LECTURE NOTES

On

FUNCTIONS

Semester: 2

Subject: Mathematics (Generic)

Paper: GE-2T , Unit-2

Dr. Sangita Chakraborty
Associate Professor
Department of Mathematics
Kharagpur College

Dear Students,

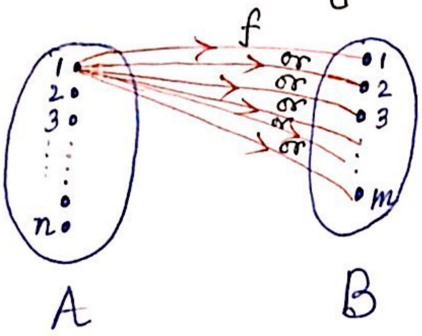
If you have any query, then contact me through my

WhatsUp No. : **9434455417**

GE-2T Unit-II FUNCTIONS.

Definition of Function :-

Let A and B be two non-empty sets. If every element of A is related to an unique element of B by a definite rule f, then f is called a function from A to B, and is denoted by $f: A \to B$.



Let the set A have n-elements and " " B " m - " . Then every element of A may have a relation f with all the m elements of B, i.e., the total number of functions from A to B is given by:

$m \times m \times m \times \ldots \times m$ (n numbers)

$= m^n$.

Examples:

① Let $A = \{1, 2, 3, 4, 5\}$, $B = \{p, q, r, s, t\}$ and $f = \{(1, r), (2, p), (3, s), (4, q), (5, t)\}$.
Here every element of A is uniquely related to an element (infact, distinct element) of B by a definite rule f and hence $f: A \to B$ is a function (one-one). f is a relation also.

② For above A & B, let $f = \{(1, r), (3, q), (4, s), (5, t)\}$
Here the element $2 \in A$ is not related to any element of B by the relation f and hence $f: A \to B$ is not a function. But f is a relation.

③ For above A & B, let $f = \{(1, r), (2, p), (2, q), (3, s), (4, t), (5, t)\}$. Here the element $2 \in A$ is related to two different elements p and q of B by f. Hence $f: A \to B$ is NOT a function. But f is a relation.

Remark: From the examples ② & ③, we see that—

⊛ Every function is a relation but every relation is NOT a function.

Image of a function :-

Let $f: A \to B$ be a function and $x \in A$ be an arbitrary element which is related to an arbitrary element $y \in B$ by the definite rule f. Then the element $y \in B$ is called <u>f-images</u> or <u>image</u> of x under f and is denoted by $f(x)$.

i.e, f-images of x : $y = f(x)$.

Image set of f :- The set $f(A) = \{f(x) : x \in A\}$ is called image set of f and is denoted by $Im(f)$.

<u>Example</u>: Let us take $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8, 10\}$. Let us consider a function $f: A \to B$ defined by $f(x) = 2x, \ \forall x \in A$.

Here $f(1) = 2$, $f(2) = 4$, $f(3) = 6$, $f(4) = 8$.

$\therefore$ Image of 1 is 2 ; Image of 2 is 4, ... and so on.

$Im(f) = \{2, 4, 6, 8\}$

Pre-image of a function :-

Let A and B be two non-empty sets and $f: A \to B$ be a function.

If $x \in A$ be an arbitrary element and is related to an arbitrary element $y \in B$ by the definite rule f, then $x \in A$ is called <u>pre-image</u> of $y \in B$.

<u>Example</u>: From the above example, we see—

1 is the pre-image of 2,

2 " " " " 4, <u>etc.</u>

Domain set, co-domain set, Range set of a function :-

Let A and B be two non-empty sets and $f: A \to B$ be a function.

Then, the domain set of $f \equiv D(f) = A$,

the co-domain " " " $\equiv Cod(f) = B$,

$f(A)$ = the Range of f is the set of all f-images of A

i.e, $f(A) = \{f(x) : x \in A\} \equiv R(f)$

<u>Example</u>: In the above example,

$D(f) = \{1, 2, 3, 4\}$, $Cod(f) = \{2, 4, 6, 8, 10\}$, $f(A) = \{2, 4, 6, 8\} \equiv R(f)$.

Different Types of Function :-

(i) Injective function (one-one function, one to one function).
(ii) Surjective function (onto function)
(iii) Bijective function (one-one and onto function).

Definions :-

Let A and B be two non-empty sets, and $f: A \to B$ be a function.

(i) f is said to be <u>injective</u> if each pair of distinct elements in A have their distinct f-images in B.

i.e, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, $\forall x_1, x_2 \in A$.

<u>OR</u>, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, $\forall x_1, x_2 \in A$.

(ii) f is said to be <u>surjective</u> if every element $y \in B$ has at least one pre-image $x \in A$ such that $f(x) = y$.

<u>OR</u>, the set of all f-images of A is equal to B.

<u>i.e</u>, $f(A) = B$.

(iii) f is said to be <u>bijective</u> if f is both injective and surjective.

<u>Examples</u>:

① Let us consider a function $f: Z \to Z$ be defined by $f(x) = 2x+1$, $\forall x \in Z$.

Here, f is injective, but <u>NOT</u> surjective.

For, let $x_1, x_2 \in Z$ (domain set), and let $f(x_1) = f(x_2)$. $\Rightarrow 2x_1 + 1 = 2x_2 + 1 \Rightarrow x_1 = x_2$, $\forall x_1, x_2 \in Z$.

$\therefore$ f is injective.

But f is <u>NOT</u> surjective.

For, if we take an arbitrary element $y \in Z$ (co-domain), then $\exists\, x \in Z$ (domain set) such that $f(x) = y$.

i.e., $2x+1 = y \Rightarrow x = \frac{y-1}{2} \notin Z$ (domain) always for an arbitrary $y \in Z$ (co-domain); for example, if $y = 6$, then $x = \frac{6-1}{2} = \frac{5}{2} \notin Z$.

$\therefore$ y does not have a pre-image x in Z (domain).

Remark: f is hence <u>NOT</u> bijective.

(2) Let us consider a function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$, $\forall x \in \mathbb{R}$.

f is NOT injective.

For, let us take $x_1 = 4$ and $x_2 = -4$ in $\mathbb{R}$ (domain)

$\therefore f(x_1) = |x_1| = 4$, $f(x_2) = |x_2| = |-4| = 4$

$\Rightarrow f(x_1) = f(x_2)$ in $\mathbb{R}$ (co-domain) for $x_1 \neq x_2$ in domain $\mathbb{R}$.

f is NOT surjective, because, $f(\mathbb{R})$ is the set of all positive real numbers $\neq \mathbb{R}$ (co-domain).

Let us take $-1 \in$ codomain $\mathbb{R}$, then $\nexists x \in$ domain $\mathbb{R}$ s.t. $|x| = -1$.

Hence f is NOT bijective.

(3) Let us consider a function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1$, $\forall x \in \mathbb{R}$.

Here f is surjective. For, if we take an arbitrary $y \in \mathbb{R}$ (co-domain), then $\exists x \in \mathbb{R}$ (domain) s.t.

$f(x) = y \Rightarrow x + 1 = y \Rightarrow x = y - 1 \in \mathbb{R}$ (always) in the domain. $\therefore y$ (arbitrary) $\in \mathbb{R}$ (codomain) has its pre-image $y - 1 (= x)$ in $\mathbb{R}$ (domain).

f is also injective. (Prove it!).

Therefore f is bijective.

(4) Let us consider a function $f: \mathbb{N} \to \mathbb{N}$ defined by $f(n) = \left[\frac{n+1}{2}\right]$, $\forall n \in \mathbb{N}$. Then

[Note: $f(x) = [x]$, is the greatest integer function]

f is NOT bijective. For, let us take $1, 2 \in \mathbb{N}$ (domain). Then $f(1) = \left[\frac{1+1}{2}\right] = [1] = 1$

& $f(2) = \left[\frac{2+1}{2}\right] = [1.5] = 1$

$\therefore 1 \neq 2 \Rightarrow f(1) = f(2)$, for $1, 2 \in \mathbb{N}$ (domain)

$\therefore f$ is NOT injective.

For surjection, let us take $m \in \mathbb{N}$ (co-domain), then $\exists n \in \mathbb{N}$ (domain) s.t. $f(n) = m$

Here $f(2m) = \left[\frac{2m+1}{2}\right] = \left[m + \frac{1}{2}\right] = m$, $\forall m \in \mathbb{N}$ (codomain)

m is arbitrary in the co-domain $\mathbb{N}$, and m has its pre-image $(2m)$ in domain $\mathbb{N}$

$\therefore f$ is surjective but NOT injective, hence NOT bijective.

The following questions may arise :-

Q.1. Give an example of a function which is injective and surjective.

Ans: $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x+1$, $\forall x \in \mathbb{R}$.

Q.2. Give an example of a function which is injective but not surjective.

Ans: $f: \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 2x+1$, $\forall x \in \mathbb{Z}$.

Q.3. Give an example of a function which is not injective but surjective.

Ans: $f: \mathbb{N} \to \mathbb{N}$ defined by $f(n) = \left[\frac{(n+1)}{2}\right]$, $\forall n \in \mathbb{N}$.

Remark: The greatest integer function:

$f: \mathbb{R} \to \mathbb{Z}$ defined by $f(x) = [x] =$ greatest integer $\leq x$.

i.e., for every real x, $\exists$ an integer n s.t.

$n-1 \leq x < n$.

So, $[x] = n-1$, for $n-1 \leq x < n$.

