SEMESTER-IV

HONOURS

CORE COURSE---C 8T

UNIT-I (MARKS-19)

UNIT-I (Riemann Integration) **Dr. Pradip Kumar Gain**

Syllabus for Unit-I : Riemann Integration: Inequalities of upper and lower sums, Darbaux Integation, Darboux Theorem, Riemann Conditions of Integrability, Riemann sum and definition of Riemann Integral through Riemann sums, Equivalence of two definitions, Riemann Integrability of monotone and continuous functions, Properties of the Riemann Integral, Definition and integrability of piecewise continuous and monotone functions.

Intermediate Value Theorem for Integrals, Fundamental Theorem of Integral Calculus.

The famous German Mathematician B. Riemann was the First to remove the concept of definite integral from a geometrical basis and give an arithmetical approach to it.

SOME DEFINITIONS AND NOTATIONS

DEFINITION : (Division or Partition) By a ***division or partition*** D (or P) of a closed interval $[a,b]$ we shall mean a finite set of numbers $D=\{x_o, x_1, x_2, ..., x_{r-1}, x_r, ..., x_{n-1}, x_n = b\}$ satisfying $a = x_o < x_1 < x_2 < .... < x_{r-1} < x_r < .... < x_{n-1} < x_n = b$.

The rth subinterval of the division D is denoted by δ_r. That is, $\delta_r = [x_{r-1}, x_r]$. The length of rth subinterval of the division D is also denoted by δ_r. That is, $\delta_r = x_r - x_{r-1}$.

DEFINITION : (Norm) By the ***Norm of the division*** D we shall mean the length of greatest of subintervals created by the division D. The Norm of the division D is denoted by $\|D\|$ or by δ.

DEFINITION : (Upper and Lower Sums) The sums

$$S(D) = U(D, f) = M_1\delta_1 + M_2\delta_2 + ... + M_{r-1}\delta_{r-1} + M_r\delta_r + ... + M_{n-1}\delta_{n-1} + M_n\delta_n,$$

$$s(D) = L(D, f) = m_1\delta_1 + m_2\delta_2 + ... + m_{r-1}\delta_{r-1} + m_r\delta_r + ... + m_{n-1}\delta_{n-1} + m_n\delta_n$$

are respectively, called the ***Upper Integral Sum*** (**or** ***Upper Sum***) and ***Lower Integral Sum*** (**or** ***Lower Sum***) of $f(x)$ for the division/Partition D where M_r is the supremum of the function

$f(x)$ for the subinterval $\delta_r = [x_{r-1}, x_r]$ and m_r is the infimum of the function $f(x)$ for the subinterval $\delta_r = [x_{r-1}, x_r]$.

DEFINITION : (Oscillatory Sum)

The difference $S(D) - s(D) = \sum_{r=1}^{n} M_r\delta_r - m_r\delta_r = \sum_{r=1}^{n} (M_r - m_r)\delta_r = \sum_{r=1}^{n} O_r\delta_r$ is called the ***oscillatory sum*** and $O_r = M_r - m_r$ is called the oscillation of the function in $\delta_r = [x_{r-1}, x_r]$.

DEFINITION : (Refinement of division/Partition) If a division/Partition D' be constructed from D by distributing a few additional division points between those already occurring we shall say that D' is a refinement of D.

NOTE : If there are two refinements D_1 and D_2 the their common refinement will be $D = D_1 \cup D_2$.

RIEMANN INTEGRABILITY

Let f be a bounded function defined in the closed interval $[a,b]$.

Let $D = \{a = x_0, x_1, x_2, ..., x_r, ..., x_n = b\}$ be a division/partition of $[a,b]$. Then $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$,................,$[x_{r-1}, x_r]$,......................,$[x_{n-1}, x_n]$ are the subintervals in which the interval $[a,b]$ is divided. Let the length the rth interval ,i.e., $\delta_r = [x_{r-1}, x_r]$ be δ_r. Since the function is bounded in $[a,b]$ is also necessarily bounded in each of the subintervals . Let M_r and m_r be the supremum and infimum of f in $\delta_r = [x_{r-1}, x_r]$.

If M and m be the supremum and infimum of f in $[a,b]$ then for every value of r, we have, $m \le m_r \le M_r \le M$

$\Rightarrow m\delta_r \le m_r\delta_r \le M_r\delta_r \le M\delta_r$. Putting $r = 1,2,3,.........., n$ we have

$$m\delta_1 \le m_1\delta_1 \le M_1\delta_1 \le M\delta_1 \text{ ,}$$

$$m\delta_2 \le m_2\delta_2 \le M_2\delta_2 \le M\delta_2 \text{,}$$

.... ,

.... ,

.... ,

$$m\delta_n \le m_n\delta_n \le M_n\delta_n \le M\delta_n$$ Adding these, we get,

$$m(\delta_1 + \delta_2 + ... + \delta_n) \le m_1\delta_1 + m_2\delta_2 + ... + m_n\delta_n \le M_1\delta_1 + M_2\delta_2 + ... + M_n\delta_n \le M(\delta_1 + \delta_2 ... + \delta_n)$$

$\Rightarrow m(b-a) \le s(D) \le S(D) \le M(b-a)$.

This is true for all possible divisions/partitions $D_1, D_2, D_3, ...$. Therefore,

$m(b-a) \le s(D), s(D_1), s(D_2), \ldots\ldots\ldots\ldots\ldots\ldots \le S(D), S(D_1), S(D_2), \ldots\ldots\ldots\ldots\ldots\ldots \le M(b-a)$

Therefore, the set of all lower sums $s(D), s(D_1), s(D_2), \ldots\ldots\ldots$ and the set of all lower sums $S(D), S(D_1), S(D_2), \ldots\ldots\ldots$ are bounded.

The infimum of the set of all upper sums $S(D), S(D_1), S(D_2), \ldots\ldots\ldots$ is called the upper integral of f over $[a,b]$ and is denoted by $U = \int_a^{\bar{b}} f(x)dx$.

The supremum of the set of all lower sums $s(D), s(D_1), s(D_2), \ldots\ldots\ldots$ is called the lower integral of f over $[a,b]$ and is denoted by $L = \int_{\underline{a}}^{b} f(x)dx$.

A bounded function f is said to be Riemann Integrable or simply integrable over $[a,b]$, if its upper integral and lower integral are equal.

The common value of these integrals is called the Riemann Integral and is denoted by $I = \int_a^b f(x)dx$.

DARBOUX'S THEOREM

THEOREM 1: ***To every positive quantity ε, however small it may be, there corresponds a positive quantity δ such that*** $S(D) < \int_a^{\bar{b}} f(x)dx \quad \forall D$ ***with*** $\|D\| \le \delta$

and $\quad s(D) > \int_{\underline{a}}^{b} f(x)dx \quad \forall D$ ***with*** $\|D\| \le \delta$

RIEMANN CONDITION OF INTEGRABILITY

NECESSARY AND SUFFICIENT CONDITION FOR INTEGRABILITY

(FIRST FORM)

THEOREM 2: ***A necessary and sufficient condition for integrability of a bounded function is that to every $\varepsilon > 0$, there corresponds a $\delta > 0$ such that for every division D whose norm is $\le \delta$, the oscillatory sum $\omega(D) = S(D) - s(D) < \varepsilon$***

Proof : **The condition is necessary**

Let the given bounded function is integrable. Then we must have

$\int_a^{\bar{b}} f(x)dx = \int_{\underline{a}}^{b} f(x)dx = \int_a^b f(x)dx$. Let $\varepsilon > 0$. By darboux's theorem, there exists $\delta > 0$ such that for every division D with $\|D\| \le \delta$ $\quad S(D) < \int_a^{\bar{b}} f(x)dx + \frac{\varepsilon}{2} = \int_a^b f(x)dx + \frac{\varepsilon}{2}$ and

$$s(D) > \int_{\underline{a}}^{b} f(x)dx - \frac{\varepsilon}{2} = \int_a^b f(x)dx - \frac{\varepsilon}{2}$$

Therefore, $\int_a^b f(x)dx - \frac{\varepsilon}{2} < s(D) \le S(D) < +\frac{\varepsilon}{2}$. This implies $\omega(D) = S(D) - s(D) < \varepsilon$.

The condition is sufficient

Let $\varepsilon > 0$. There exists a division D such that $S(D) - s(D) < \varepsilon$.

