a,b)\in\mathbb{Z}\times\mathbb{Z}$
& $y=(c,d)\in\mathbb{Z}\times\mathbb{Z}$
s.t. $x\cdot y=(a,b)\cdot(c,d)$
$=(a\cdot c, b\cdot d)$
$\Rightarrow x\cdot y\in\mathbb{Z}\times\mathbb{Z}$

($a\in\mathbb{Z}$, $b\in\mathbb{Z}$ and $(\mathbb{Z},+,\cdot)$ is ring then $a\cdot c\in\mathbb{Z}$
similarly $b\cdot d\in\mathbb{Z}$

(ii) $x\cdot(y\cdot z)=(x\cdot y)\cdot z=x\cdot y\cdot z$
$\forall\ x,y,z\in\mathbb{Z}\times\mathbb{Z}$

c) left and Right distribution law :-

① $x\cdot(y+z)=x\cdot y+x\cdot z$

② $(x+y)\cdot z=x\cdot z+y\cdot z$, $\forall\ x,y,z\in\mathbb{Z}\times\mathbb{Z}$

then $(\mathbb{Z}\times\mathbb{Z},+,\cdot)$ is ring.

similarly, $(\mathbb{Q}\times\mathbb{Q},+,\cdot)$, $(\mathbb{R}\times\mathbb{R},+,\cdot)$, $(\mathbb{C}\times\mathbb{C},+,\cdot)$, $(\mathbb{Z}_n\times\mathbb{Z}_n,+,\cdot)$, $(\mathbb{Z}\times\mathbb{Q},+,\cdot)$, $(\mathbb{Q}\times\mathbb{R},+,\cdot)$, $(\mathbb{Q}\times\mathbb{C},+,\cdot)$, $(\mathbb{Z}\times\mathbb{Q}\times\mathbb{R},+,\cdot)$ are Rings.

Ques No. :- Show that $(M_n(\mathbb{R}),+,\cdot)$ is ring.

Soln $M_n(\mathbb{R})=\{[a_{ij}]_{n\times n} \mid a_{ij}\in\mathbb{R}\}$

Case 1 :- If $n=1$ then
$M_1(\mathbb{R})=\{[a_{ij}]_{1\times1} \mid a_{ij}\in\mathbb{R}\}$
$\approx\mathbb{R}$

Case 2 :- If $n\geq2$ then
$M_n(\mathbb{R})=\{[a_{ij}]_n \mid a_{ij}\in\mathbb{R}\}$

Now, let $n=2$, then $M_2(\mathbb{R})=\left\{\begin{bmatrix}a & b\\ c & d\end{bmatrix} \mid a,b,c,d\in\mathbb{R}\right\}$

Show that $(M_2(\mathbb{R}),+,\cdot)$ is ring

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(A) $(M_2(\mathbb{R}), +)$ is Commutative group :-

(I) let $x = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in M_2(\mathbb{R})$, $a_1, b_1, c_1, d_1 \in \mathbb{R}$

and $y = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in M_2(\mathbb{R})$ $a_2, b_2, c_2, d_2 \in \mathbb{R}$

s.t. $x+y = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix} \in M_2(\mathbb{R})$

$\Rightarrow x+y \in M_2(\mathbb{R})$

(II) $x+(y+z) = x+y+z = (x+y)+z$, $\forall\, x, y, z \in M_2(\mathbb{R})$

(III) $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in M_2(\mathbb{R})$ s.t. $x+0 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} = x$

$\Rightarrow x+0 = x$, $\forall\, x \in M_2(\mathbb{R})$

(IV) let $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R})$ then $-x = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} \in M_2(\mathbb{R})$

s.t. $x+(-x) = (-x)+x = 0$

(V) let $x = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in M_2(\mathbb{R})$ and $y = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in M_2(\mathbb{R})$ s.t.

$x+y = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix} = \begin{bmatrix} a_2+a_1 & b_2+b_1 \\ c_2+c_1 & d_2+d_1 \end{bmatrix}$

$a_1 \in \mathbb{R}$, $a_2 \in \mathbb{R}$ and $\mathbb{R}$ is abelian group w.r.t. addition. then $a_1+a_2 = a_2+a_1$

then $x+y = y+x$, $\forall\, x, y \in M_2(\mathbb{R})$

then $(M_2(\mathbb{R}), +)$ is commutative group.

(B) $(M_2(\mathbb{R}), \cdot)$ is Semigroup :-

let $x = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in M_2(\mathbb{R})$ and $y = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in M_2(\mathbb{R})$

s.t. $x \cdot y = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1a_2+b_1c_2 & a_1b_2+b_1d_2 \\ c_1a_2+d_1c_2 & c_1b_2+d_1d_2 \end{bmatrix}$

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(ii) $x\cdot(y\cdot z) = xyz = (x\cdot y)\cdot z$ $\forall\ x, y, z \in M_2(\mathbb{R})$

(C) left and Right distribution law.

(1) $x\cdot(y+z) = x\cdot y + x\cdot z$

(11) $(x+y)z = x\cdot z + y\cdot z$, $\forall\ x, y, z \in M_2(\mathbb{R})$

then $(M_2(\mathbb{R}), +, \cdot)$ is Ring.

Similarly:- $(M_n(\mathbb{R}), +, \cdot)$ is Ring.

Note:- $(M_n(\mathbb{R}), +)$ is abelian group but not cyclic.

(11) $(M_n(\mathbb{Z}), +)$ is cyclic ?

Ans Need Not

(1) If $n=1$ then $M_1(\mathbb{Z}) \approx \mathbb{Z}$ and $(\mathbb{Z}, +)$ is cyclic.
then $(M_1(\mathbb{Z}))$ is cyclic.

(2) if $n \geq 2$ then $M_n(\mathbb{Z})$ is not cyclic.

Q. No. $(M_n(\mathbb{Z}_p), +, \cdot)$ is cyclic? Need not

Soln $M_n(\mathbb{Z}_p) = \{ [a_{ij}]_{n\times n} \mid a_{ij} \in \mathbb{Z}_p \}$

Case 1: at $n=1$ then

$$M_1(\mathbb{Z}_p) = \{ [a_{ij}]_{1\times 1} \mid a_{ij} \in \mathbb{Z}_p \} \approx \mathbb{Z}_p$$

and $\mathbb{Z}_p$ is cyclic. Hence $M_1(\mathbb{Z}_p)$ is cyclic.

Case 2 $n \geq 2$ then

$M_n(\mathbb{Z}_p)$ is not cyclic

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Q. $(M_2(\mathbb{R}), +)$ is cyclic?

Soln No;

$M_2(\mathbb{R}) = \{ A = [a_{ij}]_{2\times 2} \mid a_{ij} \in \mathbb{R} \}$

$M_2(\mathbb{R})$ is not abelian, $\therefore$ not cyclic.

($\because$ it does not satisfy the commutative prop.)

i.e.

$$a = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$ab = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$ba = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$ab \neq ba$

$\therefore$ $M_2(\mathbb{R})$ is not abelian $\Rightarrow$ not cyclic.

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Gaussian Integers:-

$\mathbb{Z}[i] = \{a+ib \mid a \in \mathbb{Z}, b \in \mathbb{Z}\}$ is called set of Gaussian Integers.

Now, $O(\mathbb{Z}[i]) = \infty$

Ques: Show that $(\mathbb{Z}[i], +, \cdot)$ is ring

Soln $\mathbb{Z}[i] = \{a+ib \mid a \in \mathbb{Z}, b \in \mathbb{Z}\}$

(A) $(\mathbb{Z}[i], +)$ is commutative group:-

(i) let $x = a_1 + ib_1 \in \mathbb{Z}[i]$

& $y = a_2 + ib_2 \in \mathbb{Z}[i]$

s.t. $x+y = (a_1+a_2) + i(b_1+b_2) \in \mathbb{Z}[i]$

$\Rightarrow x+y \in \mathbb{Z}[i]$

(ii) $x+(y+z) = x+y+z = (x+y)+z \quad \forall\, x, y, z \in \mathbb{Z}[i]$

(iii) $0 = 0+0i \in \mathbb{Z}[i]$ s.t. $x+0 = (a_1+ib_1) + (0+0i)$

$= a_1 + 0 + ib_1 + 0$

$= a_1 + ib_1$

$= x, \quad \forall\, x \in \mathbb{Z}[i]$

(iv) let $x = a_1 + ib_1 \in \mathbb{Z}[i]$, $a \in \mathbb{Z}$, $b \in \mathbb{Z}$

then $-x = (-a_1) + i(-b_1) \in \mathbb{Z}[i]$

s.t. $x + (-x) = -x + x = 0$

(v) let $x = a_1 + ib_1 \in \mathbb{Z}[i]$, $a_1 \in \mathbb{Z}$ and $b \in \mathbb{Z}$

and $y = a_2 + ib_2 \in \mathbb{Z}[i]$, $a_2 \in \mathbb{Z}$, $b_2 \in \mathbb{Z}$

s.t. $x+y = (a_1+ib_1) + (a_2+ib_2)$

$= (a_1+a_2) + i(b_1+b_2)$

$= (a_2+a_1) + i(b_2+b_1)$

$= (a_2+ib_2) + (a_1+ib_1)$

$x+y = y+x, \quad \forall\, x, y \in \mathbb{Z}[i]$

then $(\mathbb{Z}[i], +)$ is ...

Q. [illegible] B) $(Z'[i], \cdot)$ is Semi-Group &-

Soln:- (i) let $x = a_1 + ib_1 \in Z'[i]$, $a_1, b_1 \in Z$

& $y = a_2 + ib_2 \in Z'[i]$, $a_2, b_2 \in Z$

s.t. $x \cdot y = (a_1 + ib_1)(a_2 + ib_2)$

$= (a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2) \in Z'[i]$

$\Rightarrow x \cdot y \in Z'[i]$

(ii) $x \cdot (y \cdot z) = xyz = (x \cdot y) \cdot z$, $\forall\ x, y, z \in Z'[i]$

c) left and Right distribution law :-

$x \cdot (y+z) = x \cdot y + x \cdot z$

(ii) $(x+y) \cdot z = xz + y \cdot z$, $\forall\ x, y, z \in Z'[i]$

then $(Z'[i], +, \cdot)$ is ring.

Gaussian Integer Modulo n :-

$Z_n[i] = \{ a+ib \mid a \in Z_n, b \in Z_n \}$

is called set of Gaussian integers Modulo n.