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$-3 \leq x < -2$, $f(x) = -3$

$-2 \leq x < -1$, $f(x) = -2$

$-1 \leq x < 0$, $f(x) = -1$

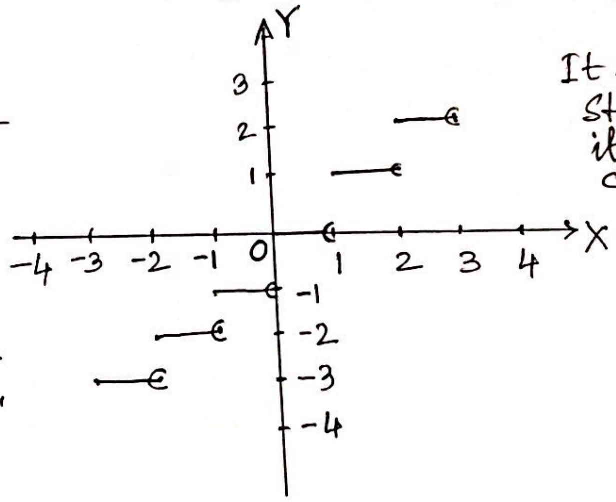
$0 \leq x < 1$, $f(x) = 0$

$1 \leq x < 2$, $f(x) = 1$

$2 \leq x < 3$, $f(x) = 2$

$- - - - - - - - -$

etc.



It looks like steps, hence it is also called Step-function

Q.4. Give an example of a function which is neither injective nor surjective.

Ans: $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$, $\forall x \in \mathbb{R}$.

Let $x_1, x_2 \in \mathbb{R}$ and let $f(x_1) = f(x_2) \Rightarrow |x_1| = |x_2|$

$\Rightarrow x_1 = \pm x_2 \Rightarrow f$ is not injective.

For example, let $x_1 = 2$, $x_2 = -2$, so $x_1 \neq x_2$, but $f(x_1) = |2| = 2 = |-2| = f(x_2)$.

Let $y \in \mathbb{R}$ be arbitrary element and $y = f(x) = |x|$

For negative real y, we do not find any $x \in$ domain $\mathbb{R}$ so that the absolute value of x becomes negative $y \in \mathbb{R}$.

$\therefore f$ is not surjective.

Many-one function:

A function $f: A \to B$ is said to be many-one if two or more than two distinct elements in A have the same f-images in B.

i.e., $x_1 \neq x_2 \Rightarrow f(x_1) = f(x_2)$, $\forall\, x_1, x_2 \in A$

Example: Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$, $\forall x \in \mathbb{R}$. f is many-one. For, let us take $a, -a \in \mathbb{R}$ (domain set), then $f(a) = a^2 = f(-a)$.

Constant function:

A function $f: A \to B$ is said to be the constant function if $f(x) = a$, $\forall\, x \in A$; a being the only f-images in B.

Example: Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = c$, $\forall\, x \in \mathbb{R}$, c is a constant

Identity function:

A function $f: A \to A$ is said to be identity function if $f(x) = x$, $\forall\, x \in A$.

It is denoted by I_A.

Example: Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x$, $\forall x \in \mathbb{R}$.

Remark: The identity function is a bijective function

For, if $x_1, x_2 \in A$ s.t. $x_1 \neq x_2$, then $f(x_1) = x_1$, $f(x_2) = x_2$.

$\therefore x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ in A.

$\therefore f$ is a injection.

Again, let $x \in A$ (co-domain) be arbitrary, then it has a pre-image $x \in$ domain A.

$\therefore$ f is a surjection.

Hence f is a bijection.

Equal functions:

Two functions $f: A \to B$ and $g: A \to C$, having same domain, are said to be equal if $f(x) = g(x)$, $\forall\, x \in A$.

Examples:

① Let $f: S \to \mathbb{R}$ and $g: S \to \mathbb{R}$, be defined by
$f(x) = \frac{|x|}{x}$, $x \in S$; $g(x) = 1$, $x \in S$; where $S = \{x \in \mathbb{R} : x > 0\}$
Then $f(x) = 1 = g(x)$, $\forall x \in S$.
$\therefore$ $f = g$.

② Let $f: S \to \mathbb{R}$ and $g: S \to \mathbb{R}$ be defined by
$f(x) = [x] - x$, $x \in S$; $g(x) = 1 - x$, $x \in S$, $S = \{x \in \mathbb{R} : 1 \leq x < 2\}$
Then $f(x) = 1 - x = g(x)$, $\forall x \in S$.
$\therefore$ $f = g$.

Restriction of a function:-

Let $f: A \to B$ be a function and $D \subseteq A$. Then the function $g: D \to B$ defined by $g(x) = f(x)$, $x \in D$, is said to be the restriction of f to D, and is denoted by f/D.
On the otherhand, f is said to be the extension of g to A.

Examples:

① Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ " irrational.} \end{cases}$
[Called Dirichlet's function]
Here f is neither injective nor surjective. —[Show it]
BUT, A restriction of f to $\mathbb{Q}$ s.t. $f|_{\mathbb{Q}}: \mathbb{Q} \to \mathbb{R}$ defined by $f|\mathbb{Q}(x) = 1$, $x \in \mathbb{Q}$, which is a Constant function.
Another restriction function defined by $f|_{(\mathbb{R}-\mathbb{Q})}: (\mathbb{R}-\mathbb{Q}) \to \mathbb{R}$ by $f|_{(\mathbb{R}-\mathbb{Q})}(x) = 0$, $x \in \mathbb{R}-\mathbb{Q}$,
Which is also a constant function.

② Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sin x$, $x \in \mathbb{R}$.
f is neither injective nor surjective, because,
for $x = 0, \pi$, $f(0) = 0 = f(\pi)$. $\Rightarrow$ f is not injective.
Also, $f(\mathbb{R}) = \{x \in \mathbb{R} : -1 \leq x \leq 1\} \neq \mathbb{R}$; so, f is not surjective.
A restriction function $f|_S: S \to \mathbb{R}$ defined by $f|_S(x) = \sin x$, $x \in S = \{x \in \mathbb{R} : -\pi/2 \leq x \leq \pi/2\}$. Then it is a injection.

Remarks:

Let $f: A \to B$ be a function s.t. for $x \in A$, $\exists$ a unique $y \in B$ s.t. $y = f(x)$.

It may happen that —

(i) an element $y \in B$ has only one pre-image, [eg. $f: \mathbb{R} \to \mathbb{R}$, $y = f(x) = 2x$, $x \in \mathbb{R}$; $f^{-1}(y) = \{\frac{1}{2}y\}$]

(ii) an " $y \in B$ " no pre-image, [eg. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$. $f^{-1}(-1) \notin \mathbb{R}$ [domain]]

(iii) " " $y \in B$ " many " [eg. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \sin x$; $f^{-1}(1) = 2n\pi + \pi/2$, $n = 0, 1, \cdots$]

Thus, the pre-images of an element $y \in B$ form a subset of A, which many be singleton set [only one pre-image], null set [no pre-image], or a set containing more than one element [many pre-images].

Here $f^{-1}(y)$ is defined to be the pre-image set of $y (\in B)$.

Note:

① If f is injective, each element of B has atmost one pre-image.

② " " " surjective, " " " " " atleast one "

③ " " " bijective, " " " " " exactly one ".

Example:

Show that $f: \mathbb{R} \to \mathbb{Z}$ defined by $f(x) = [x]$, $x \in \mathbb{R}$ is not injective but surjective.

$[x]$ is the greatest integer (~~function~~) $\leq x$.

$f(1.5) = 1$, $f(1) = 1 \Rightarrow f$ is not injective.

Now let us take $2 \in \mathbb{Z}$, then $f^{-1}(2) = \{x \in \mathbb{R} : 2 \leq x < 3\}$

$\therefore$ $2 \in \mathbb{Z}$ has many pre-images in $\mathbb{R}$.

Infact, if $y \in \mathbb{Z}$ be arbitrary, then

$$f^{-1}(y) = \{x \in \mathbb{R} : y \leq x < y+1\}.$$

$\therefore$ f is surjective.

Composition of Functions

Let $f: A \to B$ and $g: C \to D$ be two functions such that $f(A) \subseteq C$. Then a function $h: A \to D$ defined by $h(x) = g(f(x))$, $x \in A$, is called the composite (or, product) of f and g, and is denoted by gof or gf.

So, $gof: A \to D$ is defined **by** $(gof)(x) = g(f(x))$, $x \in A$ only if $R(f) \subseteq D(g)$.

Similarly, $fog: C \to B$ is defined by $(fog)(x) = f(g(x))$, $x \in C$ only if $R(g) \subseteq D(f)$.

In particulars, if $f: A \to B$ and $g: B \to C$, then $gof: A \to C$ defined by $gof(x) = g(f(x))$, $x \in A$

Again, if $f: A \to B$ and $g: B \to A$, then $gof: A \to A$ defined by $gof(x) = g(f(x))$, $x \in A$. Also $fog: B \to B$ defined by $fog(x) = f(g(x))$, $x \in B$.

$\therefore$ In this case, both the composite functions fog and gof are defined.

However, in general, fog and gof both may not exist. Also if they both exist, they may not equal, i.e., $fog \neq gof$, in general.

(i) Since $fog \neq gof$ (in general), the composition of functions is not commutative.

(ii) The composition of functions is associative.

Proof: Let $f: A \to B$, $g: B \to C$, $h: C \to D$ be three functions. Then $ho(gof): A \to D$, $(hog)of: A \to D$.

To prove: $ho(gof) = (hog)of$.

Let $x \in A$. Then $\exists\, y \in B$, $z \in C$ and $w \in D$ such that $f(x) = y$, $g(y) = z$, $h(z) = w$.