That is, $\left\{S(D) - \overline{\int_a^b} f(x)dx\right\} + \left\{\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx\right\} + \left\{\underline{\int_a^b} f(x)dx - s(D)\right\} < \varepsilon$. Since each of the three brackets is non-negative, we have $0 \le \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx < \varepsilon$. As $\varepsilon > 0$ is arbitrary, we see that the non-negative number $\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx$ is less than every positive number, however small that number may be, and hence $\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx = 0 \Rightarrow \overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx \Rightarrow f(x)$ is integrable.

NECESSARY AND SUFFICIENT CONDITION FOR INTEGRABILITY

(SECOND FORM)

THEOREM 3: ***A necessary and sufficient condition that a bounded real valued function*** $f(x)$ ***be integrable in the closed interval*** $[a,b]$ ***is that for each*** $\varepsilon > 0$***, however small, there exists a division/partition*** D ***of*** $[a,b]$ ***such that*** $0 \le S(D) - s(D) < \varepsilon$***, where*** $S(D)$ ***and*** $s(D)$ ***are the upper sum and lower sum of*** $f(x)$ ***corresponding to the division/partition*** D***.***

Proof : **The condition is necessary**

Since $f(x)$ is integrable, $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$. Also we can find a division/partition D' for which the upper sum $S(D')$ (say) such that $S(D') < \overline{\int_a^b} f(x)dx + \frac{\varepsilon}{2}$ and for a division/partition D'' the lower sum $s(D'')$ (say) such that $s(D'') > \underline{\int_a^b} f(x)dx - \frac{\varepsilon}{2}$. Let D be the common refinement of D' and D'' . Then $S(D) < \overline{\int_a^b} f(x)dx + \frac{\varepsilon}{2}$ and $s(D) > \underline{\int_a^b} f(x)dx - \frac{\varepsilon}{2}$.

Hence $S(D) - \frac{\varepsilon}{2} < \overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx < s(D) + \frac{\varepsilon}{2} \Rightarrow 0 \le S(D) - s(D) < \varepsilon$.

The condition is sufficient

Let $0 \le S(D) - s(D) < \varepsilon$. Since $\overline{\int_a^b} f(x)dx$ is the infimum of the set of all upper sums corresponding to every possible divisions and $\underline{\int_a^b} f(x)dx$ is the supremum of the set of all lower sums corresponding to every possible divisions, we must have, $S(D) \ge \overline{\int_a^b} f(x)dx$ and $s(D) \le \underline{\int_a^b} f(x)dx$. This implies $\varepsilon > S(D) - s(D) \ge \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx$. That is, $\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx < \varepsilon$. Since is arbitrary positive quantity, however small, it follows that $\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx$ is less than every positive quantity, however small. So $\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx = 0$. That is, $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$. Hence $f(x)$ is integrable.

RIEMANN SUM & RIEMANN INTEGRABILITY IN TERMS OF RIEMANN SUM

DEFINITION : (Riemann Sum) Let $f(x)$ be a bounded function defined on the closed interval $[a,b]$. Let $D = \{x_o, x_1, x_2, \ldots, x_{r-1}, x_r, \ldots, x_{n-1}, x_n = b\}$ be a division of $[a,b]$. Let $\xi_1, \xi_2, \xi_3, \ldots\ldots, \xi_n$ are arbitrary chosen points such that $\xi_1 \in \delta_1 = [x_0, x_1], \xi_2 \in [x_1, x_2], \xi_3 \in [x_2, x_3], \ldots\ldots, \xi_n \in [x_{n-1}, x_n]$. Then the sum $f(\xi_1)\delta_1 + f(\xi_2)\delta_2 + f(\xi_3)\delta_3 + \ldots\ldots + f(\xi_n)\delta_n = \sum_{r=1}^{n} f(\xi_r)\delta_r$ is called a Riemann sum for the division D and for the chosen point $\xi_r, r = 1,2,3,\ldots\ldots, n$. It is denoted by $R(D, f, \xi)$ or by $R(D)$.

NOTE : Let M_r and m_r be the supremum and infimum of f in $\delta_r = [x_{r-1}, x_r]$. Then $m_r \le f(\xi_r) \le M_r$, $r = 1,2,3,\ldots\ldots, n$

$$\Rightarrow m_r\delta_r \le f(\xi_r)\delta_r \le M_r\delta_r$$

$$\Rightarrow \sum_{r=1}^{n} m_r\delta_r \le \sum_{r=1}^{n} f(\xi_r)\delta_r \le \sum_{r=1}^{n} M_r\delta_r$$

$\Rightarrow s(D) \le R(D) \le S(D)$.

That is, Riemann sum for a function f corresponding to a division D lies between the lower sum and the upper sum of corresponding to a division D. No matter how we select the intermediate points ξ_r

DEFINITION : (Riemann Integrability in terms of Riemann Sum) Let f be a bounded function defined on the closed interval $[a,b]$. Then f is said to be integrable on $[a,b]$ if there exists a real number A such that $\underset{\|D\|\to 0}{Lim} R(D) = A$, where $\|D\|$ is the norm of the division D of $[a,b]$, $R(D)$ is a Riemann sum for f corresponding the division D of $[a,b]$ and corresponding to an arbitrary choice of intermediate points. In this case, $A = \int_a^b f(x)dx$.

EQUIVALENCE OF TWO DEFINITIONS OF INTEGRABILITY

THEOREM 4 : (Equivalence of two definitions) Let f be a bounded function defined on the closed interval $[a,b]$, $b > a$. The necessary and sufficient condition that f be integrable over $[a,b]$ and equal to $A\left(= \int_a^b f(x)dx\right)$ is that $\underset{\|D\|\to 0}{Lim} R(D) = A\left(= \int_a^b f(x)dx\right)$.

Proof : **The condition is necessary**

Let f be integrable over $[a,b]$. That is, $\int_a^b f(x)dx$ exists. Since f is integrable over $[a,b]$, for any $\varepsilon > 0$, there exists a positive δ such that $S(D) - s(D) < \varepsilon$ for all possible division D of $[a,b]$ with $\|D\| < \delta$ [i.e., $\|D\| \to 0$]. For every division D of $[a,b]$, $s(D) \le \int_a^b f(x)dx \le S(D)$ and for every division D of $[a,b]$, $s(D) \le R(D) \le S(D)$ where $R(D)$ is a Riemann sum for f corresponding the division D of $[a,b]$ and corresponding to an arbitrary choice of intermediate points . Therefore, for every division D of $[a,b]$ $\left|R(D) - \int_a^b f(x)dx\right| \le S(D) - s(D) \Rightarrow \left|R(D) - \int_a^b f(x)dx\right| < \varepsilon$ for all division D of $[a,b]$ with $\|D\| < \delta$ [i.e., $\|D\| \to 0$]. Hence $\underset{\|D\|\to 0}{Lim} R(D) = A\left(= \int_a^b f(x)dx\right)$.

The condition is sufficient

Let $\underset{\|D\|\to 0}{Lim}\, R(D) = A\left(=\int_a^b f(x)dx\right)$. Thus for each $\varepsilon > 0$ there exists $\delta > 0$ with $\|D\| < \delta$, $\left|R(D) - \int_a^b f(x)dx\right| < \frac{\varepsilon}{2}$. That is, $\int_a^b f(x)dx - \frac{\varepsilon}{2} < R(D) < \int_a^b f(x)dx + \frac{\varepsilon}{2}$.

That is, $A - \frac{\varepsilon}{2} < \sum_{r=1}^{n} f(\xi_r)\delta_r < A + \frac{\varepsilon}{2}$ for any choice of ξ_r in δ_r ....(1)

Then for each subinterval δ_r of D, there exists α_r and β_r such that $M_r - \frac{\varepsilon}{2(b-a)} < f(\alpha_r) \le M_r$ and $m_r \le f(\beta_r) < m_r + \frac{\varepsilon}{2(b-a)}$. That is, $f(\alpha_r) > M_r - \frac{\varepsilon}{2(b-a)}$ and $f(\beta_r) < m_r + \frac{\varepsilon}{2(b-a)}$.

Then $\sum_{r=1}^{n} f(\alpha_r)\delta_r > \sum_{r=1}^{n} M_r\delta_r - \frac{\varepsilon}{2(b-a)}\sum_{r=1}^{n}\delta_r$ and $\sum_{r=1}^{n} f(\beta_r)\delta_r < \sum_{r=1}^{n} m_r\delta_r + \frac{\varepsilon}{2(b-a)}\sum_{r=1}^{n}\delta_r$.