$O(Z_n[i]) = n^2$

($a \downarrow n$, $b \downarrow n$)

$n+n+n+\dots+n$ (n times) $= n \cdot n = n^2$

Construction of $Z_1[i]$

$Z_1[i] = \{ a+ib / a \in Z_1, b \in Z_1 \}$

$= \{0\}$

$\Rightarrow Z_1[i] = \{0\}$

$\Rightarrow O(Z_1[i]) = (1)^2 = 1$

Construction of $Z_2[i]$

$Z_2[i] = \{ a+ib / a \in Z_2, b \in Z_2 \}$ where $Z_2 = \{0, 1\}$

$\Rightarrow Z_2[i] = \{0, 1, i, 1+i\}$

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$\Rightarrow O[Z_2[i]] = 2^2 = 4$

Construction of $Z_3[i]$

$Z_3[i] = \{a+ib \mid a \in Z_3, b \in Z_3\}$ where $Z_3 = \{0,1,2\}$

$= \{0, 1, 2, i, 2i, 1+i, 2+2i, 1+2i, 2+i\}$

$\Rightarrow O[Z_3[i]] = 3^2 = 9$

H.W.

Q.No. Construct $Z_5[i]$

$Z_5[i] = \{a+ib \mid a \in Z_5, b \in Z_5\}$ where $Z_5 = \{0,1,2,3,4\}$

$= \{0, 1, 2, 3, 4, i, 2i, 3i, 4i, 1+i, 1+2i, 1+3i, 1+4i, 2+i, 2+2i, 2+3i, 2+4i, 3+i, 3+2i, 3+3i, 3+4i, 4+i, 4+2i, 4+3i, 4+4i\}$

$O[Z_5[i]] = 5^2 = 25$

Q) Show that $(Z_n[i], +, \cdot)$ is ring.

Soln $Z_n[i] = \{a+ib \mid a \in Z_n, b \in Z_n\}$

Similarly, for $(Z[i], +, \cdot)$

A Q.No. $(Z_n[i], +)$ is cyclic group?

Soln need not

(Ans on last page)

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Q. Show that $(\mathbb{Q}\times\{0\}, +, \cdot)$ is ring.

Soln $\mathbb{Q}\times\{0\} = \{(a,0) \mid a\in\mathbb{Q}\}$ —— (1)

(A) $(\mathbb{Q}\times\{0\}, +)$ is commutative group.

1) let $x = (a,0)\in\mathbb{Q}\times\{0\}$, $a\in\mathbb{Q}$
& $y = (b,0)\in\mathbb{Q}\times\{0\}$, $b\in\mathbb{Q}$
s.t. $x+y = (a+b,0)\in\mathbb{Q}\times\{0\}$
$\Rightarrow x+y\in\mathbb{Q}\times\{0\}$

2) $x+(y+z) = x+y+z = (x+y)+z$ $\forall\ x,y,z\in\mathbb{Q}\times\{0\}$

3) $e = (0,0)\in\mathbb{Q}\times\{0\}$ s.t.
$x+e = e+x = x$, $\forall\ x\in\mathbb{Q}\times\{0\}$

4) let $x = (a,0)\in\mathbb{Q}\times\{0\}$, $a\in\mathbb{Q}$
then $-a\in\mathbb{Q}$
$\Rightarrow -x = (-a,0)\in\mathbb{Q}\times\{0\}$
s.t. $x+(-x) = (-x)+x = (0,0)$

5) let $x = (a,0)\in\mathbb{Q}\times\{0\}$ and $y = (b,0)\in\mathbb{Q}\times\{0\}$
s.t. $x+y = (a+b, 0+0) = (b+a, 0+0)$
$= (b,0)+(a,0)$
$= y+x$
$\Rightarrow x+y = y+x$ $\forall\ x,y\in\mathbb{Q}\times\{0\}$

then $(\mathbb{Q}\times\{0\}, +)$ is abeliangroup.

(B) $(\mathbb{Q}\times\{0\}, \cdot)$ is semigroup?

let $x = (a,0)\in\mathbb{Q}\times\{0\}$, $a\in\mathbb{Q}$, and
$y = (b,0)\in\mathbb{Q}\times\{0\}$, $b\in\mathbb{Q}$
s.t. $x\cdot y = (a\cdot b, 0)\in\mathbb{Q}-\{0\}$
$\Rightarrow x\cdot y\in\mathbb{Q}\times\{0\}$

$x \cdot (y \cdot z) = xyz = (x \cdot y) \cdot z$, $\forall\ x, y, z \in \mathbb{Q} \times \{0\}$

© Left and Right distribution Law:-

(1) $x \cdot (y+z) = x \cdot y + x \cdot z$

(2) $(x+y) \cdot z = x \cdot z + y \cdot z$, $\forall\ x, y, z \in \mathbb{Q} \times \{0\}$

then $(\mathbb{Q} \times \{0\}, +, \cdot)$ is Ring.

Similarly, $(\mathbb{R} \times \{0\}, \mathbb{C} \times \{0\}, \mathbb{Z}/Y \times \{0\}, \mathbb{Z}/X \times \mathbb{Q} \times \{0\}, \mathbb{Z}/X \times \{0\} \times \{0\}, \mathbb{R} \times \mathbb{C} \times \{0\}$ etc. are rings.

Commutative Ring :- A Ring $(R, +, \cdot)$ is said to be commutative if $ab = ba$ $\forall\ a, b \in R$

Q. $R = M_2(\mathbb{R})$ is commutative ring?

Soln No, let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in M_2(\mathbb{R})$

& $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in M_2(\mathbb{R})$

s.t. $A \cdot B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$B \cdot A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$A \cdot B \neq B \cdot A$ then $M_2(\mathbb{R})$ is a commutative Ring.

Q. $\mathbb{R} = \mathbb{Z}$ is commutative Ring.

Soln :- $a \cdot b = b \cdot a$, $\forall\ a, b \in \mathbb{Z}$ then

$\mathbb{R} = \mathbb{Z}$ is commutative Ring.

Similarly, $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}_n, \mathbb{Z}[i], \mathbb{Z}_n[i]$ are commutative Ring.

Q. $R = \mathbb{Q} \times \mathbb{R}$ is commutative Ring?

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Soln let $x = (a,b) \in Q \times R$, $a \in Q$, $b \in R$
& $y = (c,d) \in Q \times R$, $c \in Q$, $d \in R$
s.t. $x \cdot y = (a,b) \cdot (c,d)$
$= (ac, bd)$
$= (ca, db)$
$= ((c,d) \cdot (a,b))$
$= y \cdot x$
$x \cdot y = y \cdot x$ $\forall$ $x, y \in Q \times R$

Similarly $Z \times Q$, $Z \times Z$, $Q \times Q$, $Q \times \phi$, $Q \times R \times \phi$, $Z_n \times Z$, $Z[i] \times Z[i]$, $Z_n[i] \times Z \times \{0\}$ etc. are commutative ring.

Boolean Ring :- A ring $(R, +, \cdot)$ is said to be Boolean Ring if $a^2 = a$ $\forall$ $a \in R$

for example :-
$R = Z_2 = \{0, 1\}$
$0^2 = 0$ & $1^2 = 1$
then $a^2 = a$ $\forall$ $a \in Z_2$
then Z_2 is Boolean Ring.

Q. Give the example of Boolean ring of order 4 and ∞.

Soln $Z_2 \times Z_2 = \{(0,0), (0,1), (1,0), (1,1)\}$
$(0,0)^2 = (0,0)$
$(0,1)^2 = (0,1)$
$(1,0)^2 = (1,0)$
$(1,1)^2 = (1,1)$
then $a^2 = a$, $\forall$ $a \in Z_2 \times Z_2$
then $Z_2 \times Z_2$ is Boolean Ring of order 4.
Now $Z_2 \times Z_2 \times$ —— is Boolean Ring of order ∞.

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Ring with unity:- A Ring $(R,+,\cdot)$ is said to be a ring with unity, if $\exists\ 0\neq e\in R$ s.t. $a\cdot e = e\cdot a\ \forall\ a\in R$

Q. $R = \mathbb{Z}/\mathbb{Q}/\mathbb{R}/\mathbb{C}/\mathbb{Z}[i]$ is ring with unity?

Soln $1\in\mathbb{R}$ s.t. $a\cdot 1 = 1\cdot a = a\ \forall\ a\in R$

Then $R = \mathbb{Z}/\mathbb{Q}/\mathbb{R}/\mathbb{C}/\mathbb{Z}[i]$ is ring with unity 1.

Q. $R = \mathbb{Q}\times\mathbb{R}$ is Ring with unity?

Soln $(1,1)\in\mathbb{Q}\times\mathbb{R}$ s.t.

$$(a,b)\cdot(1,1) = (a\cdot 1, b\cdot 1)$$
$$= (a,b)$$
$$\forall\ (a,b)\in\mathbb{Q}\times\mathbb{R}$$

then $\mathbb{Q}\times\mathbb{R}$ is ring with unity $(1,1)$.

Similarly, $\mathbb{Z}\times\mathbb{Z}$, $\mathbb{Q}\times\mathbb{Z}$, $\mathbb{Q}\times\mathbb{C}$, $\mathbb{Z}[i]\times\mathbb{Z}$, $\mathbb{Z}[i]\times\mathbb{Z}[i]$, is ring with unity $(1,1)$.

Q. $R = \mathbb{Q}\times\{0\}$ is ring with unity?

Soln $\mathbb{Q}\times\{0\} = \{(a,0)\mid a\in\mathbb{Q}\}$

$(0,0)\neq(1,0)\in\mathbb{Q}\times\{0\}$

s.t. $(a,0)\cdot(1,0) = (a\cdot 1, 0)$
$= (a,0)$

$\Rightarrow (1,0)$ is unity of $\mathbb{Q}\times\{0\}$

then $\mathbb{Q}\times\{0\}$ is ring with unity.

Similarly

$\mathbb{R}\times\{0\}$, $\mathbb{C}\times\{0\}$, $\mathbb{Z}\times\{0\}$, $\mathbb{Z}[i]\times\{0\}$ etc. are ring with unity $(1,0)$.

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Q)) $R = Z_n$ is ring with unity ?

Soln:- Need not.

(1) If $n=1$ then $Z_1 = \{0\}$ is not Ring with unity.

2. If $n>1$ then $1 \in Z_n$
s.t. $a \cdot 1 = a$, $\forall a \in Z_n$

Then Z_n is ring with unity.

Q. $R = Z_n \times Z_n$ is ring with unity ?

Soln Need Not.

Q. $R = Z_n[i]$ is ring with unity ?

Soln need not
Similarly for Z_n.

Zero-Divisor:

let $(R, +, \cdot)$ is a commutative Ring, a non zero element $a \in R$ is said to be zero divisor if $\exists$ an element $0 \neq b \in R$ s.t $a \cdot b = 0$

Then we can say that
$(R, +, \cdot)$ zero divisors.

e.g. $R = Z_{15}$

$0 \neq 3 \in Z_{15}$, then $\exists\ 5 \in Z_{15}$ s.t. $3 \cdot 5 = 0$ then 3 is zero divisor.