Now $gof(x) = g(f(x)) = g(y) = z$; $[ho(gof)](x) = h[gof(x)] = h(z) = w$, $x \in A$.

Also, $[(hog)of](x) = (hog)(f(x)) = (hog)(y) = h(g(y)) = h(z) = w$, $x \in A$.

This proves that $ho(gof) = (hog)of$.

Theorems on Composite Functions:

<u>Th.1.</u> If $f: A \to B$ and $g: B \to C$ be both bijective functions, then $gof: A \to C$ is bijective.

<u>Proof:</u> Let $x_1, x_2 \in A$ such that $x_1 \neq x_2$.
Then there exist $y_1, y_2 \in B$ s.t. $f(x_1) = y_1, f(x_2) = y_2$.
Since $y_1, y_2 \in B$, then there exist $z_1, z_2 \in C$ s.t.
$g(y_1) = z_1$, $g(y_2) = z_2$.
<u>Given:</u> f and g are both bijective, hence they are injective.
$\therefore x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2), \Rightarrow y_1 \neq y_2$, as f is injective.
Again, $y_1 \neq y_2 \Rightarrow g(y_1) \neq g(y_2) \Rightarrow z_1 \neq z_2$, as g " "
$\therefore$ For $x_1 \neq x_2$ in A, $gof(x_1) = g(f(x_1)) = g(y_1) = z_1$, and
$gof(x_2) = g(f(x_2)) = g(y_2) = z_2$, where $z_1 \neq z_2$.
$\Rightarrow (gof)(x_1) \neq (gof)(x_2)$ in C.
$\therefore$ $gof: A \to C$ is <u>injective</u>, if f & g <u>both injective</u>.

Now let $z \in C$ be an arbitrary element.
<u>Given:</u> f & g both bijective $\Rightarrow$ f & g both surjective.
Since $g: B \to C$ is surjective & $z \in C$, then $\exists$ at least one pre-image of z in B, say, $y \in B$ s.t. $g(y) = z$.
Again, since $y \in B$ & $f: A \to B$ is surjective, there exists at least one pre-image of y in A, say, $x \in A$ s.t. $f(x) = y$.
$\therefore (gof)(x) = g(f(x)) = g(y) = z$.
$\Rightarrow z \in C$ has a pre-image $x \in A$ under the composite function gof.
Since $z \in C$ is arbitrary, all the elements of C has at least one pre-image in A under gof.
This proves that $gof: A \to C$ is <u>surjective</u>, if f & g <u>both surjective</u>.

$\therefore$ gof is both injective & surjective, hence it is bijective.
(proved)

Remarks:

① The composite function gof is <u>injective</u>, if f & g both <u>injective</u>.
② " " " gof " <u>surjective</u>, " " " " <u>surjective</u>.

Scanned with CamScanner

Theorem 2. If $f: A \to B$ and $g: B \to C$ be two functions, then (i) $g \circ f: A \to C$ is injective $\Rightarrow f$ is injective, and
(ii) $g \circ f: A \to C$ is surjective $\Rightarrow g$ is surjective.

Proof:

(i) Let $x_1, x_2 \in A$ s.t. $f(x_1) = f(x_2)$ in B.
Then $g(f(x_1)) = g(f(x_2))$, since $g: B \to C$ is a function.
$\therefore$ $(g \circ f)(x_1) = (g \circ f)(x_2)$.
$\Rightarrow$ $x_1 = x_2$, since $g \circ f: A \to C$ is injective
Thus, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
$\therefore f: A \to B$ is injective, if $g \circ f: A \to C$ is injective.

(ii) Given: $g \circ f$ is surjective, where $g \circ f: A \to C$.
Let $z \in C$ be arbitrary. Then since $g \circ f$ is surjective, there exists at least one pre-image of z, say x, in A s.t. $(g \circ f)(x) = z$.
$\Rightarrow g(f(x)) = z$, where $f(x) \in B$, and $g: B \to C$. Therefore, z has a pre-image $f(x)$ in B under the function g.
Since $z \in C$, is arbitrary, all elements of C have at least one preimage in B under the function $g: B \to C$.
$\therefore$ $g: B \to C$ is surjective, if $g \circ f: A \to C$ is surjective.

Remark: If $g \circ f: A \to C$ be bijective then $f: A \to B$ is injective and $g: B \to C$ is surjective.

Remark: In order to exist the composite function
(i) $g \circ f$, it is necessary to make sure that $R(f) \subseteq D(g)$.
(ii) $f \circ g$, " " " " " " $R(g) \subseteq D(f)$.
For example, if $f(x) = 1 - x^2$ and $g(x) = \sqrt{x}$, then since $D(g) = \{x ; x \in \mathbb{R} \text{ and } x \geq 0\}$, $(g \circ f)(x) = g(f(x)) = g(1-x^2) = \sqrt{1-x^2}$, only for $x \in D(f)$ that satisfy $f(x) \geq 0$, i.e., for x satisfying $1 - x^2 \geq 0$, i.e., $-1 \leq x \leq 1$. $\boxed{R(f) \subseteq D(g)}$
But the composition $f \circ g$ is given by
$(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = 1 - (\sqrt{x})^2 = 1 - x$, only for x is in the domain $D(g) = \{x \in \mathbb{R} ; x \geq 0\}$.

Exercises / S.K. Mapa Book.

7. Find $g \circ f$ and $f \circ g$, if

(i) $f: \mathbb{Z} \to \mathbb{Z}$ and $g: \mathbb{Z} \to \mathbb{Z}$ are defined by
$f(n) = n^2, n \in \mathbb{Z}$; and $g(n) = 2n, n \in \mathbb{Z}$;

(ii) $f(n) = (-1)^n, n \in \mathbb{Z}$; and $g(n) = 2n, n \in \mathbb{Z}$;

(iii) $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are defined by
$f(x) = |x| + x, x \in \mathbb{R}$; and $g(x) = |x| - x, x \in \mathbb{R}$.

Solution: $g \circ f: \mathbb{Z} \to \mathbb{Z}$, $f \circ g: \mathbb{Z} \to \mathbb{Z}$;

(i) $(g \circ f)(n) = g(f(n)) = g(n^2) = 2n^2, n \in \mathbb{Z}$.

$(f \circ g)(n) = f(g(n)) = f(2n) = (2n)^2 = 4n^2, n \in \mathbb{Z}$.

(ii) Here $g \circ f: \mathbb{Z} \to \mathbb{Z}$, $f \circ g: \mathbb{Z} \to \mathbb{Z}$.

$(g \circ f)(n) = g(f(n)) = g((-1)^n) = 2(-1)^n, n \in \mathbb{Z}$.

$(f \circ g)(n) = f(g(n)) = f(2n) = (-1)^{2n} = 1, n \in \mathbb{Z}$.

(iii) Here $g \circ f: \mathbb{R} \to \mathbb{R}$ and $f \circ g: \mathbb{R} \to \mathbb{R}$.

$(g \circ f)(x) = g(f(x)) = g(|x| + x) = ||x| + x| - (|x| + x)$

$= |x + x| - (x + x)$, when $x \geq 0$

$= |2x| - 2x = 2x - 2x = 0, x \geq 0$

And $(g \circ f)(x) = |-x + x| - (-x + x)$, when $x < 0$

$= 0 - 0 = 0, x < 0$.

$\therefore (g \circ f)(x) = 0, x \in \mathbb{R}$. (Ans)

$(f \circ g)(x) = f(g(x)) = f(|x| - x) = ||x| - x| + (|x| - x)$

$= |x - x| + (x - x)$, when $x \geq 0$

$= 0, x \geq 0$

And $(f \circ g)(x) = |-x - x| + (-x - x)$, when $x < 0$

$= |-2x| - 2x, x < 0$

$= |-2||x| - 2x, x < 0$

$= 2|x| - 2x = -2x - 2x, x < 0$

$= -4x, x < 0$

$$\therefore (f \circ g)(x) = \left.\begin{array}{l} 0, \; x \geq 0 \\ -4x, \; x < 0. \end{array}\right\} \text{(Ans)}$$

Scanned with CamScanner

8. Let $f: A \to B$, $g: B \to C$, $h: B \to C$ be mappings (functions) such that $g \circ f = h \circ f$ and f is surjective. Prove that $g = h$.

$g \circ f : A \to C$, $h \circ f : A \to C$, $(g \circ f)(x) = (h \circ f)(x)$, $x \in A$.

Let $y \in B$ be an arbitrary element. Then since $f: A \to B$ is surjective, $\exists$ a pre-image $x \in A$ of $y \in B$ s.t. $f(x) = y$.

$\therefore (g \circ f)(x) = (h \circ f)(x) \Rightarrow g(f(x)) = h(f(x))$, $f(x) \in B$.

$\Rightarrow g(y) = h(y)$, $y \in B$.

$\Rightarrow \underline{g = h}$. (proved)

9. Let $g: A \to B$, $h: A \to B$, $f: B \to C$ be functions such that $f \circ g = f \circ h$ and f is injective. Prove that $g = h$.

$f \circ g : A \to C$, $f \circ h : A \to C$, and $f \circ g(x) = f \circ h(x)$, $x \in A$.

$\Rightarrow f(g(x)) = f(h(x))$; $g(x), h(x)$ are in B.

$\Rightarrow g(x) = h(x)$, $\forall x \in A$, since f is injective.

$\therefore \underline{g = h}$. (proved)

10. Let $f: A \to B$ and $g: B \to C$ be two functions. Prove that
(i) if $g \circ f$ is injective and f is surjective then g is injective;
(ii) if $g \circ f$ " surjective " g " injective " f " surjective.