That is, $\sum_{r=1}^{n} f(\alpha_r)\delta_r > S(D) - \frac{\varepsilon}{2}$ and $\sum_{r=1}^{n} f(\beta_r)\delta_r < s(D) + \frac{\varepsilon}{2}$ ....(2).

Since (1) holds for any choice of ξ_r in δ_r, let $\xi_r = \alpha_r$, we have from (1) $\sum_{r=1}^{n} f(\alpha_r)\delta_r < A + \frac{\varepsilon}{2}$ whereby from (2) we have $S(D) - \frac{\varepsilon}{2} < \sum_{r=1}^{n} f(\alpha_r)\delta_r < A + \frac{\varepsilon}{2} \quad \Rightarrow S(D) < A + \varepsilon$ ....(3).

Similarly taking $\xi_r = \beta_r$ we have from (1), $\sum_{r=1}^{n} f(\beta_r)\delta_r > A - \frac{\varepsilon}{2}$ whereby from (2) we have $s(D) + \frac{\varepsilon}{2} > \sum_{r=1}^{n} f(\beta_r)\delta_r > A - \frac{\varepsilon}{2} \quad \Rightarrow s(D) > A - \varepsilon$ ....(4).

Since $s(D) \le \underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx \le S(D)$, we have, $A - \varepsilon < \underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx < A + \varepsilon$

Whereby $\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \le 2\varepsilon$ and $\left|\underline{\int_a^b} f(x)dx - A\right| < \varepsilon$. Since ε is arbitrary small positive quantity, we must have $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$ and $\underline{\int_a^b} f(x)dx = A$

Therefore, $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx = A = \int_a^b f(x)dx$. Hence f is Riemann integrable.

INTEGRABILITY OF CONTINUOUS FUNCTION

THEOREM 5 : ***Every continuous function is integrable.***

Proof : Let a continuous function f is defined on the interval $[a,b]$. Let D be a division of $[a,b]$ which divides the interval $[a,b]$ into a finite number of sub-intervals $\delta_r = [x_{r-1}, x_r], r = 1,2,3,\ldots\ldots, n$. Since f is continuous in $[a,b]$ it is bounded in $[a,b]$. So f is bounded in every sub-intervals $\delta_r = [x_{r-1}, x_r], r = 1,2,3,\ldots\ldots, n$ of $[a,b]$. Again since f is continuous, it is uniformly continuous in $[a,b]$. That is, for any $\varepsilon > 0$, there exists a positive δ such that $|f(x_1) - f(x_2)| < \dfrac{\varepsilon}{b-a}$ whenever $|x_1 - x_2| < \delta$, $x_1, x_2 \in [a,b]$..........................(1).

Again since f is continuous in $[a,b]$, it is continuous in every sub-interval $\delta_r = [x_{r-1}, x_r], r = 1,2,3,\ldots, n$. Therefore, there exist α_r and β_r in $\delta_r = [x_{r-1}, x_r], r = 1,2,3,\ldots, n$ such that $f(\alpha_r) = M_r$ and $f(\beta_r) = m_r$ where M_r and m_r are respectively, the supremum and infimum of the function f in $\delta_r = [x_{r-1}, x_r], r = 1,2,3,\ldots, n$. Then by (1), we have $|M_r - m_r| = |f(\alpha_r) - f(\beta_r)| < \dfrac{\varepsilon}{b-a}$. The oscillatory sum of f for the division D, i.e., is

$$S(D) - s(D) = \sum_{r=1}^{n} (M_r - m_r)\delta_r < \sum_{r=1}^{n} \frac{\varepsilon}{(b-a)}\delta_r = \left(\frac{\varepsilon}{(b-a)} \sum_{r=1}^{n} \delta_r\right) = \frac{\varepsilon}{(b-a)}(b-a) = \varepsilon$$

$\Rightarrow S(D) - s(D) < \varepsilon$. Hence f is integrable in $[a,b]$.

INTEGRABILITY OF MONOTONIC FUNCTION

THEOREM 6 : ***If a function f is monotonic in $[a,b]$ then it is integrable in $[a,b]$.***

Proof : Since f is monotone in $[a,b]$ it is bounded in $[a,b]$. Let $f(a)$ and $f(b)$ are the bounds. For the sake of definiteness, let us suppose that the function f is monotonic increasing. Let $\varepsilon > 0$. Let $D = \{a = x_0, x_1, x_2, \ldots\ldots\ldots, x_r, \ldots\ldots, x_n = b\}$ be a division/partition of $[a,b]$ such that the length of each subinterval is $< \dfrac{\varepsilon}{f(b) - f(a) + 1}$. Let the length of the r^th^ subinterval $[x_{r-1}, x_r]$ is $\delta_r = x_r - x_{r-1}$. Let $f(x_r)(= M_r say)$ and $f(x_{r-1})(= m_r say)$ are the bounds of f in $\delta_r = [x_{r-1}, x_r]$.

Now $S(D) - s(D) = \sum_{r=1}^{n} (M_r - m_r)\delta_r = \sum_{r=1}^{n} ((f(x_r) - f(x_{r-1}))\delta_r$

$$< \frac{\varepsilon}{f(b) - f(a) + 1} \sum_{r=1}^{n} ((f(x_r) - f(x_{r-1}))$$

$$= \frac{\varepsilon}{f(b) - f(a) + 1}(f(b) - f(a)) < \varepsilon$$

$\Rightarrow S(D) - s(D) < \varepsilon$. $\Rightarrow$ f is integrable in $[a,b]$.

Properties of the Riemann Integral

Prop-1: ***If $f(x)$ is integrable in $a \le x \le b$, then it is integrable in $c \le x \le d$ where $a \le c < d \le b$. That is, $f(x)$ is integrable in every subinterval.***

Prop-2: ***If $f(x)$ is integrable in $a \le x \le c$ and in $c \le x \le b$, then it is integrable in $a \le x \le b$.***

Prop-3: ***If $f(x)$ is integrable in $a \le x \le b$, so also is $\lambda f(x)$ where λ is any real number.***

Prop-4: ***If $f(x)$ and $g(x)$ are both integrable in $a \le x \le b$, then $f(x) \pm g(x)$ are also integrable in $a \le x \le b$.***

Prop-5: ***If $f(x)$ and $g(x)$ are both integrable in $a \le x \le b$, then $f(x) \bullet g(x)$ is also integrable in $a \le x \le b$.***

Prop-6: ***If $f(x)$ and $g(x)$ are both bounded and integrable in $a \le x \le b$, then $\dfrac{f(x)}{g(x)}$ is also integrable in $a \le x \le b$ provided $|g(x)| > 0$.***

Prop-7: ***If $f(x)$ is bounded and integrable in $[a,b]$, then $|f(x)|$ is also bounded and integrable in $[a,b]$.***

Proof : Evidently, there exists a positive real number k such that $|f(x)| \le k$, $\forall x \in [a,b]$. Therefore, $|f(x)|$ is bounded. Next let $\varepsilon > 0$. Since $f(x)$ is integrable, there exists a division $D = \{a = x_0, x_1, x_2, \ldots\ldots\ldots, x_r, \ldots\ldots, x_n = b\}$ such that the corresponding oscillatory sum for $f(x)$ is less than ε, i.e., $S(D) - s(D) < \varepsilon$. Let M'_r, M_r are the supremums and m'_r, m_r are the infimums of $f(x)$ and $|f(x)|$ respectively, in $\delta_r = [x_{r-1}, x_r]$. Now $\forall x_1, x_2 \in \delta_r$, we have $\left||f(x_2)| - |f(x_1)|\right| \le |f(x_2) - f(x_1)| \le M'_r - m'_r$.(See justification at the end of the proof)

$M_r - m_r \le M'_r - m'_r$.

So $\sum_{r=1}^{n}(M_r - m_r)\delta_r \le \sum_{r=1}^{n}(M'_r - m')\delta_r < \varepsilon$.

$\Rightarrow \sum_{r=1}^{n}(M_r - m_r)\delta_r < \varepsilon$. Hence $|f(x)|$ is also integrable in $[a,b]$.

Converse of the above theorem is not true.

Example : Let $f = [a,b] \to R$ be defined by $f(x) = 1,\ x \in [a,b] \cap Q$

$= -1,\ x \in [a,b] - Q$ then f is not integrable on $[a,b]$. But $|f(x)| = 1$ for all $x \in [a,b]$. $\therefore |f|$ is integrable on $[a,b]$.