Hence Z_{15} has zero divisors.

Q)) $R = Z_5[i]$ has zero divisor ?

Soln $0 \neq (1+2i) \in Z_5[i]$
& $0 \neq (1+3i) \in Z_5[i]$
s.t. $(1+2i)(1+3i) = 0$

then $Z_5[i]$ has zero divisors.

Q. $R = Z_3[i]$ has zero divisors.

Soln No

Q. Show that Z_p has no zero divisors?

Soln let $a \in Z_p$, $b \in Z_p$ s.t. $a \cdot b = 0$

$\Rightarrow p \mid 0$

$\Rightarrow p \mid a \cdot b \Rightarrow p \mid a$ or $p \mid b$

If $p \mid a$ then $a = 0$ done

If $p \nmid a$ then $\gcd(p, a) = 1$

$\Rightarrow p \mid b$

$\Rightarrow b = 0$

$\Rightarrow a = 0$ or $b = 0$

hence Z_p has no zero divisor.

e.g. $Z_2, Z_3, Z_5, \ldots$ has no zero divisors. (prime has no zero divisor.)

Note :- If $n > 1$ and $n \neq p$ then Z_n has zero divisors.

Hint:- If $n > 1$ and $n \neq p$ then $n = \underbrace{p_1^{\gamma_1}}_{a} \underbrace{p_2^{\gamma_2} \cdots p_K^{\gamma_K}}_{b}$

where $a = p_1^{\gamma_1} \neq 0$

and $b = p_2^{\gamma_2} \cdots p_K^{\gamma_K} \neq 0$

s.t. $a \cdot b = n = 0$

$\Rightarrow a \cdot b = 0$

but neither $a = 0$ nor $b = 0$

hence Z_n has zero divisors.

e.g. $Z_4, Z_6, Z_8, Z_9, Z_{10}, Z_{12}$ etc. has zero divisors.

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Q) 1) $R = Z_2[i]$ has zero divisors ?

$i \in Z_2[i]$, $1+i \in Z_2[i]$

$i(1+i) = 1-1 = 0$

2) $R = Z_7[i]$ has zero divisors ?

No.

[illegible]

$(1+3i)(1+4i) = 1+7i-12 \neq 0$

zero divisor = 1

I.D.

Integral domain :- A commutative Ring with unity $(R, +, \cdot)$ is said to be an integral domain if $a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$ where $a \in R$, $b \in R$

OR

$0 \neq a \in R$ and $0 \neq b \in R$ then $a \cdot b \neq 0$

Q)) $R = Z$ is an integral domain ?

Soln let $a \in Z$, $b \in Z$ and $a \cdot b = 0$

$\Rightarrow a = 0$ or $b = 0$

then Z is an integral

Similarly $R = \phi / R / \phi / Z[i]$ is an integral domain.

Q. $R = Z \times Q$ is commutative ring with unity

Q)) $R = Z \times Q$ is an integral domain.

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Soln $\mathbb{Z} \times \mathbb{Q}$ is commutative ring with unity.

but $(1,0)(0,1) = (0,0)$

But $(0,0) \neq (1,0) \in \mathbb{Z} \times \mathbb{Q}$

& $(0,0) \neq (0,1) \in \mathbb{Z} \times \mathbb{Q}$

then $\mathbb{Z} \times \mathbb{Q}$ is not integral domain.

Q. $R = \mathbb{R} \times \mathbb{R}$ is an integral domain?

Soln No

Similarly $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Q} \times \mathbb{Q}$, $\mathbb{R} \times \phi$, $\phi \times \phi$, $\mathbb{Z}[i] \times \mathbb{Z}[i]$

Q. $R = \mathbb{Z} \times \mathbb{Q} \times \mathbb{R}$ is an integral domain?

Soln $(1,0,0) \cdot (0,1,0) = (0,0,0)$

But $(1,0,0) \neq (0,0,0)$

& $(0,1,0) \neq (0,0,0)$

then $R = \mathbb{Z} \times \mathbb{Q} \times R$ is not integral domain.

Q. $R = \mathbb{Q} \times \{0\}$ is an integral domain?

Soln $\mathbb{Q} \times \{0\} = \{ (a,0) \mid a \in \mathbb{Q} \}$

let $x = (a,0) \in \mathbb{Q} \times \{0\}$; $a \in \mathbb{Q}$

$y = (b,0) \in \mathbb{Q} \times \{0\}$; $b \in \mathbb{Q}$

s.t. $x \cdot y = 0$

$\Rightarrow (a,0) \cdot (b,0) = (0,0)$

$\Rightarrow (a \cdot b, 0) = (0,0)$

$\Rightarrow ab = 0$ ——— (*)

$a \in \mathbb{Q}, b \in \mathbb{Q}$ and $\mathbb{Q}$ is an integral domain.

then from eqn (*)

Let $x = (a,0) \in \mathbb{Q} \times \{0\}$, $a \in \mathbb{Q}$
& $y = (b,0) \in \mathbb{Q} \times \{0\}$, $b \in \mathbb{Q}$
} $a = 0$ or $b = 0$

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Case 1 if $a=0$ then $x=(0,0)$

Case II if $b=0$ then $y=(0,0)$

Case III if $a=0$ & $b=0$ then $x=0$ & $y=0$

for case 1, II, III, we get $x=0$ or $y=0$

Similarly $\mathbb{R}\times\{0\}$, $\mathbb{C}\times\{0\}$, $\mathbb{Z}\times\{0\}$, $\mathbb{Z}[i]\times\{0\}$ are integral domain.

Q. $R=\mathbb{Q}\times\mathbb{R}\times\{0\}$ is an integral domain?

$x=(1,0,0)\in \mathbb{Q}\times\mathbb{R}\times\{0\}$

& $y=(0,1,0)\in \mathbb{Q}\times\mathbb{R}\times\{0\}$

s.t. $x\cdot y=(1,0,0)\cdot(0,1,0)=(0,0,0)$

then $\mathbb{Q}\times\mathbb{R}\times\{0\}$ is not integral domain.

Q.)) $R=M_2(\mathbb{R})$ is an integral domain?

Soln $M_2(\mathbb{R})$ is non commutative ring then $M_2(\mathbb{R})$ is not integral domain.

[Q. $\mathbb{Z}_1$ has no unity]

Q.)) $R=\mathbb{Z}_1$ is an integral domain?

Soln $\mathbb{Z}_1$ has no unity then $\mathbb{Z}_1$ is not integral domain

Q. $\mathbb{Z}_n$ is an integral domain iff $n=p$.

Soln let $n=p$ then $\mathbb{Z}_p$ is commutative ring with unity 1.

Let $a\in\mathbb{Z}_p$ and $b\in\mathbb{Z}_p$ s.t. $a\cdot b=0$

$\Rightarrow p\mid 0 \Rightarrow p\mid a\cdot b \Rightarrow p\mid a$ or $p\mid b$

if $p\mid a$ then $a=0$

if $p\nmid a$ then $b=0$

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then Z_p is an Integral domain.

Conversly:- If $n \neq p$ then Z_n is not integral domain.

Case I if $n=1$ then Z_1 is not integral domain.

Case II if $n>1$ and $n \neq p$ then

$$n = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_K^{\gamma_K}$$

$$n = a \cdot b$$

where $0 \neq a = p_1^{\gamma_1} \in Z_n$ & $0 \neq b = p_2^{\gamma_2} \cdots p_K^{\gamma_K} \in Z_n$

s.t. $a \cdot b = 0$

then Z_n is not integral domain.

Hence from case I, II. if $n \neq p$ then Z_n is not integral domain.

Hence Z_n is an integral domain if $n = p$.

For ex:- $Z_2, Z_3, Z_5 \cdots$ are integral domain and $Z_1, Z_4, Z_6, Z_8 \cdots$ are not integral domain.

Q. $R = Z_n[i]$ is an integral domain?

Soln Need Not

Case I :- If $n \neq p$

then $Z_n[i]$ is not integral domain.

because $Z_n \subseteq Z_n[i]$ and Z_n is not integral domain if $n \neq p$ when $n>1$

Case II :- If $n=1$ then $Z_1[i] = \{0\}$

has no unity then $Z_1[i]$ is not integral domain.

From case I & II we get

$Z_n[i]$ is not integral domain if $n \neq p$.

Case III :-

if $n=p$ then $Z_p[i]$ is need not be an integral domain.

e.g.

$$Z_5[i] = \{ a+ib \mid a, b \in Z_5 \}$$

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$(2+i)(2+4i) = 0$ but
$(2+i) \neq 0$ and $(2+4i) \neq 0$

$Z_3[i] = \{a+ib \mid a, b \in Z_3\}$ is an integral domain because $0 \neq a \in Z_3[i]$ and $0 \neq b \in Z_3[i]$ then $a \cdot b \neq 0$

Note :- If $4/p-3$ then $Z_p[i]$ is an integral domain.

For example $4 \nmid 2-3$ then $Z_2[i]$ is not integral domain.

(2) $4/3-3$ then $Z_3[i]$ is an integral domain.

(3) $4 \nmid 5-3$ then $Z_5[i]$ is not integral domain.

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Q. $R = \{(a,b,c) \mid a,b,c \in Z\}$ is an integral domain.

$= Z \times Z \times Z$

Soln :- (Need Not)

$x = (1,0,0) \in R$

$y = (0,1,0) \in R$

s.t. $x \cdot y = (1,0,0)(0,1,0)$

$= (0,0,0)$

then $R = \{(a,b,c) \mid a,b,c \in Z\}$ is Need not integral domain.

H.W. $R = \{a,b,c \mid a,b \in R \text{ & } c = 0\}$ is an integral domain.

$= \mathbb{R} \times \mathbb{R} \times \{0\} \Rightarrow$ Ans: Need Not

ut $x = (1,0,0) \in R$

$y = (0,1,0) \in R$

$x \cdot y = (0,0,0)$

But $(1,0,0) \neq (0,0,0)$

or $(0,1,0) \neq (0,0,0)$

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Sub-Ring

Let S be a non empty subset of R. $(S,+,\cdot)$ is said to be subring of R if

(i) $\forall a \in S, \forall b \in S \Rightarrow a-b \in S$ ⋮ it is sub. subgroup

(ii) $\forall a \in S, \forall b \in S \Rightarrow a \cdot b \in S$

Subgroup $\Rightarrow$ Subring ✓

But Subring $\not\Rightarrow$ Subgroup (need not)

OR

$(S,+,\cdot)$ itself ring.

e.g.