(i) Let $y_1, y_2 \in B$. Then since $f: A \to B$ is surjective, y_1, y_2 have pre-images x_1, x_2 (say) respectively in A such that $f(x_1) = y_1$ and $f(x_2) = y_2$.

Now $g(y_1) = g(y_2) \Rightarrow g(f(x_1)) = g(f(x_2))$

$\Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2)$

$\Rightarrow x_1 = x_2$, since $g \circ f$ is injective.

$\Rightarrow f(x_1) = f(x_2)$, since f is a function.

$\Rightarrow y_1 = y_2$.

$$\left.\begin{array}{l} \therefore g(y_1) = g(y_2) \text{ in } C \\ \Rightarrow y_1 = y_2 \text{ in } B \end{array}\right\} \therefore g \text{ is injective.}$$

(proved)

(ii) Let $y \in B$ be an arbitrary element. Then there exists an element $z \in C$ s.t. $g(y) = z$.
Now $gof : A \to C$ and gof is surjective (given), then $z \in C$ has a pre-image, say x in A under the composite function gof s.t.

$$gof(x) = z. \Rightarrow g(f(x)) = g(y) \quad [\because g(y) = z]$$
$$\Rightarrow f(x) = y \text{ in } B \ [\because g \text{ is injective}]$$
$$\Rightarrow y \in B \text{ has a pre-image } x \in A \text{ under the function } f.$$

Since $y \in B$ is arbitrary, all elements of B has a pre-image in A under f.
$\therefore f$ is surjective. (proved)

11. Let $f : A \to B$ be a function. A relation ρ is defined on A by "$x \rho y$ if and only if $f(x) = f(y)$, $x, y \in A$". Show that ρ is an equivalence relation.

Let $x \in A$, then $x \rho x$ as $f(x) = f(x)$, $\forall x \in A$
$\therefore \rho$ is reflexive.
Let $x, y \in A$ and $x \rho y$, then $f(x) = f(y) \Rightarrow f(y) = f(x)$
$\Rightarrow y \rho x$.
$\therefore \rho$ is symmetric.
Let $x, y, z \in A$ and $x \rho y$ & $y \rho z$, then
$$f(x) = f(y) \text{ and } f(y) = f(z)$$
$$\Rightarrow f(x) = f(z) \Rightarrow x \rho z.$$
$\therefore \rho$ is transitive.
$\therefore \rho$ is reflexive, symmetric and transitive.
Hence ρ is an equivalence relation.

Ex. Let $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ be two functions defined by $f(x) = x + 1$, $\forall x \in \mathbb{Z}$, $g(x) = 2x + 3$, $\forall x \in \mathbb{Z}$. Determine whether $gof = fog$. Is gof injective? justify.

$fog : \mathbb{Z} \to \mathbb{Z}$ and $gof : \mathbb{Z} \to \mathbb{Z}$ both exist.
$(fog)(x) = f(g(x)) = f(2x+3) = 2x+3+1 = 2x+4$, $x \in \mathbb{Z}$.
$(gof)(x) = g(f(x)) = g(x+1) = 2(x+1)+3 = 2x+5$, $x \in \mathbb{Z}$.
$\therefore fog \neq gof$.

To examine: whether $g \circ f$ is a injection?
Let us take $x_1, x_2 \in \mathbb{Z}$ (domain) and
let $(g \circ f)(x_1) = (g \circ f)(x_2)$
$\Rightarrow 2x_1 + 5 = 2x_2 + 5 \Rightarrow x_1 = x_2, \ \forall x_1, x_2 \in \mathbb{Z}$.
$\therefore$ $g \circ f : \mathbb{Z} \to \mathbb{Z}$ is injective.

Ex: Let $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ be two functions defined by $f(x) = x + 2, \ \forall x \in \mathbb{Z}$, $g(x) = x - 1, \ \forall x \in \mathbb{Z}$.
Is $g \circ f$ bijective? Justify.

$g \circ f : \mathbb{Z} \to \mathbb{Z}$ exists. $(g \circ f)(x) = g(f(x)) = g(x+2)$
$= (x+2) - 1 = x + 1, \ \forall x \in \mathbb{Z}$.
$\therefore (g \circ f)(x) = x + 1, \ \forall x \in \mathbb{Z}$.

To examine: Injectivity.
Let $x_1, x_2 \in \mathbb{Z}$ and $(g \circ f)(x_1) = (g \circ f)(x_2)$
$\Rightarrow x_1 + 1 = x_2 + 1 \Rightarrow x_1 = x_2, \ \forall x_1, x_2 \in \mathbb{Z}$.
$\therefore$ $g \circ f$ is injective.

To examine: Surjectivity.
Let $y \in \mathbb{Z}$ (co domain) be an arbitrary element such that $(g \circ f)(x) = y$ for some $x \in \mathbb{Z}$ [domain set].
i.e., $x + 1 = y \Rightarrow x = y - 1 \in \mathbb{Z}$ [domain set]
Since y is arbitrary, therefore every element of the co-domain set $\mathbb{Z}$ has its pre-image in the domain set $\mathbb{Z}$. Hence $g \circ f$ is surjective.
$\therefore$ We conclude that $g \circ f : \mathbb{Z} \to \mathbb{Z}$ is bijective.

Ex: Find $g \circ f$ and $f \circ g$ if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are defined by $f(x) = x^2, \ x \in \mathbb{R}$, and $g(x) = \begin{cases} x - 1, & x \geq 0 \\ -x, & x < 0. \end{cases}$

$g \circ f : \mathbb{R} \to \mathbb{R}$ and defined by $(g \circ f)(x) = g\{f(x)\}, \ x \in \mathbb{R}$,
or, $(g \circ f)(x) = g(x^2) = \begin{cases} x^2 - 1, & x^2 \geq 0 \\ -x^2, & x^2 < 0 \text{ (which is not possible)} \end{cases}$

$\therefore (g \circ f)(x) = x^2 - 1, \ x \in \mathbb{R}$. [Since $x^2 \geq 0 \Rightarrow \pm x \geq 0$
$\Rightarrow x \leq 0 \ \& \ x \geq 0$
$\Rightarrow \forall x \in \mathbb{R}$.]

Now $f \circ g : \mathbb{R} \to \mathbb{R}$ and is defined by:
$$(f \circ g)(x) = f(g(x)) = \begin{cases} f(x-1), & x \geq 0 \\ f(-x), & x < 0 \end{cases} = \begin{cases} (x-1)^2, & x \geq 0 \\ x^2, & x < 0. \end{cases}$$

Inverse Function:

Definition → A function $f: A \to B$ is said to be invertible if there exists a function $g: B \to A$ such that $g \circ f = i_A$ and $f \circ g = i_B$.

Note: Here $g: B \to A$ is the inverse of the function $f: A \to B$, and is denoted by $f^{-1}: B \to A$.

So, $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.

Here i_A, i_B are the identity functions.

If $f: A \to A$ be an invertible function, then
$$f^{-1} \circ f = i_A = f \circ f^{-1}.$$

(i) Let $f: A \to B$ be a function. If there exists a function $g: B \to A$ s.t. $g \circ f = i_A$, then $g: B \to A$ is said to be <u>left inverse</u> of $f: A \to B$.

(ii) Let $f: A \to B$ be a function. If there exists a function $h: B \to A$ s.t. $f \circ h = i_B$, then $h: B \to A$ is said to be <u>right inverse</u> of $f: A \to B$.

(iii) If $f^{-1}: B \to A$ be the inverse function of $f: A \to B$, then f^{-1} is both left as well as right inverse of f. Because, $f^{-1} \circ f = i_A$, $f \circ f^{-1} = i_B$.

Therefore, a function $f: A \to B$ to be invertible is that it must have both left and right inverse functions to exist.

Remark: Show that the inverse of a function is unique.

Let $f: A \to B$ be a function and it is invertible. Let us assume that there exist two inverse functions of the $f: A \to B$ say, $g: B \to A$ and $h: B \to A$

s.t. $g \circ f = i_A$, $f \circ g = i_B$; $h \circ f = i_A$, $f \circ h = i_B$.

<u>To prove: $g = h$</u>. By associative property of functions, we have, $h \circ (f \circ g) = (h \circ f) \circ g \Rightarrow h \circ i_B = i_A \circ g$

$\Rightarrow h = g$

$\therefore$ Inverse of a function is unique.

Lemma: Identity function is a bijective function.

Let $i_A : A \to A$ be an identity function.

Injective: Let $x_1, x_2 \in A$ and $i_A(x_1) = i_A(x_2)$
$\Rightarrow x_1 = x_2$.

$\therefore$ i_A is injective.

Surjective: Let $y \in A$ be an arbitrary element.
Then there exists $y \in A$ (domain) s.t.
$i_A(y) = y \Rightarrow y \in$ Codomain A has a pre-image $y \in$ domain A.
Since y is arbitrary, every element of the codomain A has a pre-image in domain A.

$\therefore$ i_A is surjective.

Hence i_A is a bijection.

Theorem: A function $f: A \to B$ is invertible if and only if f is bijective.

Proof: Let $f: A \to B$ be invertible. Then $f^{-1}: B \to A$ exists.

$\therefore f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.

Let $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$ in B.

Since $f^{-1}: B \to A$ be a function, $f^{-1}\{f(x_1)\} = f^{-1}\{f(x_2)\}$ in A.

$\Rightarrow f^{-1} \circ f(x_1) = f^{-1} \circ f(x_2)$

$\Rightarrow i_A(x_1) = i_A(x_2) \Rightarrow x_1 = x_2$ in A.

$\therefore f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \Rightarrow f$ is injective.