THEOREM 7 : ***If $f(x)$ be bounded functions integrable in $[a,b]$ and $F(x)=\int_a^x f(t)dt$, $a\le x\le b$, then $F(x)$ is continuous function of x in $[a,b]$. If however, $f(x)$ be continuous in $[a,b]$. Then at every point of $[a,b]$, $F(x)$ possesses a derivative and $F'(x)=f(x)$.***

MEAN VALUE THEOREM FOR INTEGRALS

FIRST MEAN VALUE THEOREM
(GENERALIZED MEAN VALUE THEOREM)

THEOREM 8 : ***Let $f(x)$ and $\phi(x)$ be two bounded functions integrable on $[a,b]$ and let $\phi(x)$ keeps same sign in $[a,b]$, then $\int_a^b f(x)\phi(x)dx=\mu\int_a^b \phi(x)dx$ where $m\le\mu\le M$, m and M are the greatest lower bound and least upper bound of f in $[a,b]$.***

Proof : For the sake of definiteness let us suppose that $\phi(x)$ is non-negative. That is, $\phi(x)\ge 0$ in $[a,b]$. In $[a,b]$, $m\le f(x)\le M$. $\therefore m\phi(x)\le f(x)\phi(x)\le M\phi(x)$. Since $m\phi(x)$, $f(x)\phi(x)$ and $M\phi(x)$ are each integrable in $[a,b]$, we have

$$\int_a^b m\phi(x)dx\le\int_a^b f(x)\phi(x)dx\le\int_a^b M\phi(x)dx.$$

$$\Rightarrow m\int_a^b \phi(x)dx\le\int_a^b f(x)\phi(x)dx\le M\int_a^b \phi(x)dx$$

$$\Rightarrow mI\le\int_a^b f(x)\phi(x)dx\le MI \text{ , where } I=\int_a^b \phi(x)dx$$

$$\therefore \int_a^b f(x)\phi(x)dx=\mu I \text{ where } m\le\mu\le M .$$

$$\Rightarrow \int_a^b f(x)\phi(x)dx=\mu\int_a^b \phi(x)dx \text{ where } m\le\mu\le M$$

COROLLARY : ***Let $f(x)$ be a bounded function integrable on $[a,b]$, then $\int_a^b f(x)dx=\mu(b-a)$ where $m\le\mu\le M$, m and M are the greatest lower bound and least upper bound of f in $[a,b]$.***

Proof : Let us put $\phi(x)=1$ in the first mean value theorem(generalized meam value theorem) . Then $\int_a^b f(x).1dx=\mu\int_a^b 1dx=\mu(b-a)$.

ABEL'S INEQUALITY :

If (1) $a_1, a_2, a_3, \dots\dots\dots, a_n$ is a non increasing sequence of n positive numbers

(2) $v_1, v_2, v_3, \dots\dots\dots, v_n$ is a set of any n numbers

and (3) h and H are two numbers such that $h < v_1 + v_2 + v_3 + \dots\dots + v_n < H$ for $1 \le p \le n$

then $a_1 h < a_1 v_1 + a_2 v_2 + a_3 v_3 + \dots\dots + a_n v_n < a_1 H$.

SECOND MEAN VALUE THEOREM (BONNET'S FORM)

THEOREM 9 **:** ***If $f(x)$ be a bounded monotonic non-increasing never negative function defined on $[a,b]$ and $\phi(x)$ be bounded function integrable on $[a,b]$.*** Then there exists a number ξ of x in $[a,b]$ such that $\int_a^b f(x)\phi(x)dx = f(a)\int_a^\xi \phi(x)dx$ where $a \le \xi \le b$.

Proof : Let $D = \{a = x_0, x_1, x_2, \dots\dots\dots, x_r, \dots\dots\dots, x_n = b\}$ be any division/partition of $[a,b]$ and let M_r and m_r are respectively, the supremum and infimum of the function $\phi(x)$ in $\delta_r = [x_{r-1}, x_r], r = 1,2,3,\dots,n$. Let $\xi_r \in [x_{r-1}, x_r]$.

Now in $\delta_r = [x_{r-1}, x_r]$, $m_r \le \phi(x) \le M_r \Rightarrow m_r(x_r - x_{r-1}) \le \int_{x_{r-1}}^{x_r} \phi(x)dx \le M_r(x_r - x_{r-1})$ ....(1)

and $m_r(x_r - x_{r-1}) \le \phi(\xi_r)(x_r - x_{r-1}) \le M_r(x_r - x_{r-1})$ ....(2). Putting $r = 1,2,3,\dots, p \le n$

and adding we get from (1) $\sum_{r=1}^p m_r\delta_r \le \int_a^{x_p} \phi(x)dx \le \sum_{r=1}^p M_r\delta_r$ ....(3)

and from (2) get, $\sum_{r=1}^p m_r\delta_r \le \sum_{r=1}^p \phi(\xi_r)\delta_r \le \sum_{r=1}^p M_r\delta_r$ ....(4)

Now from (3), we get $\int_a^{x_p} \phi(x)dx \le \sum_{r=1}^p M_r\delta_r$ and from (4) we get $-\sum_{r=1}^p \phi(\xi_r)\delta_r \le -\sum_{r=1}^p m_r\delta_r$.

Adding, $\left| -\sum_{r=1}^p \phi(\xi_r)\delta_r + \int_a^{x_p} \phi(x)dx \right| \le \sum_{r=1}^p M_r\delta_r - \sum_{r=1}^p m_r\delta_r = \sum_{r=1}^p (M_r - m_r)\delta_r \le \sum_{r=1}^n (M_r - m_r)\delta_r$.

Or, $\int_a^{x_p} \phi(x)dx - \sum_{r=1}^n (M_r - m_r)\delta_r \le \sum_{r=1}^p \phi(\xi_r)\delta_r \le \int_a^{x_p} \phi(x)dx + \sum_{r=1}^n (M_r - m_r)\delta_r$.

Now since $\phi(x)$ is integrable, $\int_a^x \phi(x)dx$ is a continuous function of x (by theorem)

and it must have its supremum (M, say) and infimum (m, say) .

Hence $m - \sum_{r=1}^n (M_r - m_r)\delta_r \le \sum_{r=1}^p \phi(\xi_r)\delta_r \le M + \sum_{r=1}^n (M_r - m_r)\delta_r$.

Let $a_r = f(\xi_r)$, $v_r = \phi(\xi_r)\delta_r$, $h = m - \sum_{r=1}^{n}(M_r - m_r)\delta_r$, $H = M + \sum_{r=1}^{n}(M_r - m_r)\delta_r$. Then using Abel's inequality we have

$f(a)\left\{m - \sum_{r=1}^{n}(M_r - m_r)\delta_r\right\} \le \sum_{r=1}^{n} f(\xi_r)\phi(\xi_r)\delta_r \le f(a)\left\{M + \sum_{r=1}^{n}(M_r - m_r)\delta_r\right\}$. Let $\|D\| \to 0$, so that $\sum_{r=1}^{n}(M_r - m_r)\delta_r \to 0$ whereby $mf(a) \le \int_a^b f(x)\phi(x)dx \le Mf(a)$. That is,

$\int_a^b f(x)\phi(x)dx \le \mu f(a)$, where $m \le \mu \le M$. Since M and m are the supremum and infimum of the continuous function $\int_a^x \phi(x)dx$, the function $\int_a^x \phi(x)dx$ must assume every intermediate value of M and m. Therefore, there must exists at least one value ξ in $[a,b]$ for which $\int_a^b f(x)\phi(x)dx = f(a)\int_a^{\xi}\phi(x)dx$.

SECOND MEAN VALUE THEOREM (WEIERSTRASS FORM)

THEOREM 10: ***If $f(x)$ be a bounded and monotonic function defined on $[a,b]$ and $\phi(x)$ be bounded function integrable on $[a,b]$. Then there exists at least value of x, say ξ, in $[a,b]$ such that $\int_a^b f(x)\phi(x)dx = f(a)\int_a^{\xi}\phi(x)dx + f(b)\int_{\xi}^{b}\phi(x)dx$ where $a \le \xi \le b$.***

Proof : Let $f(x)$ be monotonically decreasing function so that $\psi(x) = f(x) - f(b)$ is monotonically decreasing and positive. Then from S.M.V.T(Bonnet's form) we have $\int_a^b \psi(x)\phi(x)dx = \psi(a)\int_a^{\xi}\phi(x)dx$ where $a \le \xi \le b$.

