Solution of System of Linear Algebraic Equations

1. Gauss Elimination Method

It is a direct method to solve systems of linear algebraic equations. In this method, the unknowns are eliminated successively and the system is reduced to an upper triangular system. The unknowns are obtained by back substitution.

Consider the equations

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \quad \ldots(i) \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \quad \ldots(ii) \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \quad \ldots(iii) \end{aligned}\right\} \quad \ldots(1)$$

Method to Solve System of Eq. (1)

Step 1 Assuming $a_{11} \neq 0$, We eliminate x from Eqs. (ii) and (iii) with the help of Eq. (i) and then, we get new system.

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \quad \ldots(iv) \\ a'_{22}\ y + a'_{23}\ z &= b'_2 \quad \ldots(v) \\ a'_{32}\ y + a'_{33}\ z &= b'_3 \quad \ldots(vi) \end{aligned}\right\} \quad \ldots(2)$$

Here, the first Eq. (iv) is called the pivotal equation and a_{11} is called the first pivot.

Step 2 Assuming $a'_{22} \neq 0$, we eliminate y from Eq. (vi) with the help of Eq. (v).

Now, we get the new system

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \quad \ldots(vii) \\ a'_{22}\ y + a'_{23}\ z &= b'_2 \quad \ldots(viii) \\ a'_{33}\ z &= b''_3 \quad \ldots(ix) \end{aligned}\right\} \quad \ldots(3)$$

Here, second Eq. (viii) is pivotal and a'_{22} is the new pivot.

Step 3 The values of x, y and z are found from system of Eq. (3) by back substitution.

Note Solving system of Eq. (1) by above method is equivalent to solve the matrix equation, $AX = B$

i.e., $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

by converting coefficient matrix A to upper triangular matrix by elementary row transformations.

This method fail, if any one of the pivots a_{11}, a'_{22} or a''_{33} becomes zero.

In such cases, we rewrite the equations in a different order, so that the pivots are non-zero.

Example 6. *By using Gauss-elimination method, solve the equations*

$$2x + 4y + z = 3$$
$$3x + 2y - 2z = -2$$
$$x - y + z = 6$$

Solution. Given system of equations can be written as

$$x - y + z = 6 \quad \text{...(i)}$$
$$2x + 4y + z = 3 \quad \text{...(ii)}$$
$$3x + 2y - 2z = -2 \quad \text{...(iii)}$$

Step 1 To eliminate x from Eqs. (ii) and (iii) with the help of Eq. (i).

From Eq. (ii) −2 Eq. (i) and Eq. (iii) − 3 Eq. (i), we get

$$x - y + z = 6 \quad \text{...(iv)}$$
$$6y - z = -9 \quad \text{...(v)}$$
$$5y - 5z = -20 \quad \text{...(vi)}$$

Step 2 To eliminate y from Eq. (vi) with the help of Eq. (v) by $\frac{6}{5}$ Eq. (vi) — Eq. (v), we get

$$x - y + z = 6 \quad \text{...(vii)}$$
$$6y - z = -9 \quad \text{...(viii)}$$
$$-5z = -15 \quad \text{...(ix)}$$

Step 3 Finding x, y and z by back substitution

From Eq. (ix), $z = 3$

From Eq. (viii), $6y = z - 9 = 3 - 9 = -6 \Rightarrow y = -1$

From Eq. (vii), $x = y - z + 6 = -1 - 3 + 6 = 2$

Hence, $x = 2,\ y = -1,\ z = 3$

Alternative Method From given equations, we have

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & 1 \\ 3 & 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix}$$

$R_2 - 2R_1,\ R_3 - 3R_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 6 & -1 \\ 0 & 5 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -9 \\ 20 \end{bmatrix}$$

$\frac{6}{5}R_3 - R_2$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 6 & -1 \\ 0 & 0 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -9 \\ -15 \end{bmatrix}$$

Here, coefficient matrix in upper triangular form.