$(R,+,\cdot), (\mathbb{C},+,\cdot)$ itself Ring.

$\phi \neq \mathbb{Z} \subseteq \mathbb{Q} \mid \mathbb{R} \mid \mathbb{C} \mid \mathbb{Z}[i]$

If $(\mathbb{Z},+,\cdot)$ is ring then $(\mathbb{Z},+,\cdot)$ is subring of $(\mathbb{Q},+,\cdot)/(R,+,\cdot)/(\mathbb{C},+,\cdot)/(\mathbb{Z}[i],+,\cdot)$

Q.No. $\phi \neq Q \subseteq \mathbb{R} \mid \mathbb{C}$ and $(Q,+,\cdot)$ is ring then $(Q,+,\cdot)$ is subring of $(\mathbb{R},+,\cdot)/(\mathbb{C},+,\cdot)$

Soln Given $\phi = Q \subseteq \mathbb{R} \mid \mathbb{C}$

also $(Q,+,\cdot)$ is ring

Q is non empty subset of $(\mathbb{R},+,\cdot)/(\mathbb{C},+,\cdot)$ is said to be $(\mathbb{R},+,\cdot)/(\mathbb{C},+,\cdot)$

(i) $\forall a \in Q, \forall b \in Q \Rightarrow a-b \in Q$

(ii) $\forall a \in Q, \forall b \in Q \Rightarrow a \cdot b \in Q$

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Q.No. $S = \left\{ \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} \middle| \; a, c \in \mathbb{R} \right\}$ is subring of $M_2(\mathbb{R})$?

Soln $M_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \; a, b, c, d \in \mathbb{R} \right\}$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$ then $\phi \neq S \subseteq M_2(\mathbb{R})$

(i) Let $A = \begin{bmatrix} a_1 & 0 \\ c_1 & 0 \end{bmatrix} \in S$, $a_1, c_1 \in \mathbb{R}$

& $B = \begin{bmatrix} a_2 & 0 \\ c_2 & 0 \end{bmatrix} \in S$, $a_2, c_2 \in \mathbb{R}$

s.t. $A - B = \begin{bmatrix} a_1 - a_2 & 0 \\ c_1 - c_2 & 0 \end{bmatrix} \in S$, $a_1 - a_2 \in \mathbb{R}$, $c_1 - c_2 \in \mathbb{R}$

(ii) $A \cdot B = \begin{bmatrix} a_1 & 0 \\ c_1 & 0 \end{bmatrix} \begin{bmatrix} a_2 & 0 \\ c_2 & 0 \end{bmatrix} = \begin{bmatrix} a_1 a_2 & 0 \\ c_1 a_2 & 0 \end{bmatrix} \in S$, $a_1 a_2 \in \mathbb{R}$, $c_1, a_2 \in \mathbb{R}$

then $(S, +, \cdot)$ is subring of $M_2(\mathbb{R})$

Q.No. $S = \left\{ \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} \middle| \; b \in \mathbb{Z} \right\}$ is subring of $M_2(\mathbb{Z})$?

Soln $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$ then $\phi \neq S \subseteq M_2(\mathbb{R})$

i.e. then S is non empty subset of $M_2(\mathbb{R})$

(i) let $A = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix} \in S$, $a_1 \in \mathbb{Z}$

& $B = \begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix} \in S$, $a_2 \in \mathbb{Z}$

s.t. $A - B = \begin{bmatrix} a_1 - a_2 & 0 \\ 0 & 0 \end{bmatrix} \in S$, $a_1 - a_2 \in \mathbb{Z}$

(ii) $A \cdot B = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1 a_2 & 0 \\ 0 & 0 \end{bmatrix} \in S$

then $(S, +, \cdot)$ is subring of $M_2(\mathbb{Z})$

C.S.I.R

$S = \left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \middle| \; b \in \mathbb{R} \right\}$ is subring of $M_2(\mathbb{R})$?

Soln

$M_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \; a, b, c, d \in \mathbb{R} \right\}$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$ then $\phi \neq S \subseteq M_2(\mathbb{R})$

(i) let $A = \begin{bmatrix} a & b_1 \\ 0 & 0 \end{bmatrix} \in S, \; b_1 \in \mathbb{R}$

$B = \begin{bmatrix} 0 & b_2 \\ 0 & 0 \end{bmatrix} \in S, \; b_2 \in \mathbb{R}$

$A - B = \begin{bmatrix} 0 & b_1 - b_2 \\ 0 & 0 \end{bmatrix} \in S, \; b_1 - b_2 \in \mathbb{R}$

(ii) $A \cdot B = \begin{bmatrix} 0 & b_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & b_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S, \; 0 \in \mathbb{R}$

then $(S, +, \cdot)$ is subring of $M_2(\mathbb{R})$

Q. $S = \left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \middle| \; 0 \neq b \in \mathbb{R} \right\}$ is subring of $M_2(\mathbb{R})$?

$M_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \; a, b, c, d \in \mathbb{R} \right\}$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$ then $\phi \neq S \subseteq M_2(\mathbb{R})$

(i) Let $A = \begin{bmatrix} 0 & b_1 \\ 0 & 0 \end{bmatrix} \in S, \; b_1 \in \mathbb{R}$

$B = \begin{bmatrix} 0 & b_2 \\ 0 & 0 \end{bmatrix} \in S, \; b_2 \in \mathbb{R}$

$A - B = \begin{bmatrix} 0 & b_1 - b_2 \\ 0 & 0 \end{bmatrix} \in S, \; b_1 - b_2 \in \mathbb{R}$

(ii) $A \cdot B = \begin{bmatrix} 0 & b_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & b_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin S \; (\because b \neq 0)$

then $(S, +, \cdot)$ is not subring of $M_2(\mathbb{R})$

Q. $S = \left\{ \begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix} \middle| \; a, b \in \mathbb{R} \right\}$ is subring of $M_2(\mathbb{R})$?

(इसमें दिया है कि 1st और 4th entry का sum 2nd & 3rd position पर है।)

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Soln $M_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \; a, b, c, d \in \mathbb{R} \right\}$

$\begin{bmatrix} 0 & 0+0 \\ 0+0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$ then $\phi \neq S \subseteq M_2(\mathbb{R})$

(i) Let $A = \begin{bmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{bmatrix} \in S$ $\; a_1, b_1 \in \mathbb{R}$

$$B = \begin{bmatrix} a_2 & a_2+b_2 \\ a_2+b_2 & b_2 \end{bmatrix} \in S \quad a_2, b_2 \in \mathbb{R}$$

s.t.

$$A - B = \begin{bmatrix} a_1 - a_2 & (a_1+b_1)-(a_2+b_2) \\ (a_1+b_1)-(a_2+b_2) & b_1 - b_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 - a_2 & (a_1-a_2)+(b_1-b_2) \\ (a_1-a_2)+(b_1-b_2) & b_1 - b_2 \end{bmatrix}$$

$\Rightarrow A - B \in S$

(ii) $A \cdot B = \begin{bmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{bmatrix} \cdot \begin{bmatrix} a_2 & a_2+b_2 \\ a_2+b_2 & b_2 \end{bmatrix} \notin S$

e.g. $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \in S, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \in S$

s.t. $A \cdot B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \notin S$

it is subgroup ($\because$ 1st prop. satisfied) but not subring.

then $(S, +, \cdot)$ is not subring of $M_2(\mathbb{R})$

Note $(S, +, \cdot)$ is subgroup of $M_2(\mathbb{R})$

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H.W.

Q. $S = \left\{ \begin{bmatrix} a & a-b \\ a-b & b \end{bmatrix} \middle/ a, b \in \mathbb{R} \right\}$ is subring of $M_2(\mathbb{R})$

Imp.

(1st & 4th = 2nd or 3rd position)

$\begin{bmatrix} 0 & 0-0 \\ 0-0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$ then $\phi \neq S \subseteq M_2(\mathbb{R})$

(i) Let $A = \begin{bmatrix} a_1 & a_1-b_1 \\ a_1-b_1 & b_1 \end{bmatrix} \in S$, $a_1, b_1 \in \mathbb{R}$

$B = \begin{bmatrix} a_2 & a_2-b_2 \\ a_2-b_2 & b_2 \end{bmatrix} \in S$, $a_2, b_2 \in \mathbb{R}$

s.t.

$$A-B = \begin{bmatrix} a_1-a_2 & (a_1-a_2)-(-b_2+b_1) \\ (a_1-a_2)+(b_2+b_1) & b_1-b_2 \end{bmatrix}$$

$\Rightarrow A-B \in S$

(ii) $A \cdot B = \begin{bmatrix} a_1 & a_1-b_1 \\ a_1-b_1 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & a_2-b_2 \\ a_2-b_2 & b_2 \end{bmatrix} \in$

e.g. $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \in S$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$$A \cdot B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \in S$$

It is subgroup and also subring.

H.P.

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Q.No. $S = \{(a,b,c) \mid a+b=c,\ a,b,c \in \mathbb{Z}\}$ is subring of $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$?

1st 4

$x = (a_1\ b_1\ c_1)$

$y = (a_2\ b_2\ c_2)$

$x \cdot y = (a_1 \cdot a_2\quad b_1 \cdot b_2 \cdot\ c_1 \cdot c_2)$

$x = (1,0,1) \in S$

$y = (0,1,1) \in S$

$x \cdot y = (0,0,1) \notin S$

it is not subring

:

$x = (1,0,1) \in S$ and $y = (0,1,1) \in S$ s.t.

$x \cdot y = (1,0,1)\cdot(0,1,1) = (0,0,1) \notin S$

then S is not subring of $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$

Q. $S = \{(a,b,c) \mid a+b+c=1,\ a,b,c \in \mathbb{Z}\}$ is subgp. of $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$?

& also S is subgp. of $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$

Soln

it is not subgroup ($\because$ Additive identity not hold)

$\Rightarrow$ subgroup $\Rightarrow$ subring

$\Rightarrow$ it is not subring

$(0,0,0) \notin S$ because $0+0+0 \neq 1$

then $(S,+)$ is not subgp. of $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$

Q. Show that $S=\{0\}$ and $S=R$ are always subring of R.

(अगर $o(R) \geq 1$ तब ये दो prop. hold करेंगी | तब atleast two subring होंगी)

Group is cardinality 1 के case में atleast one ही होगी ऐसा ...

Soln:

Case 1 let $S = \{0\}$

① let $a \in S$ & $b \in S \Rightarrow a - b = 0 \in S$

($a = 0$) ($b = 0$)

② let $a \in S$, $b \in S \Rightarrow a \cdot b = 0 \cdot 0 \Rightarrow 0 \in S$

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then $S=\{0\}$ is subring of R.