Let $y \in B$ be an arbitrary element. Then since $f^{-1}: B \to A$, $\exists\, x \in A$ s.t. $f^{-1}(y) = x$. $\Rightarrow f\{f^{-1}(y)\} = f(x)$ in B,
(since f is a function)
$\Rightarrow f \circ f^{-1}(y) = f(x)$
$\Rightarrow i_B(y) = f(x) \Rightarrow y = f(x)$.

$\therefore y \in B$ has a pre-image $x \in A$ s.t. $y = f(x)$.

$\therefore f$ is surjective, since $y \in B$ is arbitrary.

Hence f is bijective if f is invertible.

Converse Part: Let $f: A \to B$ be bijective.

Let $y \in B$, then $\exists$ an unique pre-image $x \in A$ s.t. $y = f(x)$. (Note: unique pre-image exist, beacause f is bijective).

Scanned with CamScanner

Let us define a function $g: B \to A$ s.t. $g(y) = x$.

$\therefore$ $g\{f(x)\} = x \Rightarrow g\circ f(x) = x \Rightarrow \underline{g\circ f = i_A}$.

Also, $f\{g(y)\} = f(x)$, since $g(y) = x$ and f is a function.

$\Rightarrow f\circ g(y) = y \Rightarrow \underline{f\circ g = i_B}$.

Therefore, $\exists$ a function $g: B \to A$ s.t. $g\circ f = i_A$, $f\circ g = i_B$.

$\therefore$ $g: B \to A$ is the inverse of $f: A \to B$.

$\therefore$ $f: A \to B$ is invertible if $f: A \to B$ is bijective.

<u>Example</u>: Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x+4$, $x \in \mathbb{R}$. Show that f is bijective and find f^{-1}.

<u>To show: f is injective</u>.

Let $x_1, x_2 \in \mathbb{R}$ and $f(x_1) = f(x_2)$ in codomain $\mathbb{R}$.

$\Rightarrow 3x_1 + 4 = 3x_2 + 4 \Rightarrow x_1 = x_2$ in domain $\mathbb{R}$.

$\therefore$ $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$\therefore$ f is injective.

<u>To show: f is surjective</u>.

Let $y \in \mathbb{R}$ (codomain) be an arbitrary element and also let $y = f(x)$ for some $x \in$ domain $\mathbb{R}$.

$\Rightarrow y = 3x + 4 \Rightarrow x = \frac{y-4}{3} \in \mathbb{R}$ (domain).

$\therefore$ y has a pre-image $\frac{y-4}{3}$ in domain $\mathbb{R}$.

Since y is arbitrary in B, every element of B has a pre-image in domain $\mathbb{R}$.

$\therefore$ f is surjective.

Hence <u>f is bijective.</u>

Since f is bijective, $f^{-1}: \mathbb{R} \to \mathbb{R}$ exists and is defined by $f^{-1}(y) = x$, $y \in$ domain $\mathbb{R}$, for some $x \in$ Codomain $\mathbb{R}$; where $y = f(x) \Rightarrow x = \frac{y-4}{3} \in \mathbb{R}$.

$f^{-1}(y) = x$, $y \in \mathbb{R}$

$\Rightarrow f^{-1}(y) = \frac{y-4}{3}$, $y \in \mathbb{R}$.

Theorem:- If $f: A \to B$ be a bijective function, then its inverse $f^{-1}: B \to A$ is also bijective, and $(f^{-1})^{-1} = f$.

Proof: Since $f: A \to B$ is bijective, $f^{-1}: B \to A$ exists, and $f^{-1} \circ f = i_A$, $f \circ f^{-1} = i_B$.

To show f^{-1} is injective: Let $y_1, y_2 \in B$ and $f^{-1}(y_1) = f^{-1}(y_2)$ in A.

$\Rightarrow f(f^{-1}(y_1)) = f(f^{-1}(y_2))$ in B, since f is a function.

$\Rightarrow f \circ f^{-1}(y_1) = f \circ f^{-1}(y_2) \Rightarrow i_B(y_1) = i_B(y_2) \Rightarrow y_1 = y_2$ in B.

$\therefore f^{-1}$ is injective.

To show f^{-1} surjective: Let $x \in A$ be arbitrary element. Since $f: A \to B$ is bijective, $x \in A$ has exactly one image, say $y \in B$ s.t. $y = f(x)$ in B.

Since $f^{-1}: B \to A$ is a function, $\Rightarrow f^{-1}(y) = f^{-1}(f(x))$ in A

$\Rightarrow f^{-1}(y) = f^{-1} \circ f(x) = i_A(x) = x$.

$\therefore x \in A$ has a pre-image $y \in B$ under f^{-1}. Since $x \in A$ is arbitrary, every element of A has a pre-image in B under f^{-1}.

$\Rightarrow f^{-1}: B \to A$ is surjective.

$\therefore f^{-1}: B \to A$ is bijective, if $f: A \to B$ is bijective.

2nd Part: To prove $(f^{-1})^{-1} = f$.

Since $f^{-1}: B \to A$ is bijective, f^{-1} is invertible and its inverse function $(f^{-1})^{-1}: A \to B$; so that $(f^{-1})^{-1} \circ f^{-1} = i_B$ and $f^{-1} \circ (f^{-1})^{-1} = i_A$.

Again, $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$. [$\because f^{-1}$ is the inverse of f.]

Let $x \in A$, then $\exists$ exactly one image $y \in B$ under f s.t. $y = f(x)$.

$\Rightarrow f^{-1}(y) = f^{-1}(f(x))$, since f^{-1} is a function.

$\Rightarrow f^{-1}(y) = f^{-1} \circ f(x) = i_A(x) = x$

$\Rightarrow (f^{-1})^{-1}(x) = (f^{-1})^{-1}\{f^{-1}(y)\} = (f^{-1})^{-1} \circ f^{-1}(y) = i_B(y) = y$

$\therefore (f^{-1})^{-1}(x) = y$; Also we have $f(x) = y$.

$\therefore (f^{-1})^{-1}(x) = f(x), \forall x \in A$. $\Rightarrow (f^{-1})^{-1} = f$. (proved)

Theorem: Let $f: A \to B$ and $g: B \to C$ be both bijective functions. Then the composite function $gof: A \to C$ is invertible and $(gof)^{-1} = f^{-1} o g^{-1}$.

Proof: Since $f: A \to B$ & $g: B \to C$ are bijective, then we have $gof: A \to C$ is also bijective, and hence, gof is invertible.

So, $(gof)^{-1}: C \to A$ exists.

Again, f bijective $\Rightarrow f^{-1}: B \to A$ exists,
and g " $\Rightarrow g^{-1}: C \to B$ " , so
$f^{-1} o g^{-1}: C \to A$ also exists.

Now to prove: $(gof)^{-1} = f^{-1} o g^{-1}$.

Let $z \in C$ be an arbitrary element s.t. $g(y) = z$, for some $y \in B$, and consequently $f(x) = y$ for $x \in A$, since g and f both are bijective, hence surjective.

Now $f(x) = y \Rightarrow f^{-1}\{f(x)\} = f^{-1}(y)$, $\because f^{-1}$ is a function.

$\Rightarrow (f^{-1} o f)(x) = f^{-1}(y) \Rightarrow i_A(x) = f^{-1}(y)$

$\Rightarrow f^{-1}(y) = x$.

Also, $g(y) = z \Rightarrow (g^{-1} o g)(y) = g^{-1}(z) \Rightarrow i_B(y) = g^{-1}(z)$

$\Rightarrow g^{-1}(z) = y$

Hence $f^{-1} o g^{-1}(z) = f^{-1}(g^{-1}(z)) = f^{-1}(y) = x$, $\forall z \in C$.

And $(gof)(x) = g\{f(x)\} = g(y) = z$

$\Rightarrow (gof)^{-1}\{(gof)(x)\} = (gof)^{-1}(z)$.

$\Rightarrow i_A(x) = (gof)^{-1}(z) \Rightarrow (gof)^{-1}(z) = x$, $\forall z \in C$.

$\therefore (gof)^{-1}(z) = x = (f^{-1} o g^{-1})(z)$, $\forall z \in C$

$\Rightarrow (gof)^{-1} = f^{-1} o g^{-1}$. (proved)

Example: Let $f: Z \to Z$ and $g: Z \to Z$ are defined by $f(x) = x+2$, $\forall x \in Z$, and $g(x) = x-1$, $\forall x \in Z$.

Here f & g be both bijective (show it). Hence f^{-1} & g^{-1} exist.
Also $gof: Z \to Z$ is bijective (show it). Hence $(gof)^{-1}: Z \to Z$ exists.
$f^{-1} o g^{-1}: Z \to Z$ also exists.

Let $x \in Z$, $(gof)(x) = g(f(x)) = g(x+2) = (x+2) - 1 = x+1$

Let $(gof)(x) = y$. then $x+1 = y \Rightarrow x = y-1$, $y \in Z$

$\therefore (gof)^{-1}(y) = x = y-1$, $y \in Z$.

Again, let $y_1 \in Z$ s.t. $f(x) = y_1 \Rightarrow x+2 = y_1 \Rightarrow x = y_1 - 2$

$\therefore f^{-1}(y_1) = x = y_1 - 2$, $y_1 \in Z$.

Also, let $y_2 \in Z$ s.t. $g(x) = y_2 \Rightarrow x-1 = y_2 \Rightarrow x = y_2 + 1$

$\therefore g^{-1}(y_2) = x = y_2 + 1$, $y_2 \in Z$.

$\therefore$ we have $f^{-1}(x) = x-2$, $g^{-1}(x) = x+1$, and

$(g \circ f)^{-1}(x) = x - 1$, $\forall x \in Z$.