Or, $\int_a^b [f(x) - f(b)]\phi(x)dx = [f(a) - f(b)]\int_a^{\xi}\phi(x)dx$

$$\Rightarrow \int_a^b f(x)\phi(x)dx = f(b)\int_a^b \phi(x)dx + f(a)\int_a^{\xi}\phi(x)dx - f(b)\int_a^{\xi}\phi(x)dx$$

$$= f(a)\int_a^{\xi}\phi(x)dx + f(b)\left[\int_a^b \phi(x)dx - \int_a^{\xi}\phi(x)dx\right]$$

$$= f(a)\int_a^{\xi}\phi(x)dx + f(b)\left[\int_{\xi}^{a}\phi(x)dx + \int_a^b \phi(x)dx\right]$$

$$= f(a)\int_a^{\xi}\phi(x)dx + f(b)\int_{\xi}^{b}\phi(x)dx .$$

That is, $\int_a^b f(x)\phi(x)dx = f(a)\int_a^{\xi}\phi(x)dx + f(b)\int_{\xi}^{b}\phi(x)dx$ where $a \le \xi \le b$.

FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

THEOREM 11: ***If*** $f(x)$ ***is integrable on*** $[a,b]$ ***and if there exists a function*** $\phi(x)$ ***such that*** $\phi'(x) = f(x)$ ***on*** (a,b)***, then*** $\int_a^b f(x)dx = \phi(b) - \phi(a)$. ($\phi(x)$ is called primitive of $f(x)$ and $f(x)$ is called derivative of $\phi(x)$).

Proof : Let $\varepsilon > 0$. Since $\phi'(x) = f(x)$ is bounded and integrable on $[a,b]$, there exists a division/partition $D = \{a = x_0, x_1, x_2, \ldots\ldots\ldots, x_r, \ldots\ldots, x_n = b\}$ of $[a,b]$ such that

$\left|\sum_{r=1}^{n}\phi'(\xi_r)\delta_r - \int_a^b \phi'(x)dx\right| < \varepsilon$ ....(1) Where $\sum_{r=1}^{n}\varphi'(\xi_r)\delta_r$ is the Riemann sum for the function $\phi'(x)$ corresponding to a division D. Considering the r^th^ subinterval $\delta_r = [x_{r-1}, x_r]$, by the Lagrange's Mean-Value Theorem of differential calculus, we have $\phi(x_r) - \phi(x_{r-1}) = (x_r - x_{r-1})\phi'(\xi_r) = \phi'(\xi_r)\delta_r$ where $\xi_r \in [x_{r-1}, x_r]$.

Therefore, $\sum_{r=1}^{n}\phi'(\xi_r)\delta_r = \sum_{r=1}^{n}[\phi(x_r) - \phi(x_{r-1})] = \phi(b) - \phi(a)$ ....(2). Then by (1) and (2), it follows that $\left|\phi(b) - \phi(a) - \int_a^b \phi'(x)dx\right| < \varepsilon$. As ε is an arbitrary positive number, we conclude that $\phi(b) - \phi(a) - \int_a^b \phi'(x)dx = 0$ or $\int_a^b f(x)dx = \phi(b) - \phi(a)$.

SOME IMPORTANT RESULTS

RESULT 1: ***If*** $f(x)$ ***be bounded in*** $[a,b]$ ***and if*** M ***and*** m ***be the supremum and infimum of*** $f(x)$ ***in*** $[a,b]$***, then*** $m(b-a) \le \int_{\underline{a}}^{b} f(x)dx \le \int_a^{\bar{b}} f(x)dx \le M(b-a)$.

Proof : Let $D = \{a = x_0, x_1, x_2, \ldots\ldots\ldots, x_r, \ldots\ldots, x_n = b\}$ be a division/partition of $[a,b]$. Then $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$,..............., $[x_{r-1}, x_r]$,...................., $[x_{n-1}, x_n]$ are the subintervals in which the interval $[a,b]$ is divided. Let the length the r^th^ subinterval ,i.e., $\delta_r = [x_{r-1}, x_r]$ be δ_r. Let norm of the division/partition D is $\|D\|$. Since the function is bounded in $[a,b]$ is also necessarily bounded in each of the subintervals . Let M_r and m_r be the supremum and infimum of f in $\delta_r = [x_{r-1}, x_r]$.

If M and m be the supremum and infimum of f in $[a,b]$ then for every value of r, we have, $m \le m_r \le M_r \le M$

$\Rightarrow m\delta_r \le m_r\delta_r \le M_r\delta_r \le M\delta_r$. Putting $r = 1,2,3,\ldots\ldots\ldots,n$ and adding, we have

$$\sum_{r=1}^{n} m\delta_r \le \sum_{r=1}^{n} m_r\delta_r \le \sum_{r=1}^{n} M_r\delta_r \le \sum_{r=1}^{n} M\delta_r .$$

$\Rightarrow m(b-a) \le s(D) \le S(D) \le M(b-a)$.

Now if $\|D\| \to 0$, then $m(b-a) \le \underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx \le M(b-a)$.

RESULT 2: ***If*** $f(x)$ ***be bounded and integrable in*** $[a,b]$ ***and if*** M ***and*** m ***be the supremum and infimum of*** $f(x)$ ***in*** $[a,b]$***, then*** $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$.

Proof : Since $f(x)$ is integrable in $[a,b]$ we have $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx = \int_a^b f(x)dx$. The form the conclusion of the result-1 we get $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$.

RESULT 3: ***If*** $f(x)$ ***be integrable in*** $[a,b]$***, then there exists a number*** μ ***where*** $m \le \mu \le M$***,*** M ***and*** m ***are the supremum and infimum of*** $f(x)$ ***in*** $[a,b]$***, such that*** $\int_a^b f(x)dx = \mu(b-a)$.

Proof : Since $m \le \mu \le M$ the result follows from the conclusion of result-2.

RESULT 4: ***If*** $f(x)$ ***be integrable in*** $[a,b]$ ***and*** $f(x) \ge 0$***,then*** $\int_a^b f(x)dx \ge 0$.

Proof : By result-2, we have $\int_a^b f(x)dx \ge m(b-a)$. As $f(x) \ge 0$, $m \ge 0$ and as $(b-a)$ is the length ot the interval $[a,b]$, we have $(b-a) \ge 0$. Therefore, $\int_a^b f(x)dx \ge m(b-a) \ge 0$.

RESULT 5: ***If*** $f(x)$ ***and*** $g(x)$ ***be both bounded and integrable in*** $[a,b]$ ***and*** $f(x) \ge g(x)$, ***then*** $\int_a^b f(x)dx \ge \int_a^b g(x)dx$.

Proof : Since $f(x)$ and $g(x)$ be both bounded and integrable in $[a,b]$, $f(x) - g(x)$ is also bounded and integrable in $[a,b]$. Then by result-4, we have $\int_a^b \{f(x) - g(x)\}dx \ge 0$. Hence $\int_a^b f(x)dx \ge \int_a^b g(x)dx$.

RESULT 6: ***If*** $f(x)$ ***be integrable in*** $[a,b]$ ***then*** $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$.

Proof : Since $f(x)$ is integrable in $[a,b]$, $\int_a^b f(x)dx$ exists and also $\int_a^b |f(x)|dx$ exists (by Prop-7).

Now $-|f(x)| \le f(x) \le |f(x)|$ ($\therefore x \le |x|$)

$$\Rightarrow -\int_a^b |f(x)|dx \le \int_a^b f(x)dx \le \int_a^b |f(x)|dx$$

$$\Rightarrow -\int_a^b f(x)dx \le \int_a^b |f(x)|dx \text{ and } \int_a^b f(x)dx \le \int_a^b |f(x)|dx$$

$$\Rightarrow \left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx.$$

EXAMPLES ON RIEMANN INTEGRATION

EX 1: Show by an example that if $|f(x)|$ is integrable then $f(x)$ may not be integrable.

Solution : Let $f(x)=1$, when x is rational.

$=-1$, when x is irrational. be defined in $[a,b]$, $b>a$. Clearly $f(x)$ is bounded in $[a,b]$. Let Let $D=\{a=x_0,x_1,x_2,.........,x_r,......., x_n=b\}$ be a division/partition of $[a,b]$. Let the length the r[th] subinterval ,i.e., $\delta_r=[x_{r-1},x_r]$ be δ_r. Since the function is bounded in $[a,b]$ is also necessarily bounded in each of the subintervals . Let M_r and m_r be the supremum and infimum of f in $\delta_r=[x_{r-1},x_r]$.