Thus, we have $-5z = -15 \Rightarrow z = 3$

$6y - z = -9 \Rightarrow 6y = z - 9$

$= 3 - 9 = -6$

$\Rightarrow y = -1$

and $x - y + z = 6 \Rightarrow x = y - z + 6$

$\Rightarrow x = -1 - 3 + 6 = 2$

$\Rightarrow x = 2,\ y = -1,\ z = 3.$

Example 7. *Using Gauss-elimination method, solve the equations*

$$x + 2y + 3z - u = 10$$
$$2x + 3y - 3z - u = 1$$
$$2x - y + 2z + 3u = 7$$
$$3x + 2y - 4z + 3u = 2$$

Solution. Writing the given system of equation in matrix form $AX = B$, we have

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 3 & -3 & -1 \\ 2 & -1 & 2 & 3 \\ 3 & 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \\ 7 \\ 2 \end{bmatrix}$$

$(R_2 - 2R_1,\ R_3 - 2R_1,\ R_4 - 3R_1)$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -9 & 1 \\ 0 & -5 & -4 & 5 \\ 0 & -4 & -13 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 10 \\ -19 \\ -13 \\ -28 \end{bmatrix}$$

$(R_3 - 5R_2,\ R_4 - 4R_2)$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -9 & 1 \\ 0 & 0 & 41 & 0 \\ 0 & 0 & 23 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 10 \\ -19 \\ 82 \\ 48 \end{bmatrix}$$

$\left(R_4 - \frac{23}{41}R_3\right)$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -9 & 1 \\ 0 & 0 & 41 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 10 \\ -19 \\ 82 \\ 2 \end{bmatrix}$$

Now, the coefficient matrix in the upper triangular form.

Now, from backward substitution, we get

$2u = 2 \Rightarrow u = 1$

$41z = 82 \Rightarrow z = 2$

$-y - 9z + u = -19 \Rightarrow y = -9z + u + 19$

$\Rightarrow y = -18 + 1 + 19 = 2$

$x + 2y + 3z - u = 10 \Rightarrow x = -2y - 3z + u + 10$

$\Rightarrow x = -4 - 6 + 1 + 10 = 1$

Hence, $x = 1,\ y = 2,\ z = 2,\ u = 1$

5.5 Gauss Elimination Method

In this method, the variables are eliminated by a process of systematic elimination. Suppose the system has n variables and n equations of the form (5.1). This procedure reduces the system of linear equations to an equivalent upper triangular system which can be solved by back-substitution. To convert an upper triangular system, x_1 is eliminated from second equation to nth equation, x_2 is eliminated from third equation to nth equation, x_3 is eliminated from fourth equation to nth equation, and so on and finally, x_{n-1} is eliminated from nth equation.

To eliminate x_1, from second, third, $\cdots$, and nth equations the first equation is multiplied by $-\frac{a_{21}}{a_{11}}, -\frac{a_{31}}{a_{11}}, \ldots, -\frac{a_{n1}}{a_{11}}$ respectively and successively added with the second, third, $\cdots$, nth equations (assuming that $a_{11} \neq 0$). This gives

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{22}^{(1)}x_2 + a_{23}^{(1)}x_3 + \cdots + a_{2n}^{(1)}x_n &= b_2^{(1)} \\
a_{32}^{(1)}x_2 + a_{33}^{(1)}x_3 + \cdots + a_{3n}^{(1)}x_n &= b_3^{(1)} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots &\cdots \\
a_{n2}^{(1)}x_2 + a_{n3}^{(1)}x_3 + \cdots + a_{nn}^{(1)}x_n &= b_n^{(1)},
\end{aligned} \tag{5.12}$$

where

$$a_{ij}^{(1)} = a_{ij} - \frac{a_{i1}}{a_{11}}a_{1j}; \quad i, j = 2, 3, \ldots, n.$$

Again, to eliminate x_2 from the third, forth, ..., and nth equations the second equation is multiplied by $-\frac{a_{32}^{(1)}}{a_{22}^{(1)}}, -\frac{a_{42}^{(1)}}{a_{22}^{(1)}}, \ldots, -\frac{a_{n2}^{(1)}}{a_{22}^{(1)}}$ respectively (assuming that $a_{22}^{(1)} \neq 0$), and successively added to the third, fourth, ..., and nth equations to get the new system of equations as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{22}^{(1)}x_2 + a_{23}^{(1)}x_3 + \cdots + a_{2n}^{(1)}x_n &= b_2^{(1)} \\ a_{33}^{(2)}x_3 + \cdots + a_{3n}^{(2)}x_n &= b_3^{(2)} \\ \cdots\cdots\cdots\cdots\cdots\cdots &\cdots \\ a_{n3}^{(2)}x_3 + \cdots + a_{nn}^{(2)}x_n &= b_n^{(2)}, \end{aligned} \tag{5.13}$$

where

$$a_{ij}^{(2)} = a_{ij}^{(1)} - \frac{a_{i2}^{(1)}}{a_{22}^{(1)}}a_{2j}^{(1)}; \quad i, j = 3, 4, \ldots, n.$$