Now $(f^{-1} \circ g^{-1})(x) = f^{-1}(x+1) = (x+1) - 2 = x - 1 = (g \circ f)^{-1}(x)$, $\forall x \in Z$.

$\therefore$ $\underline{(g \circ f)^{-1} = f^{-1} \circ g^{-1}}$. The result is established.

Ex: Let a function $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 + 1$, $x \in \mathbb{R}$. Is f bijective? Justify your answer. Find $f^{-1}(-8)$.

Solution: $f(x) = x^2 + 1$, $x \in \mathbb{R}$.

Let us take $x_1, x_2 \in \mathbb{R}$, and $f(x_1) = f(x_2)$

$\Rightarrow x_1^2 + 1 = x_2^2 + 1 \Rightarrow x_1^2 = x_2^2$

$\Rightarrow x_1 = \pm x_2$

i.e., $f(x_1) = f(x_2) \Rightarrow$ either $x_1 = x_2$ or $x_1 = -x_2$.

$\therefore$ f is not injective.

Let $y \in \mathbb{R}$ (Codomain) be an arbitrary element s.t. $y = f(x) = x^2 + 1 \Rightarrow x^2 = y - 1 \Rightarrow x = \pm\sqrt{y-1}$.

Now, $\sqrt{y-1} \in \mathbb{R}$ if $y \geq 1$. When $y \geq 1$, then $x = \pm\sqrt{y-1}$ exists

i.e., y has a pre-image in domain $\mathbb{R}$ if $y \geq 1$.

$\therefore$ For other values of y, i.e., for $\{-\infty < y < 1\}$, y do not have any pre-images in domain $\mathbb{R}$.

$\therefore$ f is not surjective.

$\therefore$ f is not a bijective function. So, f is not invertible

$\therefore$ f^{-1} does not exist. $\Rightarrow f^{-1}(-8)$ does not exist.

Scanned with CamScanner

Some important Points to remember:

1. Total number of <u>subsets</u> of a non-empty set A with $|A| = n$ is $\underline{2^n}$.

2. Total number of elements in the <u>power set</u> of A with $|A| = n$ is $\underline{2^n}$.

3. Total number of <u>relations</u> between two non-empty set A & B with $|A| = m$, $|B| = n$ is $\underline{2^{mn}}$.

4. Total number of <u>functions</u> from set A to set B with $|A| = m$, $|B| = n$ is $\underline{n^m}$.

5. Total number of <u>relations which are not functions</u> $= \underline{2^{mn} - n^m}$.

6. Total number of <u>injective</u> functions from set A to set B with $|A| = m$, $|B| = n$ is
$$\begin{cases} {}^nC_m \cdot m!, & \text{if } m \le n \\ 0, & \text{if } m > n \end{cases} \quad \underline{\text{OR}} \quad \begin{cases} {}^nP_m, & \text{if } m \le n \\ 0, & \text{if } m > n. \end{cases}$$

7. Total number of <u>surjective</u> (onto) functions $f: A \to B$ with $|A| = m$, $|B| = n$ is given by
$$n^m - \left[{}^nC_1 (n-1)^m - {}^nC_2 (n-2)^m + {}^nC_3 (n-3)^m - \cdots (-1)^{n-1}\, {}^nC_{n-1} \cdot 1^m\right].$$
$$= \begin{cases} \sum_{r=0}^{n-1} (-1)^r \, {}^nC_r (n-r)^m, & \text{if } m \ge n \\ 0, & \text{if } m < n \end{cases}$$

8. Total number of bijections $f: A \to B$ with $|A| = n = |B|$ is given by $\underline{n!}$.

9. <u>Every function is a relation</u>, <u>but not every relation is a function</u>.

For example, let $A = \{2, 3, 4\}$ and f be a relation s.t. $f = \{(a, b) \in A \times A : a | b\} = \{(2,2), (2,4), (3,3), (4,4)\}$
f is not a function, since $2 \in A$ has two images 2 & 4 in A (co-domain).

Scanned with CamScanner

(10) For a surjective (onto) function $f: A \to B$, $f(A) = B$. (How?)

For any function $f: A \to B$, we have $f(A) \subseteq B \longrightarrow$ (i)

In onto function, for each element of B, there exists at least one pre-image in A. So, if $y \in B$, then $\exists\ x \in A$ s.t. $y = f(x) \in f(A)$.

$\therefore\ y \in B \Rightarrow y \in f(A)$

Hence $B \subseteq f(A) \longrightarrow$ (ii)

(i) & (ii) $\Rightarrow f(A) = B$ for an onto function.

(11) If a function $f: A \to B$ be invertible, then $f^{-1}: B \to A$ exists. And if $y = f(x)$, $x \in A$, $y \in B$, then $f^{-1}(y) = x$, $y \in B$. (How?).

Let $y = f(x) \Rightarrow f^{-1}(y) = f^{-1}\{f(x)\}$, since $f^{-1}: B \to A$ is a function.

$$\Rightarrow f^{-1}(y) = f^{-1} \circ f(x) = (f^{-1} \circ f)(x)$$
$$= i_A(x) = x.$$

$\therefore\ y = f(x) \Rightarrow f^{-1}(y) = x$, if f is invertible

(12) Let $f: Z^+ \to Z$ be a function defined by

$f(x) = \frac{x}{2}$, if x is an even integer.

$\quad = \frac{1-x}{2}$, if x " " odd "

Z^+ is the set of all non-negative integers.

Is f bijective? Justify.

Let us consider x_1 and x_2 are even integers $\in Z^+$.

Let $f(x_1) = f(x_2)$. $\Rightarrow \frac{x_1}{2} = \frac{x_2}{2} \Rightarrow x_1 = x_2$

Now let us consider x_3, x_4 are odd integers $\in Z^+$.

Let $f(x_3) = f(x_4)$. $\Rightarrow \frac{1-x_3}{2} = \frac{1-x_4}{2} \Rightarrow x_3 = x_4$.

$\therefore$ f is injective.

Let $y \in Z$ be an arbitrary negative integer.

If y is an even negative integer, then $y = x/2 \Rightarrow x = 2y$, x is an even negative integer $\notin Z^+$

Again if y is an odd ~~negative~~ positive integer, then $y = \frac{1-x}{2} \Rightarrow x = 1-2y$ then x is an negative integer $\notin Z^+$ $\therefore$ f is not onto. $\therefore$ f is not bijective

Scanned with CamScanner

Problems on Function :-

Ex. ① Let $f: \mathbb{Z} \to \mathbb{Z}$ and $g: \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x+2$, $g(x) = |x|$. Which is one-one (injective)? Is $f \circ g = g \circ f$?

f is one-one, since for $x_1, x_2 \in \mathbb{Z}$, $f(x_1) = f(x_2) \Rightarrow x_1 + 2 = x_2 + 2$
$\Rightarrow x_1 = x_2$.

g is not one-one, since for $2 \in \mathbb{Z}$, and $-2 \in \mathbb{Z}$, $(2 \neq -2)$,
$g(2) = 2 = g(-2)$.

Now $f \circ g: \mathbb{Z} \to \mathbb{Z}$; $f \circ g(x) = f\{g(x)\} = f(|x|) = |x| + 2$, $x \in \mathbb{Z}$.

And $g \circ f: \mathbb{Z} \to \mathbb{Z}$, $g \circ f(x) = g\{f(x)\} = g(x+2) = |x+2|$, $x \in \mathbb{Z}$.

$\therefore$ $f \circ g \neq g \circ f$, as $|x| + 2 \neq |x+2|$, for some $x \in \mathbb{Z}$.

② Show that the function $f: \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n+2$ is one-one (injective) but not onto (surjective).

For $n_1, n_2 \in \mathbb{N}$, let $f(n_1) = f(n_2) \Rightarrow n_1 + 2 = n_2 + 2 \Rightarrow n_1 = n_2$; $\forall n_1, n_2 \in \mathbb{N}$.
$\therefore$ f is one-one

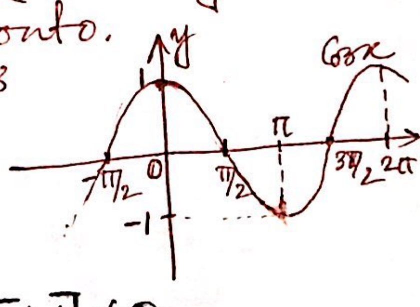
Let m be an arbitrary element of the co-domain $\mathbb{N}$ such that $m = f(n) = n+2 \Rightarrow n = m-2 \notin \mathbb{N}$ if $m \leq 2$,
i.e., if $m = 1$ and $m = 2$.
$\therefore$ Elements 1 and 2 in the co domain $\mathbb{N}$ have no pre-image in the domain $\mathbb{N}$.
$\Rightarrow$ f is not onto.

③ A function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1$, if x is rational
$= -1$, if x is irrational.
Find $R(f)$. Find $f(\sqrt{7})$. Is f one-one & onto?

$R(f) = \{1, -1\}$. $f(\sqrt{7}) = -1$, since $\sqrt{7}$ is irrational.

For $x_1, x_2 \in \mathbb{R}$, $f(x_1) = f(x_2) \not\Rightarrow x_1 = x_2$, since
$f(2) = 1 = f(3)$, but $2 \neq 3$.
$\therefore$ f is not one-one.
f is not onto also, since $R(f) = \{1, -1\} \neq \mathbb{R}$ (co-domain).

④ Show that the function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = \cos x$, $\forall x \in \mathbb{R}$ is neither one-one nor onto.
However, $f: [-\pi/2, \pi/2] \to [-1, 1]$ defined as $f(x) = \cos x$ is bijective.