Then $S(D)=U(P,f)=\sum_{r=1}^{n} M_r\delta_r=\sum_{r=1}^{n}1.\delta_r=(b-a)$ and the same will be result for every possible division/partition of $[a,b]$. Hence the infimum of the set of all upper sums is clearly $(b-a)$. That is, $\overline{\int_a^b} f(x)dx=(b-a)$. Again, $s(D)=L(P,f)=\sum_{r=1}^{n} m_r\delta_r=\sum_{r=1}^{n}-1.\delta_r=-(b-a)$ and the same will be result for every possible division/partition of $[a,b]$. Hence the supremum of the set of all lower sums is clearly $-(b-a)$. That is, $\underline{\int_a^b} f(x)dx=-(b-a)$.

Therefore, $\underline{\int_a^b} f(x)dx \ne \overline{\int_a^b} f(x)dx$. So $f(x)$ is not integrable.

Where as for $|f(x)|$, $S(D)=s(D)=\sum_{r=1}^{n}1.\delta_r=(b-a)$ ad it is true for every possible division. So in that case $\underline{\int_a^b}|f(x)|dx=\overline{\int_a^b}|f(x)|dx$ and consequently $|f(x)|$ is integrable.

EX 2: If $f(x)=x^2$, when $0\le x\le 1$.

$=\sqrt{x}$, when $1\le x\le 2$. Evaluate $\int_0^2 f(x)dx$

Solution : Since $f(x)=x^2$ and $f(x)=\sqrt{x}$ are both continuous in $[0,1]$ and $[1,2]$ respectively, they are integrable in their respective interval.

Now $\int_0^2 f(x)dx=\int_0^1 f(x)dx+\int_1^2 f(x)dx$

$$=\int_0^1 x^2dx+\int_1^2\sqrt{x}dx=\frac{4\sqrt{2}}{3}-\frac{1}{3}.$$

EX 3: If $f(x)=(1-x)$, when $0\le x\le 1$.

$=(x-1)$, when $1\le x\le 2$, Evaluate $\int_0^2 |1-x|dx$

Solution : Since $f(x)=(1-x)$ and $f(x)=(x-1)$ are both continuous in $[0,1]$ and $[1,2]$ respectively, they are integrable in their respective interval.

Now $\int_0^2 f(x)dx=\int_0^2|1-x|dx=\int_0^1 f(x)dx+\int_1^2 f(x)dx$

$$=\int_0^1|1-x|dx+\int_1^2|x-1|dx.$$

$$=\left[x-\frac{x^2}{2}\right]_0^1+\left[\frac{x^2}{2}-x\right]_1^2$$

$$=1$$

EX 4: Show that $\int_a^b \frac{|x|}{x}dx=|b|-|a|$, $(a<b)$

Solution : Case-1. When $0<a<b$. Then for all $x\in[a,b]$, $f(x)=\frac{|x|}{x}=\frac{x}{x}=1$ is continuous in $[a,b]$and hence integrable in $[a,b]$. Therefore, $\int_a^b f(x)dx=\int_a^b\frac{|x|}{x}dx=\int_a^b dx=b-a=|b|-|a|$ $(\because 0<a<b)$.

Case-2. When $a<0<b$. Then the function $f(x)=\frac{|x|}{x}$ has only one point of discontinuity at $x=0$ and hence integrable in $[a,b]$.

Now if $a\le x<0$, then $f(x)=\frac{|x|}{x}=\frac{-x}{x}=-1$ and if $0<x\le b$, then $f(x)=\frac{|x|}{x}=\frac{x}{x}=1$.

Therefore, $\int_a^b f(x)dx=\int_a^b\frac{|x|}{x}dx=\int_a^0\frac{|x|}{x}dx+\int_0^b\frac{|x|}{x}=-[x]_a^0+[x]_0^b=a+b=|b|-|a|$ $(\because b>0,a<0)$.

Case-3. When $a<x<b<0$. Then for all $x\in[a,b]$, $f(x)=\frac{|x|}{x}$ is continuous in $[a,b]$and hence integrable in $[a,b]$. Therefore, $\int_a^b f(x)dx=\int_a^b\frac{|x|}{x}dx=-\int_a^b dx=-[b-a]=a-b=|b|-|a|$. So in any case, $\int_a^b\frac{|x|}{x}dx=|b|-|a|$.

EX 5: Using the relation $m(b-a)\le\int_a^b f(x)dx\le M(b-a)$ estimate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sin x}{x}dx$.

Solution : Since $f(x)=\dfrac{\sin x}{x}$ is continuous in $\left[\dfrac{\pi}{4},\dfrac{\pi}{3}\right]$, $f(x)$ is integrable in $\left[\dfrac{\pi}{4},\dfrac{\pi}{3}\right]$. That is, $\int\limits_{\frac{\pi}{4}}^{\frac{\pi}{3}}\dfrac{\sin x}{x}dx$ exists. Also $f(x)$ is bounded and monotonically decreasing in $\left[\dfrac{\pi}{4},\dfrac{\pi}{3}\right]$. The greatest lower bound (m) of is $f(\dfrac{\pi}{3})=\dfrac{3\sqrt{3}}{2\pi}$ and the least upper bound (M) of is $f(\dfrac{\pi}{4})=\dfrac{4\sqrt{2}}{2\pi}$.

Therefore, $m(b-a)\le\int_a^b f(x)dx\le M(b-a)$

$$\Rightarrow\frac{3\sqrt{3}}{2\pi}\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\le\int\limits_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sin x}{x}dx\le\frac{4\sqrt{2}}{2\pi}\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$$

$$\Rightarrow\frac{\sqrt{3}}{8}\le\int\limits_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sin x}{x}dx\le\frac{\sqrt{2}}{6}.$$

EX 6: Show that $\int\limits_{-1}^{1}|x|dx=1$ and also show that $\int\limits_{-1}^{1}\{x+|x|\}dx=1$.

Solution : $f(x)=|x|=-x$ when $-1\le x\le 0$

$=x$ when $0\le x\le 1.$

Since $f(x)=-x$ and $f(x)=x$ are both continuous in $[-1,0]$ and $[0,1]$ respectively, they are integrable in their respective interval.

Therefore, $\int\limits_{-1}^{1}|x|dx=\int\limits_{-1}^{0}-xdx+\int\limits_{0}^{1}xdx$

$=1$

Also $f(x)=\{x+|x|\}=x+(-x)=0$ when $-1\le x\le 0$

$=x+x=2x$ when $0\le x\le 1.$

Since $f(x)=0$ and $f(x)=2x$ are both continuous in $[-1,0]$ and $[0,1]$ respectively, they are integrable in their respective interval.

Therefore, $\int\limits_{-1}^{1}f(x)dx=\int\limits_{-1}^{1}\{x+|x|\}dx=\int\limits_{-1}^{0}0.dx+\int\limits_{0}^{1}2xdx$

$=0+\left[x^2\right]_0^1$

$=1$

EX 7: Show that $4\le\int\limits_{1}^{3}\sqrt{3+x^3}dx\le 2\sqrt{30}$.

Solution : Let $f(x)=\sqrt{3+x^3}$. Clearly, $f(x)$ is monotonically increasing in $[1,3]$. Therefore, supremum (M) of $f(x)$ is $f(3)=\sqrt{30}$ and infimum (m) of $f(x)$ is $f(1)=\sqrt{4}=2$. Since $f(x)$ is monotonically increasing in $[1,3]$, it is integrable in $[1,3]$. That is, $\int_1^3 \sqrt{3+x^3}dx$ exists.

Therefore, using the relation $m(b-a)\le \int_a^b f(x)dx \le M(b-a)$, we can have

$$2(3-1)\le \int_1^3 \sqrt{3+x^3}dx \le \sqrt{30}(3-1)$$

$$\Rightarrow 4\le \int_1^3 \sqrt{3+x^3}dx \le 2\sqrt{30}.$$

EX 8: If $0\le x\le 1$ show that $\frac{x^2}{\sqrt{2}}\le \frac{x^2}{\sqrt{1+x}}\le x^2$ and hence show that $\frac{1}{3\sqrt{2}}\le \int_0^1 \frac{x^2}{\sqrt{1+x}}dx \le \frac{1}{3}$.

Solution : Let $f(x)=\frac{1}{\sqrt{1+x}}$. Clearly, $f(x)$ is monotonically decreasing in $[0,1]$. Therefore, supremum (M) of $f(x)$ is $f(0)=1$ and infimum (m) of $f(x)$ is $f(1)=\frac{1}{\sqrt{2}}$.