Finally, after eliminating x_{n-1}, the above system of equations become

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{22}^{(1)}x_2 + a_{23}^{(1)}x_3 + \cdots + a_{2n}^{(1)}x_n &= b_2^{(1)} \\ a_{33}^{(2)}x_3 + \cdots + a_{3n}^{(2)}x_n &= b_3^{(2)} \\ \cdots\cdots\cdots\cdots &\cdots \\ a_{nn}^{(n-1)}x_n &= b_n^{(n-1)}, \end{aligned} \tag{5.14}$$

where,

$$a_{ij}^{(k)} = a_{ij}^{(k-1)} - \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}}a_{kj}^{(k-1)};$$

$i, j = k+1, \ldots, n$; $k = 1, 2, \ldots, n-1$, and $a_{pq}^{(0)} = a_{pq}$; $p, q = 1, 2, \ldots, n$.

Now, by back substitution, the values of the variables can be found as follows:

From last equation we have, $x_n = \dfrac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$, from the last but one equation, i.e., $(n-1)$th equation, one can find the value of x_{n-1} and so on. Finally, from the first equation we obtain the value of x_1.

The evaluation of the elements $a_{ij}^{(k)}$'s is a **forward substitution** and the determination of the values of the variables x_i's is a **back substitution** since we first determine the value of the last variable x_n.

Note 5.5.1 The method described above assumes that the diagonal elements are non-zero. If they are zero or nearly zero then the above simple method is not applicable to solve a linear system though it may have a solution. If any diagonal element is zero or very small then partial pivoting should be used to get a solution or a better solution.

It is mentioned earlier that if the system is diagonally dominant or real symmetric and positive definite then no pivoting is necessary.

Example 5.5.1 Solve the equations by Gauss elimination method.
$2x_1 + x_2 + x_3 = 4,\ x_1 - x_2 + 2x_3 = 2,\ 2x_1 + 2x_2 - x_3 = 3.$

Solution. Multiplying the second and third equations by 2 and 1 respectively and subtracting them from first equation we get

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 4 \\ 3x_2 - 3x_3 &= 0 \\ -x_2 + 2x_3 &= 1. \end{aligned}$$

Multiplying third equation by -3 and subtracting from second equation we obtain

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 4 \\ 3x_2 - 3x_3 &= 0 \\ 3x_3 &= 3. \end{aligned}$$

From the third equation $x_3 = 1$, from the second equations $x_2 = x_3 = 1$ and from the first equation $2x_1 = 4 - x_2 - x_3 = 2$ or, $x_1 = 1$.
Therefore the solution is $x_1 = 1, x_2 = 1, x_3 = 1$.

Example 5.5.2 Solve the following system of equations by Gauss elimination method (use partial pivoting).

$$\begin{aligned} x_2 + 2x_3 &= 5 \\ x_1 + 2x_2 + 4x_3 &= 11 \\ -3x_1 + x_2 - 5x_3 &= -12. \end{aligned}$$

Solution. The largest element (the pivot) in the coefficients of the variable x_1 is -3, attained at the third equation. So we interchange first and third equations

$$\begin{aligned} -3x_1 + x_2 - 5x_3 &= -12 \\ x_1 + 2x_2 + 4x_3 &= 11 \\ x_2 + 2x_3 &= 5. \end{aligned}$$

Multiplying the second equation by 3 and adding with the first equation we get,

$$\begin{aligned} -3x_1 + x_2 - 5x_3 &= -12 \\ x_2 + x_3 &= 3 \\ x_2 + 2x_3 &= 5 \end{aligned}$$

The second pivot is 1, which is at the positions a_{22} and a_{32}. Taking $a_{22} = 1$ as pivot to avoid interchange of rows. Now, subtracting the third equation from second equation, we obtain

$$\begin{aligned} -3x_1 + x_2 - 5x_3 &= -12 \\ x_2 + x_3 &= 3 \\ -x_3 &= -2. \end{aligned}$$

Now by back substitution, the values of x_3, x_2, x_1 are obtained as

$x_3 = 2,\ x_2 = 3 - x_3 = 1, x_1 = -\frac{1}{3}(-12 - x_2 + 5x_3) = 1.$

Hence the solution is $x_1 = 1,\ x_2 = 1,\ x_3 = 2.$