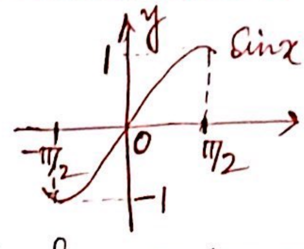


First case: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \cos x$,
For $x = 0$, $x = 2\pi$, $f(0) = 1 = f(2\pi)$, $0 \neq 2\pi$.
$\Rightarrow$ f is not one-one.

Again, $-1 \leq \cos x \leq 1$, $\forall x \in \mathbb{R}$. Then $R(f) = [-1, 1] \neq \mathbb{R}$.
$\therefore$ f is not onto.

For 2nd case,

$f : [-\pi/2, \pi/2] \to [-1, 1]$, $f(x) = \sin x$.



In the interval $[-\pi/2, \pi/2]$, $\sin x$ assumes only one value between -1 and 1. $\therefore$ $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, for $-\pi/2 \le x_1, x_2 \le \pi/2$

$\therefore$ $f(x) = \sin x$ is one-one.

Also, $-1 \le \sin x \le 1$, $\forall x \in \mathbb{R}$.

$\therefore$ $R(f) = [-1, 1] =$ co-domain of f.

$\therefore$ $\sin x$ is onto.

$\therefore$ $\sin x$ is bijective.

⑤ Let $f : \mathbb{R} \to \mathbb{Z}$ be a function defined by $f(x) = [x]$, $x \in \mathbb{R}$. Show that f is surjective but not injective.

$f(x) = [x] =$ the greatest integer $\le x$.

$f(1) = 1$, $f(1.5) = 1$, where $1 \neq 1.5$,

but $f(1) = f(1.5)$ in $\mathbb{Z}$.

$\therefore$ f is not injective.

Since $R(f) = \mathbb{Z}$, therefore f is surjective.

For, if $y \in \mathbb{Z}$ be arbitrary element, then there exists pre-images $y \le x < y+1$ in $\mathbb{R}$.

⑥ Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$, $x \in \mathbb{R}$ is a bijection. Determine f^{-1}.

Let $x_1, x_2 \in \mathbb{R}$, and $f(x_1) = f(x_2) \Rightarrow x_1^3 = x_2^3$.

$\Rightarrow (x_1 - x_2)(x_1^2 + x_1 x_2 + x_2^2) = 0$

$\Rightarrow (x_1 - x_2)[x_1^2 \omega^3 - (\omega^2 + \omega) x_1 x_2 + x_2^2 \omega^3] = 0$, where $\omega^3 = 1$ & $1 + \omega + \omega^2 = 0$

$\Rightarrow (x_1 - x_2)[x_1 \omega^2 (x_1 \omega - x_2 \omega^2) - x_2 \omega (x_1 \omega - x_2 \omega^2)] = 0$

$\Rightarrow (x_1 - x_2)(x_1 \omega^2 - x_2 \omega)(x_1 \omega - x_2 \omega^2) = 0$

$\Rightarrow (x_1 - x_2)$, since $x_1 \neq x_2 \omega$ & $x_2 \neq x_1 \omega$, ~~for $x_1, x_2 \in \mathbb{R}$~~

$\Rightarrow x_1 = x_2$, for $x_1, x_2 \in \mathbb{R}$.

$\therefore$ f is injective.

Let $y \in \mathbb{R}$ (Codomain) be arbitrary s.t.

$y = f(x) = x^3$, then $x = \sqrt[3]{y}, \sqrt[3]{y}\,\omega, \sqrt[3]{y}\,\omega^2$.

But $\sqrt[3]{y}\,\omega$ and $\sqrt[3]{y}\,\omega^2$ do not belong to $\mathbb{R}$,

By $x = \sqrt[3]{y} \in \mathbb{R}$ for $y \in \mathbb{R}$.

$\therefore$ y has a pre-image $\sqrt[3]{y}$ in domain $\mathbb{R}$.

Since y is arbitrary, f is surjective.

$\therefore$ f is a bijection.
Hence f is invertible, and $f^{-1}: \mathbb{R} \to \mathbb{R}$ exists,
if $f(x) = y$ then $f^{-1}(f(x)) = f^{-1}(y)$

$$\Rightarrow f^{-1} \circ f(x) = f^{-1}(y) \Rightarrow f^{-1}(y) = i_{\mathbb{R}}(x) = x.$$

or, $f^{-1}(y) = x = \sqrt[3]{y} \in \mathbb{R}$.

$\therefore$ $\boxed{f^{-1}(x) = \sqrt[3]{x}, \ x \in \mathbb{R}}$.

Ex:⑦ Show that the function $f: S \to \mathbb{R}$ defined by
$f(x) = \dfrac{x}{1-|x|}$, $x \in S$, where $S = \{x \in \mathbb{R} : -1 < x < 1\}$,
is a bijection. Determine f^{-1}.

Let $x_1, x_2 \in S$ and $f(x_1) = f(x_2) \Rightarrow \dfrac{x_1}{1-|x_1|} = \dfrac{x_2}{1-|x_2|}$

$\Rightarrow x_1, x_2$ are either both positive, or both negative.

Let $x_1 > 0, x_2 > 0$,
then $f(x_1) = f(x_2) \Rightarrow \dfrac{x_1}{1-x_1} = \dfrac{x_2}{1-x_2} \Rightarrow x_1 = x_2$.

Let $x_1 < 0, x_2 < 0$, then
$f(x_1) = f(x_2) \Rightarrow \dfrac{x_1}{1+x_1} = \dfrac{x_2}{1+x_2} \Rightarrow x_1 = x_2$.

$\therefore$ $f: S \to \mathbb{R}$ is injective.

Let $y \in \mathbb{R}$ be arbitrary and let $y > 0$ and
$y = f(x)$ for $x > 0$.
$\Rightarrow y = \dfrac{x}{1-x} \Rightarrow x = \dfrac{y}{1+y}$ where $0 < x < 1$.

$\therefore$ $\dfrac{y}{1+y}$ is a pre-image of $y \in \mathbb{R}$.

Now, let $y < 0$ & let $y = f(x)$ for $x < 0$.
Then $y = \dfrac{x}{1+x} \Rightarrow x = \dfrac{y}{1-y}$; $-1 < x < 0$.

$\therefore$ $\dfrac{y}{1-y}$ is a pre-image of y.

Again, if $y = 0 \in \mathbb{R}$, then $x = 0 \in S$ is its pre-image.

$\therefore$ Each $y \in \mathbb{R}$ has a pre-image in S.
So f is surjective. Hence f is a bijection.

$\therefore$ $f^{-1}: \mathbb{R} \to S$ exists.

We have obtained:
For $y > 0$, the pre-image is $\dfrac{y}{1+y} = \dfrac{y}{1+|y|}$
For $y < 0$, " " " " $\dfrac{y}{1-y} = \dfrac{y}{1+|y|}$
For $y = 0$, " " " " $0 = \dfrac{y}{1+|y|}$

$\therefore$ For each $y \in \mathbb{R}$, the pre-image $= \dfrac{y}{1+|y|}$.
$\therefore$ $f^{-1}: \mathbb{R} \to S$ is defined by $f^{-1}(x) = \dfrac{x}{1+|x|}$, $x \in \mathbb{R}$.

(8) Find whether the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = |x-3|$, $x \in \mathbb{Z}$ is onto or not.

Let $y \in \mathbb{Z}$ be an arbitrary element and $y = f(x)$

$\Rightarrow y = |x-3| \Rightarrow x-3 = \pm y \Rightarrow x = 3 \pm y \in \mathbb{Z}$.

$\therefore$ y has two pre-images $3+y$ and $3-y$ in $\mathbb{Z}$.

Since y is arbitrary, then every element in the co-domain $\mathbb{Z}$ has at least one pre-image in the domain $\mathbb{Z}$. This proves that f is onto.

Note that f is not injective. Because, for $x=1$ and $x=5$, $f(1) = |1-3| = 2 = f(5) = |5-3|$

$\therefore 1 \neq 5 \Rightarrow f(1) = f(5)$.

(9) Let $f : \mathbb{R}-\{1\} \to \mathbb{R}-\{1\}$ be defined by $f(x) = \frac{x+1}{x-1}$, $x \in \mathbb{R}-\{1\}$. Show that f is a bijection. Determine f^{-1}.

Let $x_1, x_2 \in \mathbb{R}-\{1\}$. Then $f(x_1) = f(x_2) \Rightarrow \frac{x_1+1}{x_1-1} = \frac{x_2+1}{x_2-1}$

$\Rightarrow (x_1+1)(x_2-1) = (x_1-1)(x_2+1)$

$\Rightarrow \cancel{x_1x_2} + x_2 - x_1 - \cancel{1} = \cancel{x_1x_2} - x_2 + x_1 - \cancel{1}$

$\Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$

$\therefore f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Hence f is injective.

Let $y \in \mathbb{R}-\{1\}$ be arbitrary element and

$y = f(x) = \frac{x+1}{x-1} \Rightarrow xy - y = x+1 \Rightarrow x(y-1) = y+1$

$\Rightarrow x = \frac{y+1}{y-1}$, $y \in \mathbb{R}-\{1\}$.

$\therefore$ y has a pre-image $\frac{y+1}{y-1}$ in the domain.

Since y is arbitrary, f is surjective.

$\therefore$ f is a bijection.

So, f is invertible & $f^{-1} : \mathbb{R}-\{1\} \to \mathbb{R}-\{1\}$ exists.