Now $m\le f(x)\le M$

$$\Rightarrow \frac{1}{\sqrt{2}}\le \frac{1}{\sqrt{1+x}}\le 1$$

$$\Rightarrow \frac{x^2}{\sqrt{2}}\le \frac{x^2}{\sqrt{1+x}}\le x^2 \quad (\because x\ge 0).$$

Now for second part, let $\phi(x)=\frac{x^2}{\sqrt{1+x}}$. x^2 is continuous and hence integrable in $[0,1]$ and $\frac{1}{\sqrt{1+x}}$ is monotonically decreasing in $[0,1]$ and hence integrable in $[0,1]$. Therefore, $\phi(x)=\frac{x^2}{\sqrt{1+x}}$ is integrable in $[0,1]$. Thus $\int_0^1 \frac{x^2}{\sqrt{1+x}}dx$ and $\int_0^1 x^2 dx$ exist. Now we have from the first part $\frac{x^2}{\sqrt{2}}\le \frac{x^2}{\sqrt{1+x}}\le x^2$

$$\Rightarrow \int_0^1 \frac{x^2}{2}\le \int_0^1 \frac{x^2}{\sqrt{1+x}}dx \le \int_0^1 x^2 dx$$

$$\Rightarrow \frac{1}{3\sqrt{2}}\le \int_0^1 \frac{x^2}{\sqrt{1+x}}dx \le \frac{1}{3}.$$

EX 9: Show that $\frac{2\pi^2}{9}\le \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{2x}{\sin x}dx \le \frac{\pi^2}{3}$.

Solution : Let $f(x)=\frac{2x}{\sin x}$. Clearly, $f(x)$ is monotonically increasing in $\left[\frac{\pi}{6},\frac{\pi}{2}\right]$. Therefore, supremum (M) of $f(x)$ is $f(\frac{\pi}{2})=\pi$ and infimum (m) of $f(x)$ is $f(\frac{\pi}{6})=\sqrt{4}=\frac{2\pi}{3}$. Since $f(x)$ is monotonically increasing in $\left[\frac{\pi}{6},\frac{\pi}{2}\right]$, it is integrable in $\left[\frac{\pi}{6},\frac{\pi}{2}\right]$. That is, $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{2x}{\sin x}dx$ exists. Now $m\le f(x)\le M$

$$\Rightarrow \frac{2\pi}{3}\le\frac{2x}{\sin x}\le\pi$$

$$\Rightarrow \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{2\pi}{3}dx\le\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{2x}{\sin x}dx\le\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\pi dx$$

$$\Rightarrow \frac{2\pi^2}{9}\le\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{2x}{\sin x}dx\le\frac{\pi^2}{3}.$$

EX 10: Show that $\frac{1}{2}\le\int_0^1\frac{dx}{\sqrt{4-x^2+x^3}}\le\frac{\pi}{6}$.

Solution : $4-x^2+x^3>4-x^2$ $(\because 0\le x\le 1)$

$4-x^2+x^3>4-\left(x^2-x^3\right)<4$ $(\because 0\le x\le 1)$

Therefore, $\sqrt{4-x^2+x^3}>\sqrt{4-x^2}$ and $\sqrt{4-x^2+x^3}<\sqrt{4}=2$.

So, $\frac{1}{2}<\frac{1}{\sqrt{4-x^2+x^3}}<\frac{1}{\sqrt{4-x^2}}$ ....(1)

Now $\frac{1}{2}$, $\frac{1}{\sqrt{4-x^2+x^3}}$ and $\frac{1}{\sqrt{4-x^2}}$ are all continuous in $[0,1]$ and hence they are intigrable in $[0,1]$. That is, $\int_0^1\frac{1}{2}dx$, $\int_0^1\frac{dx}{\sqrt{4-x^2+x^3}}$ and $\int_0^1\frac{dx}{\sqrt{4-x^2}}$ exist.

Now from (1) we have $\frac{1}{2}<\frac{1}{\sqrt{4-x^2+x^3}}<\frac{1}{\sqrt{4-x^2}}$

$$\Rightarrow \int_0^1\frac{1}{2}dx<\int_0^1\frac{dx}{\sqrt{4-x^2+x^3}}<\int_0^1\frac{dx}{\sqrt{4-x^2}}$$

$$\Rightarrow \frac{1}{2}<\int_0^1\frac{dx}{\sqrt{4-x^2+x^3}}<\sin^{-1}\frac{1}{2}$$

$$\Rightarrow \frac{1}{2}<\int_0^1\frac{dx}{\sqrt{4-x^2+x^3}}<\frac{\pi}{6}.$$

EX 11: Prove that $\cdot 5 \le \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^{2x}}} \le 0\cdot 524$.

Solution : $1-x^{2x} < 1 \quad (\because 0 \le x \le \frac{1}{2})$ and $1-x^{2x} > 1-x^2$**.** Therefore, $\sqrt{1-x^{2x}} < 1$ and $\sqrt{1-x^{2x}} > \sqrt{1-x^2}$**.** These imply $1 < \frac{1}{\sqrt{1-x^{2x}}} < \frac{1}{\sqrt{1-x^2}}$ ....(1)

Here 1, $\frac{1}{\sqrt{1-x^{2x}}}$ and $\frac{1}{\sqrt{1-x^2}}$ are all

continuous in $\left[0, \frac{1}{2}\right]$ and hence they are integrable in $\left[0, \frac{1}{2}\right]$. That is, $\int_0^{\frac{1}{2}} 1dx$, $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^{2x}}}$ and $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$ exist. Therefore, from (1) $\int_0^{\frac{1}{2}} 1dx \le \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^{2x}}} \le \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$

$$\Rightarrow \frac{1}{2} \le \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^{2x}}} \le \sin^{-1}\frac{1}{2}$$

$$\Rightarrow \cdot 5 \le \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^{2x}}} \le \frac{\pi}{6} = \frac{3\cdot 1416}{6} = \cdot 524$$

EX 12: Prove that $\underset{x\to 3}{Lim} \frac{1}{x-3} \int_3^x e^{\sqrt{1+t^2}} dt = e^{\sqrt{10}}$.

Solution : At the point $x = 3$, $\int_3^x e^{\sqrt{1+t^2}} dt = 0$. Then by L' Hospital's rule the given limit becomes $\underset{x\to 3}{Lim} \frac{e\sqrt{1+x^2}}{1}$ (using theorem 7)

$= e^{\sqrt{10}}$.

EX 13: Prove that $\underset{x\to 0}{Lim} \frac{x}{1-e^{x^2}} \int_0^x e^{t^2} dt = -1$.

Solution : At the point $x=0$, $\int_0^x e^{t^2}dt=0$. Then by L' Hospital's rule the given limit becomes $\lim\limits_{x\to 0}\dfrac{x\dfrac{d}{dx}\left(\int_0^x e^{t^2}dt\right)+\int_0^x e^{t^2}dt}{\dfrac{d}{dx}\left(1-e^{x^2}\right)}$

$$=\lim_{x\to 0}\frac{xe^{x^2}+\int_0^x e^{t^2}dt}{-2xe^{x^2}}\quad\left[\frac{0}{0}\ form\right]$$

$$=\lim_{x\to 0}\frac{e^{x^2}+2x^2e^{x^2}+e^{x^2}}{-2e^{x^2}-4x^2e^{x^2}}$$

$$=-1$$

EX 14: Prove that $\lim\limits_{x\to 0}\dfrac{\int_0^{x^2}\sin\sqrt{x}}{x^3}=\dfrac{2}{3}$. (task).

EX 15: If $f(x)$ be continuous in $[a,b]$ and $f(x)\geq 0$ for all x in $[a,b]$ and if $\int_a^b f(x)dx=0$, prove that $f(x)=0$, $\forall x\in[a,b]$.

Solution : Let c be any point in $[a,b]$. Since $f(x)\geq 0, \forall x\in[a,b]$ we must have $f(c)\geq 0$. If $f(c)=0$ then, the result follows immediately. Next let $f(c)>0$. Since $f(x)$ is continuous in $[a,b]$ and c be any point in $[a,b]$, $f(x)$ must be continuous at c. Hence for every $\varepsilon>0$,however small, there exists a $\delta>0$ such that $|f(x)-f(c)|<\varepsilon$ whenever $|x-c|<\delta$.