Case II $\phi \neq R \subsetneq \mathbb{R}$ and $(R,+,\cdot)$ is ring then $S=R$ is subring of R.

Hence $S=\{0\}$ and $S=R$ are always ~~Subgroup~~ subring of R.

Note Minimum element required for Group = 1, Ring = 1, I.D. = 2, Field = 2.

" " " " Ring = 1

" " " " I.D = 2

" " " " Field = 2

For Example

① $S=\{0\}$ and $S=\mathbb{Z}$ are subring of $\mathbb{Z}$

② $S=\{0\}$ " $S=\mathbb{R}$ " " " $\mathbb{R}$

(3) $S=\{0\}$ " $S=\mathbb{Z}_n$ " " " $\mathbb{Z}_n$

Q.No. Show that $m\mathbb{Z}$ is subring of $\mathbb{Z}$.

Soln $m\mathbb{Z}=\{m.a \mid a\in\mathbb{Z}\}$

$0\in m\mathbb{Z}$ then $\phi \neq m\mathbb{Z} \subseteq \mathbb{Z}$

① let $x\in m\mathbb{Z}$ then $x=m.a_1,\ a_1\in\mathbb{Z}$

& $y\in m\mathbb{Z}$ " $y=m.a_2,\ a_2\in\mathbb{Z}$

s.t. $x-y=ma_1-ma_2=m(a_1-a_2)=ma_3\in m\mathbb{Z}$,

where $a_3=a_2-a_1\in\mathbb{Z}$

$\Rightarrow x-y\in m\mathbb{Z}$

② $x.y=(m.a_1)(ma_2)=m.(ma_1a_2)=ma'\in m\mathbb{Z}$

where $a'=ma_1a_2\in\mathbb{Z}$

$\Rightarrow x.y\in m\mathbb{Z}$

then $m\mathbb{Z}$ is subring of $\mathbb{Z}$

for Example:-

$\{0\}, \mathbb{Z}, 2\mathbb{Z}, 3\mathbb{Z}, \ldots$ are subrings of $\mathbb{Z}$.

Subring of Z_n :-

If $m|n$ then $S = \langle m \rangle = mZ_n$ is subgroup of Z_n. Now show that $S = mZ_n = \{m.a \mid a \in Z_n\}$ is subring of Z_n

$R = Z_{10} = \{0, \ldots, 9\}$

$\langle 2 \rangle = \{0, 2, \ldots, 8\} = \{2.a \mid a \in Z_{10}\} = 2Z_{10}$

$\langle 5 \rangle = 5Z_{10}$

mZ_n here $m|n$

① Since mZ_n is subgroup of Z_n then $\forall a \in mZ_n$, $\forall b \in mZ_n$ s.t. $a - b \in mZ_n$.

2. let $x \in mZ_n$ then $x = ma_1, a_1 \in Z_n$ & $y \in mZ_n$ then $y = ma_2, a_2 \in Z_n$

s.t.

$x.y = m(m.a_1 a_2) = m.a' \in mZ_n$, where $a' = ma_1a_2$

$\Rightarrow x.y \in mZ_n$

Hence mZ_n is subring of Z_n

($m|n$ but $0 \times 10 = 0 \Rightarrow S = \{0\} = \langle 0 \rangle = \langle 10 \rangle$

Subring of Z_{10} $S_1 = \{0\}$

$S_2 = 2Z_{10} = \langle 2 \rangle$

$S_3 = 5Z_{10} = \langle 5 \rangle$

$S_4 = Z_{10}$

Note No. of subring in Z_n = No. of subgp. of Z_n = $\tau(n)$

Q. Find all subring of Z_{12}.

Soln No. of subring in $Z_{12} = \tau(12) = \tau(2^2 \times 3^1) = (2+1)(2) = 6$

divisor of 12 = 1, 2, 3, 4, 6, 12

$S_1 = \langle 1 \rangle = 1Z_{12} = Z_{12}$

$S_2 = \langle 2 \rangle = 2Z_{12} = \{0, 2, 4, 6, 8, 10\}$

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$S_4 = \langle 4 \rangle = 4Z_{12} = \{0, 4, 8\}$

$S_5 = \langle 6 \rangle = 6Z_{12} = \{0, 6\}$

$S_6 = \langle 12 \rangle = 0Z_{12} = \{0\}$

Q. $R = Z_{49}$, How many subring of order 7.

Soln # of subring in $Z_{49} = \tau(49) = \tau(7^2) = (2+1) = 3$

Say $S_1 = \langle 1 \rangle = Z_{49}$

$S_2 = \langle 7 \rangle = 7Z_{49} = \{0, 7, 14, 21, 28, 35, 42\}$

$S_3 = \langle 49 \rangle = \langle 0 \rangle = \{0\}$

$O(S_1) = 1, \; O(S_2) = 7, \; O(S_3) = 1$

then Z_{49} has unique subring of order 7.

(subring in case [illegible] in case of prime)

Q.No. How many subrings in Z_{p^2q}, where p and q are prime numbers.

Soln of p & q are distinct element [illegible]

Case 1 If $q = p$

then # of subring in $Z_{p^3} = (3+1) = 4$

Case 2 If $q \neq p$

then # of subring in $Z_{p^2q} = (2+1)(1+1)$
$= 3 \cdot 2$
$= 6$

Sum of two subrings :-

(S.R. Ring नहीं बनेगी)

let A and B are two subring of R. The sum of two subrings is denoted by A+B and defined by

$A + B = \{a + b \mid a \in A, b \in B\}$

For example :- $2\mathbb{Z}$ and $4\mathbb{Z}$ are subring of $\mathbb{Z}$

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Then $2\mathbb{Z}+4\mathbb{Z} = \{a+b \mid a\in 2\mathbb{Z}, b\in 4\mathbb{Z}\}$

$= \{2x+4y \mid x,y\in\mathbb{Z}\}$

$= \{\gcd(2,4)\cdot k \mid k\in\mathbb{Z}\}$

$= \{2\cdot k \mid k\in\mathbb{Z}\} = 2\mathbb{Z}$

Q.No. Sum of two subring of R is subring of R?

Soln Need Not

$S_1 = \left\{ \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} \mid a,c\in\mathbb{R} \right\}$ is subring of $M_2(\mathbb{R})$

& $S_2 = \left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \mid b\in\mathbb{R} \right\}$ is also subring of $M_2(\mathbb{R})$

But $S_1+S_2 = \left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \mid a,b,c\in\mathbb{R} \right\}$ is not subring of $M_2(\mathbb{R})$

because $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \in S_1+S_2$

& $B = \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix} \in S_1+S_2$ s.t. $A\cdot B = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \notin S_1+S_2$

Hence Sum of two subring of R is not subring of R.

Q. If $a\in\mathbb{Z}[i]$ then $S = a\mathbb{Z}[i]$ is subring of $\mathbb{Z}[i]$.
where $S = a\mathbb{Z}[i] = \{a\cdot m \mid m\in\mathbb{Z}[i]\}$

Soln $S = a\mathbb{Z}[i] = \{a\cdot m \mid m\in\mathbb{Z}[i]\}$

Let $x\in S$ & $y\in S$ then $x = am_1$ & $y = am_2$, $m_1,m_2\in\mathbb{Z}[i]$

(i) $x-y = am_1 - am_2$

$= a(m_1-m_2) = am' \in S$, where $m' = m_1-m_2$

(ii) $x\cdot y = (am_1)\cdot(am_2)$

$= am_1m_2$

$= am' \in S$, where $m' = m_1m_2$

Hence, S is subring of $\mathbb{Z}[i]$

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Q.No. How many subring of $Q / \mathbb{R} / \mathbb{C} / Z[i]$?
$\infty \quad \infty \quad \infty \quad \infty$

Soln Infinite

Q.No. Unity of Ring and subring is same ?

Soln Need Not

e.g. let $R = Z_{10}$ is Ring with unity 1

& S = subring of $Z_{10} = 2Z_{10} = \{0, 2, 4, 6, 8\}$

Now ~~$6 \in 2Z_{10}$ is unity~~ Now $6 \in 2Z_{10}$ s.t. $a \cdot 6 = a$, $\forall a \in 2Z_{10}$

then 6 is unity of $2Z_{10}$.

But $6 \neq 1_{10}$

$\Rightarrow 6 \neq 1$

then the unity of Ring & subring need not be same.

~~Idempotent~~ Element :- An element $a \in R$ is said to be idempotent element of R if $a^2 = a$

Q.No. Find Idempotent Elements of Z.

Soln $Z = \{0, \pm 1, \pm 2, \pm 3, \cdots\}$

$0^2 = 0$
$1^2 = 1$

if $a \neq 0$ and 1 then $a^2 = a$ in Z
then a is not idempotent element of Z.

Hence 0 and 1 are idempotent elements of Z
then Z has exactly two idempotent elements.

Q. How many idempotent Elements in Z_8.

Soln $Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$

$0 \in Z_8$, $0^2 = 0$
$1 \in Z_8$ $1^2 = 1$

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Hence Z_8 has exactly two idempotents elements say 0 & 1.

Q. how many idempotent in Z_{30}

Soln $Z_{30} = \{0,1,2,3,\ldots\ldots 29\}$

$0^2 = 0$
$1^2 = 1$
$6^2 = 6$
$10^2 = 10$
$(15)^2 = 15$
$(16)^2 = 16$
$(21)^2 = 21$
$(25)^2 = 25$

Hence Z_{30} has exactly 8 idempotent elements.

Q.No. If $a \in R$ is an idempotent element of R then $1-a$ is also idempotent element of R
(unity)

let $a \in R$ and a is an idempotent element of R
then $a^2 = a$ — (1)

Now $(1-a)^2 = 1^2 + a^2 - 2a$
$= 1 + a - 2a$
$\Rightarrow (1-a)^2 = 1-a$

then $(1-a)$ is also idempotent element of R.

e.g. $0 \in Z_6$ is an idempotent Element then $1-0=1$ is also idempotent.
$3 \in Z_6$ " " " " $1-3 = -2 = 4$ is idempotent.

Note 0-0 in Z_6 — [Means prime divisor]
$2^{(4)} = 2^2 = 4$

in Z_8 : $2^4 = 2^1 = 2$

in Z_1 = $2^0 = 1$

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No. of Idempotent element in $Z_n = 2^d$, where d is the no. of prime divisor of n.

e.g. Z_{30}

$d = 2, 3, 5 = 3$

$2^3 = 8$

(II) No. of idempotent element in $Z_{20} = 2, 5 = 2$

$d^2 = 2^2 = 4$

Say $0, 1, 5, 16$

(III) No. of idempotent element in $Z_{10} = 2^2 = 4$

Say $0, 1, 5, 6$

Q. If R is an integral domain then R has exactly two idempotent Elements.

Soln let R is an integral domain and a is an idempotent element of R then

$a^2 = a$

$\Rightarrow a^2 - a = 0 \Rightarrow a(a-1) = 0$ — (*)

$a^2 = a$
$a^2 - a = 0$
$a(a-1)$
Unity

Since R is an integral domain then from eqn (*), we get

$a = 0$ or $a - 1 = 0$

If $a = 0$ then 0 is idempotent element of R.

If $a - 1 = 0$ then $a = 1$ " " " R.

Since R is an integral domain then R is an commutative Ring with unity.

Hence 0 and 1 are member of R.

$\Rightarrow$ R has exactly two idempotents elements.

For Example:-

$R = \mathbb{Q} \mid \mathbb{R} \mid \mathbb{C} \mid Z'[i] \mid Z_3[i]$ is an integral domain then R has exactly two idempotent elements.

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Q.No. $R = Z_p$, how many idempotents elements in R?

Soln $R = Z_p$ is an integral domain then R has exactly two idempotents elements.

Note:- converse of above statement need not be true.

$R = Z_q$ has exactly two idempotent elements But Z_q is not integral domain.

Q: How many idempotent elements in $Q \times IR$?

Soln:- Let $(x, y) \in Q \times IR$ is an idempotents elements of $Q \times IR$.

Then $(x,y)^2 = (x,y)$

$\Rightarrow (x^2, y^2) = (x, y)$

$\Rightarrow \underbrace{x^2 = x}_{\in Q}$ and $\underbrace{y^2 = y}_{\in IR}$

$\Rightarrow x$ is an idempotent element of Q & y " " " " " IR

Now $R = Q \times IR$

Idempotents Element

$0, 1 \quad 0, 1$

$\Rightarrow (0,0), (0,1), (1,0), (1,1)$ are idempotent elements of $Q \times IR$

Hence $Q \times IR$ has exactly 4 idempotent elements.

H.W.

Q.No. $R = Q \times Z_{10}$, Find all idempotent Elements.

(ii) $R = IR \times Z_7[i] \times \phi$, Find all idempotent elements.

Ans (i) & ii) = 8

Soln we know that Q has exactly two idempotent elements

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In Z_{10}, No. of prime divisor of Z_{10} are 2 and 5

$2^d = 2^2 = 4$

idempotents are 0, 1, 5 and 6

then $R = \mathbb{Q} \times Z_{10}$ has 8 idemptotes Elements.

$\downarrow 2 \times 4 = 8$

Soln: we know that $\mathbb{R}$ has exactly two idemptotes elements.

also $Z_7[i]$ is an integral domain and we know that integral domain has exactly two idemptotes elements.

then $R = \mathbb{R} \times Z_7[i] \times \mathbb{C}$ has exactly $2 \times 2 \times 2 = 8$

$\downarrow 2 \quad \downarrow 2 \quad \downarrow 2$

Elements.

Nilpotent Elements:- An element $a \in R$ is said to be nilpotent elements of R if $a^n = 0$, for some $n = 1, 2, 3, \ldots$ or $n \in \mathbb{N}$

Q. Find all Nilpotent elements of $\mathbb{Z}$

Soln $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$

$0 \in \mathbb{Z}$, s.t. $0^1 = 0$ then 0 is Nilpotent element of $\mathbb{Z}$.

$0 \neq x \in \mathbb{Z}$ then $x^n \neq 0$ because $\mathbb{Z}$ is an integral domain.

then $\mathbb{Z}$ has exactly one element is Nilpotent - say 0,

Nilpotent element.

Similarly :- $Q|R|\mathbb{C}|Z[i]|Z_p|Z_{p^3}[i]$ has exactly one nilpotent element.

Q.No. If R is an integral domain then R has exactly one nilpotent element.

Soln Let R is an integral domain and $x \in R$ is nilpotent element of R then $x^n = 0$, for some n

$$\Rightarrow x x^{n-1} = 0 \quad \text{——} (1)$$

Since R is an integral domain then from eqn (1), we get

$$x = 0 \text{ or } x^{n-1} = 0$$

If $x = 0$ then 0 is nilpotent element of R

Now if $x \neq 0$ then $\underbrace{x \cdot x \cdot x \cdots x}_{n \text{ times}} \neq 0$

But from (1) $\Rightarrow x^n \neq 0$

$x^n = 0$ then supposition is wrong.

Hence $x \neq 0$ is not possible.

Hence only 0 is nilpotent element of R.

Hence R has exactly one nilpotent element.

For Example:-

$$R = Z \,|\, Q \,|\, IR \,|\, \mathbb{C} \,|\, Z \times \{0\} \,|\, Q \times \{0\} \,|\, Z[i] \,|\, Z[i] \times \{0\} \times \{0\}$$

Q.N. $R = Z_6$ then How many nilpotent elements?

Soln

$$Z_6 = \{0, 1, 2, 3, 4, 5\}$$

$0^1 = 0$ then 0 is nilpotent element

$1^n \neq 0$ " 1 " not nilpotent "

$2^n \neq 0$ " 2 " " " "

$3^n \neq 0$ " 3 " " " "

$4^n \neq 0$ " 4 " " " "

$5^n \neq 0$ " 5 " " " "

then Z_6 has only one nilpotent element say 0

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Note:- Z_6 has exactly one Nilpotent element but Z_6 is not integral domain.

Hence converse of above statement need not be true.

for e.g. Z_6 has exactly one Nilpotent element but Z_6 is not integral domain.

Q. Find all Nilpotent element of Z_8.

$Z_8 = \{0, 1, 2, \cdots 7\}$

$0^1 = 0$

$1^n \neq 1$ then 1 is not possible similarly $3^n \neq 0,\ 5^n \neq 0,\ 7^n \neq 0$

$2^3 = 8 = 0$ then 2 is nilpotent

$4^2 = 16 = 0$ " 4 " "

$6^3 = 216 = 0$ " 6 " "

then $0, 2, 4, 6$ are Nilpotent element of Z_8.

Z_8
$(2)^3 = 2$
$2, 4, 6$
# of Nilpotent element of Z_{100}
$2^2 \cdot 5^2$
$2^{2-1}\ 5^{2-1}$
$= 10$

NOTE:- If $n > 1$ then $n = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_K^{\gamma_K}$

then # of nilpotent element in $Z_n = p_1^{\gamma_1 - 1} p_2^{\gamma_2 - 1} \cdots p_K^{\gamma_K - 1}$

for e.g.

# of nilpotent elements in $Z_{10} = 2^{1-1} \cdot 5^{1-1}$

$= 2^0 \cdot 5^0$

$= 1$

# of Nilpotent element in $Z_{20} = 2^{2-1} 5^{1-1} = 2$ say $0, 10$

H.W.

Q.No. Find all Nilpotent element of $Z_{1024} = Z_{2^{10}}$

# of nilpotent element in $Z_{2^{10}}$

$= 2^{10-1} = 2^9$

$= 512.$

i.e. $0, 2, 2^2, 2^3, 6, 8, 10, \ldots$ ——— $\ldots 1022$.

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Q.No: Find Nilpotent element of $Q \times Z'$

Soln let $(x,y) \in Q \times Z'$ is nilpotent element of $Q \times Z'$

then $\exists\ n$ s.t. $(x,y)^n = (0,0)$

$\Rightarrow (x^n, y^n) = (0,0) \Rightarrow x^n = 0$ and $y^n = 0$

$\Rightarrow x$ is nilpotent element of Q and y is nilpotent element of Z'.

Now $R = Q \times Z'$

Nilpotent Element

$0 \quad 0$

then nilpotent element of $Q \times Z' = (0,0)$

H.w. Q.No. $Q \times Z'$ has exactly one nilpotent element.

Q.No. $R = Q \times Z_8 \times R$, find all Nilpotent Element of R.

$0 \quad 0,2,4,6 \quad 0$

Q has exactly one nilpotent element

Z_8 " " 4 " "

R " " 1 " "

then All Nilpotent element of R is

$(0,0,0), (0,2,0), (0,4,0), (0,6,0)$.

Q.No. $R = Z_p$, how many nilpotent elements.

Soln Z_p is an integral domain then Z_p has exactly one nilpotent element say 0.

Imp. Q.No. If R is commutative ring, and a, and b are nilpotent element of R then $a+b$ is also Nilpotent element of R.

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Soln Let a is nilpotent element of R then $\exists$ n s.t. $a^n = 0$ —(I)

& b " " " " R " m " $a^m = 0$ —(II)

s.t. $(a+b)^{n+m} = {}^{n+m}C_0\, a^{n+m} b^0 + {}^{n+m}C_1\, a^{n+m-1} b^1 - -$

$- - - {}^{n+m}C_m\, a^n b^m + - - {}^{n+m}C_{n+m}\, a^0 b^{n+m}$

$= 0+0+0 - - - +0+ - - +0$

$= 0$

$\Rightarrow (a+b)^{n+m} = 0$

$\Rightarrow$ a+b is also nilpotent element of R

e.g. 2 is nilpotent element of Z_8

4 is " " " "

& Z_8 is commutative then 2+4 = 6 is also nilpotent element.

Note:- If R is non commutative then above Result need not be true.

e.g. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in M_2(\mathbb{R})$ s.t. $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

then A is nilpotent element of $M_2(\mathbb{R})$

& $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in M_2(\mathbb{R})$ s.t. $B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ then B is nilpotent element of $M_2(\mathbb{R})$.

s.t.

$A+B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in M_2(\mathbb{R})$ s.t. $(A+B)^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

then A+B is not nilpotent element of $M_2(\mathbb{R})$.

Q.No. If R is commutative Ring and a,b is nilpotent element of R then a·b is also nilpotent element of R. It is need not be true if R is non commutative Ring.

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Soln Let R is commutative and a is nilpotent element of R then $a^n = 0$ — ①, for some n

Now $b \in R$ s.t.

$$(a \cdot b)^n = a^n b^n \quad (\because R \text{ is commutative})$$
$$= 0 \cdot b^n = 0 \quad (\text{from equn ①})$$

$\Rightarrow (ab)^n = 0$

$\Rightarrow a \cdot b$ is also nilpotent.

Note:- it is need not be true if R is non commutative

Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in M_2(\mathbb{R}) \Rightarrow A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ then A is nilpotent element of $M_2(\mathbb{R})$

Now $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in M_2(\mathbb{R})$

s.t.

$$A \cdot B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

But $A \cdot B$ is not nilpotent element of $M_2(\mathbb{R})$.

Q.No. If $a \cdot b$ is Nilpotent element of R then $b \cdot a$ is also nilpotent element of R.