Let us take $\varepsilon=\dfrac{f(c)}{k}$ where k is a large positive quantity. That is,

$|f(x)-f(c)|<\dfrac{f(c)}{k}$ whenever $|x-c|<\delta$.

$$\Rightarrow f(x)>\frac{f(c)}{k}$$

$\Rightarrow \int_a^b f(x)dx>\dfrac{f(c)}{k}\int_a^b dx$.($\because$ $f(x)$ is continuous in $[a,b]$ and hence $f(x)$ is integrable in $[a,b]$)

$$\Rightarrow \int_a^b f(x)dx>\frac{f(c)}{k}(b-a)>\frac{f(c)}{k}\delta$$

$\Rightarrow \int_a^b f(x)dx>\dfrac{f(c)}{k}\delta>0$ ($\because$ $f(c)>0$ and $\delta>0$). But it is given that $\int_a^b f(x)dx=0$. Hence our assumption , that is, $f(c)>0$ is not true. Therefore, in any case $f(c)=0$

APPLICTIONS OF MEAN VALUE THEOREMS

EX 16: If $0 \le x' \le x''$ then show that $\left|\int_{x'}^{x''} \frac{\sin x}{x} dx\right| \le \frac{2}{x'}$.

Solution : Let $f(x) = \frac{1}{x}$ and $\phi(x) = \sin x$. Clearly, $f(x) = \frac{1}{x}$ is monotonically decreasing in $[x', x'']$ and bounded. And $\phi(x) = \sin x$ is continuous in $[x', x'']$ and hence integrable in $[x', x'']$. Then by **Second Mean Value Theorem of Bonnet's form** we have $\int_{x'}^{x''} f(x)\phi(x)dx = f(x')\int_{x'}^{\xi} \phi(x)dx$ when $x' \le \xi \le x''$.

Or, $\int_{x'}^{x''} \frac{\sin x}{x} dx = \frac{1}{x'}\int_{x'}^{\xi} \sin x dx$

$$= \frac{1}{x'}[-\cos x]_{x'}^{\xi}$$

$$= \frac{1}{x'}[\cos x' - \cos \xi]$$

Therefore, $\left|\int_{x'}^{x''} \frac{\sin x}{x} dx\right| = \left|\frac{1}{x'}(\cos x' - \cos \xi)\right| = \left|\frac{1}{x'}\right| |\cos x' - \cos \xi| \le \left|\frac{1}{x'}\right| |\cos x'| + \left|\frac{1}{x'}\right| |\cos \xi|$

$$= \left|\frac{1}{x'}\right| (|\cos x'| + |\cos \xi|) \le \left|\frac{1}{x'}\right| \cdot 2 = \frac{2}{x'}$$

($\because |\cos x'| \le 1$ and $|\cos \xi| \le 1$)

EX 17: If $0 \le x' \le x''$ then show that $\left|\int_{x'}^{x''} \frac{\sin x}{x} dx\right| \le \frac{4}{x'}$.

Solution : Let $f(x) = \frac{1}{x}$ and $\phi(x) = \sin x$. Clearly, $f(x) = \frac{1}{x}$ is monotonically decreasing in $[x', x'']$ and bounded. And $\phi(x) = \sin x$ is continuous in $[x', x'']$ and hence integrable in $[x', x'']$. Then by **Second Mean Value Theorem of Weierstrass's form** we have $\int_{x'}^{x''} f(x)\phi(x)dx = f(x')\int_{x'}^{\xi} \phi(x)dx + f(x'')\int_{\xi}^{x''} \phi(x)dx$ when $x' \le \xi \le x''$.

Or, $\int_{x'}^{x''} \frac{\sin x}{x} dx = \frac{1}{x'}\int_{x'}^{\xi} \sin x dx + \frac{1}{x''}\int_{\xi}^{x''} \sin x dx$

$$= \frac{1}{x'}[-\cos x]_{x'}^{\xi} + \frac{1}{x''}[-\cos x]_{\xi}^{x''}$$

$$= \frac{1}{x'}[\cos x' - \cos \xi] + \frac{1}{x''}[\cos \xi - \cos x'']$$

Therefore, $\left|\int_{x'}^{x''}\frac{\sin x}{x}dx\right| = \left|\frac{1}{x'}\left(\cos x' - \cos\xi\right) + \frac{1}{x''}\left(\cos\xi - \cos x''\right)\right|$

$$\le \left|\frac{1}{x'}\left(\cos x' - \cos\xi\right)\right| + \left|\frac{1}{x''}\left(\cos\xi - \cos x''\right)\right|$$

$$= \left|\frac{1}{x'}\right|\left|\cos x' - \cos\xi\right| + \left|\frac{1}{x''}\right|\left|\cos\xi - \cos x''\right|$$

$$\le \left|\frac{1}{x'}\right|\left|\cos x'\right| + \left|\frac{1}{x'}\right|\left|\cos\xi\right| + \left|\frac{1}{x''}\right|\left|\cos\xi\right| + \left|\frac{1}{x''}\right|\left|\cos x''\right|$$

$$= \left|\frac{1}{x'}\right|\left(\left|\cos x'\right| + \left|\cos\xi\right|\right) + \left|\frac{1}{x''}\right|\left(\left|\cos\xi\right| + \left|\cos x''\right|\right)$$

$\therefore \left|\int_{x'}^{x''}\frac{\sin x}{x}dx\right| \le \left|\frac{1}{x'}\right|\cdot 2 + \left|\frac{1}{x''}\right|\cdot 2 = \frac{2}{x'} + \frac{2}{x''} < \frac{2}{x'} + \frac{2}{x'} = \frac{4}{x'}$ $(\because \left|\cos x'\right| \le 1, \left|\cos x''\right| \le 1,\ \left|\cos\xi\right| \le 1)$

and ($\because x' < x'' \Rightarrow \frac{1}{x''} < \frac{1}{x'} \Rightarrow \frac{2}{x''} < \frac{2}{x'}$).

EX 18: Show for $k^2 < 1$, $\frac{\pi}{6} \le \int_0^{\frac{1}{2}}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \le \frac{\pi}{6}\cdot\frac{1}{\sqrt{1-\frac{1}{4}k^2}}$.

Solution : Let us chose ξ in $\left[0,\frac{1}{2}\right]$. Applying **First Mean Value Theorem**, we get

$$\int_0^{\frac{1}{2}}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{\sqrt{1-k^2\xi^2}}\int_0^{\frac{1}{2}}\frac{dx}{\sqrt{1-x^2}}$$

$$= \frac{1}{\sqrt{1-k^2\xi^2}}\left[\sin^{-1}x\right]_0^{\frac{1}{2}}$$

$$= \frac{\pi}{6}\frac{1}{\sqrt{1-k^2\xi^2}} \quad \text{.............(1)}$$

Now let us put $\xi = 0$ and $\xi = \frac{1}{2}$ in (1) to get the minimum and maximum values of $\int_0^{\frac{1}{2}}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$. That is, $\frac{\pi}{6} \le \int_0^{\frac{1}{2}}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \le \frac{\pi}{6}\cdot\frac{1}{\sqrt{1-\frac{1}{4}k^2}}$.

EX 19 : Verify Second Mean Value Theorem of Weierstrass form for the function $x^2\cos x$ in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

Solution : Given integrand, that is, $x^2 \cos x$ can be considered as the product of two functions $f(x)$ and $\phi(x)$ in the following way :

1) $f(x) = x^2$, $\phi(x) = \cos x$
2) $f(x) = \cos x$, $\phi(x) = x^2$
3) $f(x) = x$, $\phi(x) = x\cos x$.

Let $f(x) = x^2$, $\phi(x) = \cos x$. Then $f(x) = x^2$ is not monotonic in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and hence Second Mean Value Theorem of Weierstrass's form is not applicable for the integrand $x^2 \cos x$.

Next let $f(x) = \cos x$, $\phi(x) = x^2$. Then also $f(x) = \cos x$ is not monotonic in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and hence Second Mean Value Theorem of Weierstrass's form is not applicable for the integrand $x^2 \cos x$.

Lastly, let $f(x) = x$, $\phi(x) = x\cos x$. Then $f(x) = x$ is monotonic in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\phi(x) = x\cos x$ is integrable in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and hence Second Mean Value Theorem of Weierstrass form is applicable in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for the integrand $x^2 \cos x$.

INSTRUCTION FOR STUDENTS

All Definitions, Theorems, Results, Properties and Examples highlighted by RED COLOUR are very important.