Soln let $a \cdot b$ is nilpotent element of R then $\exists$ n s.t.

$$(ab)^n = 0 \quad — ①$$

Case I if R is commutative

Now $(ba)^n = (ab)^n = 0$

$\Rightarrow (ba)^n = 0$.

then ba is also nilpotent element of R.

Case II :- If R is non commutative

$$(ba)^{n+1} = \underbrace{(ba)(ba)\cdots(ba)}_{(n+1) \text{ times}}$$

$$= b\underbrace{(ab)(ab)\cdots(ab)}a$$

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$= b(ab)^n a = b \cdot 0 \cdot a$ (from 1)

$(ba)^{n+1} = 0$

then ba is nilpotent element of R.

From case I & II

ba is always nilpotent element of R if ab is nilpotent element of R.

Units :-

Statement :- An element $a \in R$ is said to be unit element of R if R has Multiplicative inverse of a.

The set of all units of R is denoted by $U(R)$. and defined by $U(R) = \{ a \in R \mid \bar{a}^1 \text{ is exist in } R \}$

$U(n) \rightarrow$ Unit group

Q.No. Find $U(Z)$

Soln $U(Z) = \{1, -1\} = Z^*$

Q.No. $U(Z[i]) = ?$

(2) $U(Q) = ?$

(3) $U(\mathbb{R}) = ?$

(4) $U(\phi) = ?$

(5) $U(Z_5) = ?$

Soln (i) $U(Z[i]) = \{ x \in Z[i] \mid \bar{x}^1 \text{ is exist in } Z[i] \}$

$= \{1, -1, i, -i\} = Z[i]^*$

(ii) $U(Q) = \{ x \in Q \mid \bar{x}^1 \text{ is exist w.r.t. Multiplication in } Q \}$

$= \{ x \mid 0 \neq x \in Q \}$

$= x - \{0\} = Q^*$

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Similarly $U(R) = R - \{0\} = R^*$

$U(\mathbb{C}) = \mathbb{C} - \{0\} = \mathbb{C}^*$

⑤ $U(Z_{15}) = \{ x \in Z_{15} \mid \bar{x}^{-1}$ is exist w.r.t. Multiplication modulo 15 $\}$

$= \{ x \in Z_{15} \mid \gcd(x, 15) = 1 \}$

$= \{ x \mid 1 \leq x \leq 15 \text{ and } \gcd(x, 15) = 1 \}$

$U(Z_{15}) = \{1, 2, 4, 7, 8, 11, 13, 14\}$

$= Z_{15}^*$

Note:- If $n > 1$ then # of units in $Z_n = \phi(n)$

Q. No. $R = \mathbb{Q} \times \mathbb{Q}$, find $U(\mathbb{Q} \times \mathbb{Q})$?

Soln $\mathbb{Q} \times \mathbb{Q} = \{(a, b) \mid a \in \mathbb{Q}, b \in \mathbb{Q}\}$

$(1, 1) \in \mathbb{Q} \times \mathbb{Q}$ is the unity of $\mathbb{Q} \times \mathbb{Q}$.

Let $(x_1, y_1) \in \mathbb{Q} \times \mathbb{Q}$ is unit element of $\mathbb{Q} \times \mathbb{Q}$ then $\exists$ $(x_2, y_2) \in \mathbb{Q} \times \mathbb{Q}$

s.t. $(x_1, y_1)(x_2, y_2) = (1, 1)$

$\Rightarrow (x_1 x_2, y_1 y_2) = (1, 1)$

$\Rightarrow x_1 x_2 = 1, \; y_1 y_2 = 1$

$\Rightarrow x_1$ is unit in $\mathbb{Q}$.

& y_1 is unit in $\mathbb{Q}$.

$\Rightarrow (x_1, y_1)^{-1} = (x_1^{-1}, y_1^{-1})$

Now

$U(\mathbb{Q} \times \mathbb{Q}) = \{ (x, y) \mid \bar{x}^{-1}$ is exist in $\mathbb{Q}$ & $\bar{y}^{-1}$ is exist in $\mathbb{Q} \}$

$= \{ (x, y) \mid 0 \neq x \in \mathbb{Q}, \; 0 \neq y \in \mathbb{Q} \}$

$= \{ (x, y) \mid x \in \mathbb{Q}^*, y \in \mathbb{Q}^* \}$

$= \mathbb{Q}^* \times \mathbb{Q}^*$

$= U(\mathbb{Q}) \times U(\mathbb{Q})$

Similarly $U(\mathbb{Q} \times \mathbb{R}) = U(\mathbb{Q}) \times U(\mathbb{R})$

$= \mathbb{Q}^* \times \mathbb{R}^*$

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Imp. Find unit of $\mathbb{Q}\times\{0\}$

Soln $\mathbb{Q}\times\{0\} = \{(a,0) \mid a\in\mathbb{Q}\}$

let $(x,0) \in \mathbb{Q}\times\{0\}$ is the unity of $\mathbb{Q}\times\{0\}$

Let $(x,0) \in \mathbb{Q}\times\{0\}$ is the unit element of $\mathbb{Q}\times\{0\}$

then $\exists\ (y,0) \in \mathbb{Q}\times\{0\}$ s.t.

$(x,0)(y,0) = (1,0)$

$\Rightarrow (x\cdot y, 0\cdot 0) = (1,0)$

$\Rightarrow x\cdot y = 1$

$\Rightarrow x^{-1}$ is exist in $\mathbb{Q}$.

Now $U(\mathbb{Q}\times\{0\}) = \{(x,0) \mid (x,0)^{-1} \text{ is exist in } \mathbb{Q}\times\{0\}\}$

$= \{(x,0) \mid x^{-1} \text{ is exist in } \mathbb{Q}\}$

$= \{(x,0) \mid 0\neq x\in\mathbb{Q}\}$

$= \{(x,0) \mid x\in\mathbb{Q}^*\}$

$= \mathbb{Q}^*\times\{0\}$

$\Rightarrow U(\mathbb{Q}\times\{0\}) = \mathbb{Q}^*\times\{0\} = U(\mathbb{Q})\times\{0\}$

Note: $\mathbb{Q}^*\times\{0\} = \mathbb{Q}\times\{0\} - \{(0,0)\}$

H.W.

$U(\mathbb{Q}\times\mathbb{R}\times\mathbb{C}) = ?$

$U(\mathbb{R}\times\mathbb{Q}\times\{0\}) = ?$

Ans. $\mathbb{Q}\times\mathbb{R}\times\mathbb{C} = \{(a,b,c) \mid a\in\mathbb{Q}, b\in\mathbb{R}, c\in\mathbb{C}\}$

a^{-1} exist in $\mathbb{Q}$, $0\neq a$

b^{-1} " " $\mathbb{R}$, $0\neq b$

c^{-1} " " $\mathbb{C}$, $0\neq c$

$= \{(a,b,c) \mid a\in\mathbb{Q}^*, b\in\mathbb{R}^*, c\in\mathbb{C}^*\}$

$= \mathbb{Q}^*\times\mathbb{R}^*\times\mathbb{C}^*$

$= U(\mathbb{Q})\times U(\mathbb{R})\times U(\mathbb{C})$

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$$U(\mathbb{R}\times\mathbb{Q}\times\{0\}) = \{(x,y,0)\in \mathbb{R}\times\mathbb{Q}\times\{0\} \mid (x,y,0)^{-1} \text{ exist in } \mathbb{R}\times\mathbb{Q}\times\{0\}\}$$

$$= \{(x,y,0) \mid x^{-1} \text{ exist in } \mathbb{R} \text{ \& } y^{-1} \text{ exist in } \mathbb{Q}\}$$

$$= \{(x,y,0) \mid 0\neq x\in\mathbb{R} \text{ \& } 0\neq y\in\mathbb{Q}\}$$

$$= \{(x,y,z) \mid x\in\mathbb{R}^*, y\in\mathbb{Q}^*\}$$

$$= \mathbb{R}^*\times\mathbb{Q}^*\times\{0\}$$

$$= U(\mathbb{R})\times U(\mathbb{Q})\times\{0\}$$

H.P.

Note: If $R_1, R_2, R_3 \ldots R_n$ are Commutative Ring with unity then $U(R_1\times R_2\times\cdots\times R_n) = U(R_1)\times U(R_2)\cdots\times U(R_n)$

Ideal :-

Left Ideal :- Let I is non empty subgp of R, $(I,+,\cdot)$ is an ideal of R.

If (i) $\forall\ a\in I, \forall\ b\in I \Rightarrow a-b\in I$

(ii) $\forall\ a\in I, \forall\ r\in I \Rightarrow r\cdot a\in I$

Right ideal :- $(I,+,\cdot)$ is an ideal of $(R,+,\cdot)$

(i) $\forall\ a\in I, \forall\ b\in I \Rightarrow a-b\in I$

(ii) $\forall\ a\in I, \forall\ r\in I \Rightarrow a\cdot r\in I$

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Q.No. $S=\left\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \middle| a,b \in R\right\}$ is Right ideal of $M_2(\mathbb{R})$?

(i) Let $A=\begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix} \in S$ and $B=\begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix}$

s.t. $A-B=\begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix}-\begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} a_1-a_2 & b_1-b_2 \\ 0 & 0 \end{bmatrix} \in S$

(ii) Let $A=\begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix} \in S$ and $\gamma=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R})$

s.t. $A.\gamma=\begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} a_1a+b_1c & a_1b+b_1d \\ 0 & 0 \end{bmatrix} \in S$

then S is right ideal of $M_2(\mathbb{R})$

H.W.

Q. $S=\left\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \middle| a,b \in \mathbb{R}\right\}$ is left ideal of $M_2(\mathbb{R})$?

(ii) $S=\left\{\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \middle| a,b \in \mathbb{R}\right\}$ is left ideal of $M_2(\mathbb{R})$?

Soln (i) Let $A=\begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix} \in S$ and $B=\begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix}$

s.t. $A-B=\begin{bmatrix} a_1-a_2 & b_1-b_2 \\ 0 & 0 \end{bmatrix} \in S$

(ii) Let $A=\begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix} \in S$ and $\gamma=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R})$

s.t.

$\gamma.A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} aa_1+0 & ab_1+0 \\ ca_1 & db_1 \end{bmatrix} \notin S$

then S is not left ideal of $M_2(\mathbb{R})$

Soln (ii)

Let $A=\begin{bmatrix} a_1 & 0 \\ b_1 & 0 \end{bmatrix} \in S$ and $B=\begin{bmatrix} a_2 & 0 \\ b_2 & 0 \end{bmatrix}$

s.t. $A-B=\begin{bmatrix} a_1 & 0 \\ b_1 & 0 \end{bmatrix}-\begin{bmatrix} a_2 & 0 \\ b_2 & 0 \end{bmatrix}=\begin{bmatrix} a_1-a_2 & 0 \\ b_1-b_2 & 0 \end{bmatrix} \in S$

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(ii) Let $A=\begin{bmatrix} a_1 & 0 \\ b_1 & 0 \end{bmatrix} \in S$ and $r=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R})$

$$r \cdot A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a_1 & 0 \\ b_1 & 0 \end{bmatrix} = \begin{bmatrix} aa_1+bb_1 & 0 \\ ca_1+db_1 & 0 \end{bmatrix} \in S$$

then S is left ideal of $M_2(\mathbb{R})$.

Note:- If R is commutative Ring then every left ideal of R is right ideal of R.

Ideal:-

Let I is non empty subgroup of R, $(I,+,\cdot)$ is said to be an ideal of R if

(i) $\forall\ a \in I, \forall\ b \in I$ s.t. $a-b \in I$

(ii) $\forall\ a \in I, \forall\ r \in R$ s.t. $ra \in I$ and $a \cdot r \in I$

Q.No: $I=\mathbb{Z}$ is an ideal of $\mathbb{Q}$?

Soln $2 \in \mathbb{Z}$ and $\frac{1}{3} \in \mathbb{Q}$

But $2 \cdot \frac{1}{3} = \frac{2}{3} \notin \mathbb{Z}$

then $\mathbb{Z}$ is not ideal of $\mathbb{Q}$.

Similarly, $\mathbb{Z}$ is not ideal of $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ and $\mathbb{Z}[i]$

$\mathbb{Q}$ is " " " $\mathbb{R}, \mathbb{C}$

Q.No. Show that $I=m\mathbb{Z}$ is an ideal of $\mathbb{Z}$.

Soln $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \cdots\}$

& $m\mathbb{Z} = \{ m \cdot a \mid a \in \mathbb{Z} \}$

(i) $I=m\mathbb{Z}$ is subgp. of $\mathbb{Z}$ then $\forall\ a \in m\mathbb{Z}, \forall\ b \in m\mathbb{Z}$

$\Rightarrow a-b \in m\mathbb{Z}$

(ii) Let $x \in m\mathbb{Z}$ then $x = m \cdot a$, $a \in \mathbb{Z}$

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and $r \in \mathbb{Z}$ s.t. $r.x = r.m.a = mr.a = m.a' \in m\mathbb{Z}$ where $a' = r.a \in \mathbb{Z}$

$\Rightarrow r.x \in m\mathbb{Z}$

Since $\mathbb{Z}$ is commutative then $x.r = r.x \in m\mathbb{Z}$

then $x.r \in m\mathbb{Z}$

Hence $m\mathbb{Z}$ is an ideal of $\mathbb{Z}$

Note:- Every subgp. of $\mathbb{Z}$ is an ideal of $\mathbb{Z}$

Hence $\mathbb{Z}$ has infinite no. of ideals.

e.g. $\{0\}, \mathbb{Z}, 2\mathbb{Z}, \dots$ are ideals of $\mathbb{Z}$.

Q.No. If I is an ideal of R then I is subring of R.

Soln let I is an ideal of R then

(i) $\forall a \in I, \forall b \in I \Rightarrow a-b \in I$

(ii) $\forall a \in I, \forall r \in I \Rightarrow r.a \in I$ and $a.r \in I$

Now Since I is an ideal of R then $\phi \neq I \subseteq R$

from (i) we get

(i)' $\forall a \in I, \forall b \in I \Rightarrow a-b \in I$

(ii)' $\forall a \in I, \forall b \in I \Rightarrow b \in R$, because $I \subseteq R$

s.t. $a.b \in I$

then I is subring of R. ↑ **Note** but converse of above statement cannot true.

e.g. $\mathbb{Z}$ is subring of $\mathbb{Z}[i]$

But $\mathbb{Z}$ is not ideal of $\mathbb{Z}[i]$

because $2 \in \mathbb{Z}$, $i \in \mathbb{Z}[i]$, but $2i \notin \mathbb{Z}$

Q.No.:- Show $I = \{0\}$ and $I = R$ is always ideal of R.

Soln **Case 1:-** Let $I = \{0\}$

(i) $\forall a \in I, \forall b \in I \Rightarrow a-b = 0-0 = 0 \in I \Rightarrow a-b \in I$

(ii) $\forall a \in I, \forall r \in I \Rightarrow a.r = r.a = 0 \in I$

$= r \cdot a = a \cdot r \in I$

then $I = \{0\}$ is an ideal of R.

Case II Let $I = R$ (i) $\forall a \in I, \forall b \in I$

$\Rightarrow a - b \in R$

$\Rightarrow a - b \in R = I$

$\Rightarrow a - b \in I$

(ii) $\forall a \in I = R, \forall r \in R$

s.t. $r \cdot a \in R$ (By closure prop. because R is Ring)

& $a \cdot r \in R$

then $I = R$ is an ideal of R

from Case I & II, $I = \{0\}$ & $I = R$ are alwalys an ideal of R.

For Examples-

(1) $I = \{0\}$ & $I = \mathbb{Q}$ are ideal of $\mathbb{Q}$

(2) $I = \{0\}$ & $I = \mathbb{Z}[i]$ " " " $\mathbb{Z}[i]$

(3) $I = \{0\}$ " $I = \mathbb{R}$ " " " $\mathbb{R}$

Q. If I is an ideal of R and $1 \in I$ then $I = R$

Soln let I is an ideal of R and $1 \in I$

since I is an ideal of R then $I \subseteq R$ —(1)

Now Let $r \in R$ s.t. $r \cdot 1 \in I$

$\Rightarrow r \in I$

$\Rightarrow r$ is subset of I ($\because I$ is an ideal of R)

$\Rightarrow R \subseteq I$ —(2)

from (1) and (ii), we get.

$R = I$

Q. Let $I = \{0, 1, a_1, a_2, a_3, a_4\}$ be an ideal of R then

(1) $I \subset R$

(2) $R \subset I$

(3) $I = R$

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Soln I is an ideal of R and $1 \in I$ then $I = R$

Q. Show that $I = mZ_n = \langle m \rangle$ is an ideal of Z_n.

Soln $I = \langle m \rangle = mZ_n = \{ m \cdot a \mid a \in Z_n \}$

(i) $\langle m \rangle = mZ_n$ is subring of Z_n then

$\forall x \in mZ_n, \forall y \in mZ_n \Rightarrow x - y \in mZ_n$

(ii) Let $x \in mZ_n$ then $x = m \cdot a$, $a \in Z_n$

& $r \in Z_n$ s.t. $x \cdot r = m \cdot a \cdot r = ma' \in mZ_n$, where $a' = ar \in Z_n$

$\Rightarrow x \cdot r \in mZ_n$

Since Z_n is commutative Ring then $x \cdot r = r \cdot x$

$\Rightarrow r \cdot x \in mZ_n$

then mZ_n is an ideal of Z_n.

Note: Every subring of Z_n is an ideal of Z_n.

then $\boxed{\#\text{ of ideals in } Z_n = \tau(n)}$

Q. $R = Z_{40}$, find No. of ideals in Z_{40}

Soln # of ideals in $Z_{40} = \tau(40) = 2^3 \cdot 5^1$

$= (3+1)(1+1)$

$= 8$

Say $I_1 = \{0\}$, $I_2 = \langle 2 \rangle$, $I_3 = \langle 4 \rangle$, $I_4 = \langle 5 \rangle$, $I_5 = \langle 8 \rangle$, $I_6 = \langle 10 \rangle$, $I_7 = \langle 20 \rangle$, $I_8 = Z_{40}$

Q. How many deals in Z_{100} of order 25

Ans Z_{100} has unique subgp. of order 25 then Z_{100} has unique ideal of order 25.

Becoz every Sbgp. of Z_n is an ideal of Z_n.

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Q. $R = Z_{p^2}$, how many ideals of order p in Z_{p^2} ?

Soln Z_{p^2} has unique subgp. of order p.

then Z_{p^2} has unique ideal of order p.

H.P.

Q. if $a \in \mathbb{Z}[i]$ then $I = \langle a \rangle = \{a \cdot k \mid k \in \mathbb{Z}[i]\} = a\mathbb{Z}[i]$ is an ideal of $\mathbb{Z}[i]$.

Soln $\mathbb{Z}[i] = \{a+ib \mid a, b \in \mathbb{Z}\}$ — ①

and $I = \langle a \rangle = a\mathbb{Z}[i]$

① Let $x \in a\mathbb{Z}[i]$ then $x = a \cdot k_1$, $k_1 \in \mathbb{Z}[i]$

& $y \in a\mathbb{Z}[i]$ " $y = a \cdot k_2$, $k_2 \in \mathbb{Z}[i]$

s.t. $x - y = a(k_1 - k_2)$

$= a k' \in a\mathbb{Z}[i]$, where $k' = k_1 - k_2 \in \mathbb{Z}[i]$

② Let $x \in a\mathbb{Z}[i]$

then $x = a \cdot k_1$, $k_1 \in \mathbb{Z}[i]$

and $r \in \mathbb{Z}[i]$ s.t. $x \cdot r = a k_1 r = a k' \in a\mathbb{Z}[i]$ where $k' = k_1 r$

$\Rightarrow x \cdot r \in a\mathbb{Z}[i]$

Since $\mathbb{Z}[i]$ is commutative

then $x \cdot r = r \cdot x \in a\mathbb{Z}[i]$

$\Rightarrow r \cdot x \in a\mathbb{Z}[i]$

then $a\mathbb{Z}[i]$ is an ideal of $\mathbb{Z}[i]$.

e.g. ① $i \in \mathbb{Z}[i]$ then $I = \langle i \rangle = i\mathbb{Z}[i]$ is an ideal of $\mathbb{Z}[i]$.

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(ii) $1+i \in Z[i]$ then $I = \langle (1+i) \rangle = (1+i)Z[i]$ is an ideal of $Z[i]$.

(iii) $2+3i \in Z[i]$ then $I = \langle 2+3i \rangle = (2+3i)Z[i]$ is an ideal of $Z[i]$.

Note:

$iZ[i] = Z[i]$

$iZ[i] = \{ i \cdot a \mid a \in Z[i] \}$

$-iZ[i]$ and $i \in iZ[i]$

Such that $i(-i) \in iZ[i]$, because $iZ[i]$ is an ideal of $Z[i]$

$\Rightarrow 1 \in iZ[i]$

then $iZ[i] = Z[i]$

Similarly $-iZ[i] = Z[i]$

$-1Z[i] = Z[i]$

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