$y = f(x) \Rightarrow f^{-1}(y) = f^{-1}(f(x))$, [$\because f^{-1}$ is a function]

$\Rightarrow f^{-1}(y) = (f^{-1} \circ f)(x) = x$

$= \frac{y+1}{y-1}$

$\therefore f^{-1}(x) = \frac{x+1}{x-1}$, $x \in \mathbb{R}-\{1\}$.

$\boxed{\text{Here } f = f^{-1}}$ Hence it is an identity function.

(10) Let $f = \{(x,y) \in \mathbb{R}\times\mathbb{R} : y = \frac{1}{x}\}$. Examine if f is a function from $\mathbb{R}$ to $\mathbb{R}$.

f is not a function, because, $0 \in \mathbb{R}$ does not have any image in $\mathbb{R}$.
But, if we take $S = \mathbb{R} - \{0\}$. Then $f = \{(x,y) \in S \times \mathbb{R} : y = \frac{1}{x}\}$ is a function from S to $\mathbb{R}$. So $f : S \to \mathbb{R}$ is a function defined by $y = f(x) = \frac{1}{x}$, $x \in S$.

(11) Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + x$, $\forall x \in \mathbb{R}$ is not a bijection.

Let $x_1, x_2 \in \mathbb{R}$ and $f(x_1) = f(x_2) \Rightarrow x_1^2 + x_1 = x_2^2 + x_2$
$\Rightarrow x_1^2 - x_2^2 + (x_1 - x_2) = 0$
$\Rightarrow (x_1 - x_2)(x_1 + x_2 + 1) = 0$
$\therefore$ either $x_1 = x_2$, or, $x_1 = -x_2 - 1$.
$\therefore f(x_1) = f(x_2) \not\Rightarrow x_1 = x_2$, $\forall x_1, x_2 \in \mathbb{R}$.
So, f is not injective.
Let $y \in \mathbb{R}$ be an arbitrary element and $y = f(x) \Rightarrow y = x^2 + x$ or, $x^2 + x - y = 0 \Rightarrow x = \frac{-1 \pm \sqrt{1+4y}}{2}$
If $1 + 4y < 0$, or, $y < -\frac{1}{4}$, then $x \notin \mathbb{R}$.
$\therefore$ For $y < -\frac{1}{4}$, y do not have any pre-image x in the domain $\mathbb{R}$. For example, if $y = -1$, $x = \frac{-1 \pm \sqrt{3}i}{2} \notin \mathbb{R}$.
$\therefore f$ is not surjective.
Therefore, f is not a bijection.

(12) Examine whether the function $f : \mathbb{R} \to (-1,1)$ defined by $f(x) = \frac{x}{x^2+1}$, $\forall x \in \mathbb{R}$, is bijective, i.e., in $-1 < f(x) < 1$.

Let $x_1, x_2 \in \mathbb{R}$ and $f(x_1) = f(x_2)$ in $(-1,1)$,
$\Rightarrow \frac{x_1}{x_1^2+1} = \frac{x_2}{x_2^2+1} \Rightarrow x_1 x_2^2 - x_1^2 x_2 + x_1 - x_2 = 0 \Rightarrow x_1 x_2 (x_2 - x_1) + (x_1 - x_2) = 0$
$\Rightarrow (x_1 - x_2)(1 - x_1 x_2) = 0 \Rightarrow$ either $x_1 = x_2$, or, $x_1 x_2 = 1$.
$\therefore f(x_1) = f(x_2) \Rightarrow$ either $x_1 = x_2$, or, $x_1 = 1/x_2$
$\therefore f(x_1) = f(x_2) \not\Rightarrow x_1 = x_2$ $\forall x_1, x_2 \in \mathbb{R}$.
$\therefore f$ is not injective.
Let $y \in (-1,1)$ be an arbitrary element s.t. $f(x) = y$.
$\therefore \frac{x}{x^2+1} = y \Rightarrow x^2 y - x + y = 0 \Rightarrow x = \frac{1 \pm \sqrt{1-4y^2}}{2}$.
If $1 - 4y^2 < 0$, or, $|y| > \frac{1}{2}$, $x \notin \mathbb{R}$. For example, if $y = 2/3$, then $x = \frac{1 \pm \frac{\sqrt{7}}{3} i}{2} \notin \mathbb{R}$. $\therefore$ For $|y| > \frac{1}{2}$, there do not exist any pre-image in $\mathbb{R}$. Hence f is not surjective, hence not bijective.

Functions

Direct and inverse images.

Let $f: A \to B$ be a function. Let P be a non-empty subset of A.

Direct image of P under f : $f(P) = \{f(x) ; x \in P\} \subset B$

Let S be a non-empty subset of B.

Inverse image of S under f: $f^{-1}(S) = \{x : f(x) \in S\} \subset A$

Note: f need not be a bijection so that the inverse fn $f^{-1}: B \to A$ need not exist.

If however, f^{-1} exists as the inverse fn, then $f^{-1}(S)$ is regarded as the Direct image of S under f^{-1}.

Also, $f^{-1}(S)$ " " " inverse " of S " f.

Example:

① Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2, x \in \mathbb{R}$.

Let $P = \{x \in \mathbb{R} : 0 \le x \le 3\}$. Then

$f(P) = \{f(x) : x \in P\} = \{y \in \mathbb{R} : 0 \le y \le 9\}$.

Inverse image of the set $f(P) = f^{-1}(f(P))$

$= \{x \in \mathbb{R} : f(x) \in f(P)\}$

$= \{x \in \mathbb{R} : -3 \le x \le 3\}$.

Here $f^{-1}(f(P)) \ne P$. . But if $f: \mathbb{R}^+ \to \mathbb{R}$, then $f^{-1}(f(P)) = P$

② Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x, x \in \mathbb{R}$.

Let $P = \{x \in \mathbb{R} : 0 \le x \le 3\}$. Then

$f(P) = \{f(x) : x \in P\} = \{y \in \mathbb{R} : 0 \le y \le 6\}$

$f^{-1}(f(P)) = \{x \in \mathbb{R} : f(x) \in f(P)\} = \{x \in \mathbb{R} ; 0 \le x \le 3\}$

$\therefore$ $f^{-1}(f(P)) = P$ for this problem.

Note: Here f is a bijection. $\therefore$ inverse fun f^{-1} exists

$\left.\begin{array}{l}\text{Direct image of } f(P) \text{ under } f^{-1} \text{ is } f^{-1}(f(P)) \\ \text{\& inverse " " } f(P) \text{ under } f \text{ is } f^{-1}(f(P))\end{array}\right\}$ are same

Scanned with CamScanner

Th: Let $f: A \to B$ be a fn and P, Q be nonempty subsets of A. Then

(a) $P \subset Q \Rightarrow f(P) \subset f(Q)$

(b) $f(P \cup Q) = f(P) \cup f(Q)$

(c) $f(P \cap Q) = f(P) \cap f(Q)$, if f is injective.

Proof (a) Let $y \in f(P)$. Then $\exists\, x \in P$ s.t. $f(x) = y$.

$x \in P \Rightarrow x \in Q \Rightarrow f(x) \in f(Q) \Rightarrow y \in f(Q)$

$\therefore$ $\underline{f(P) \subset f(Q)}$.

(b) $P \subset P \cup Q$, $Q \subset P \cup Q \Rightarrow f(P) \subset f(P \cup Q)$, $f(Q) \subset f(P \cup Q)$

$\Rightarrow f(P) \cup f(Q) \subset f(P \cup Q)$ ---- (i)

Let $y \in f(P \cup Q) \Rightarrow \exists\, x \in P \cup Q$ s.t. $f(x) = y$

$x \in P \cup Q \Rightarrow x \in P$ or $x \in Q$.

$\Rightarrow f(x) \in f(P)$ or $f(x) \in f(Q)$

$\Rightarrow y = f(x) \in f(P) \cup f(Q)$

$\therefore$ $f(P \cup Q) \subset f(P) \cup f(Q)$ ------- (ii).

From (i) & (ii) $\Rightarrow$ $\underline{f(P \cup Q) = f(P) \cup f(Q)}$.

(c) $P \cap Q \subset P$ and $P \cap Q \subset Q$

$\Rightarrow f(P \cap Q) \subset f(P)$ and $f(P \cap Q) \subset f(Q)$

$\Rightarrow f(P \cap Q) \subset f(P) \cap f(Q)$ ------ (i)

Let $y \in f(P) \cap f(Q) \Rightarrow y \in f(P)$ and $y \in f(Q)$

Since f is injective, y has a unique pre-image x in both P and Q.

$\therefore$ $x \in P \cap Q \Rightarrow f(x) \in f(P \cap Q) \Rightarrow y \in f(P \cap Q)$

$\therefore$ $f(P) \cap f(Q) \subset f(P \cap Q)$ ------ (ii).

From (i) & (ii) $\Rightarrow$ $\underline{f(P \cap Q) = f(P) \cap f(Q)}$.

Th: Let $f: A \to B$ be an onto function and S, T be subsets of B. Then

(a) $S \subset T \Rightarrow f^{-1}(S) \subset f^{-1}(T)$

(b) $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$

(c) $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$

(d) $f^{-1}(S') = [f^{-1}(S)]'$

(a) Let $x \in f^{-1}(S) \Rightarrow f(x) \in S$

$\Rightarrow f(x) \in S \subset T$

$\Rightarrow x \in f^{-1}(T)$

$\therefore$ $S \subset T \Rightarrow f^{-1}(S) \subset f^{-1}(T)$

(b) $S \subset S \cup T$, $T \subset S \cup T \Rightarrow f^{-1}(S) \subset f^{-1}(S \cup T)$, $f^{-1}(T) \subset f^{-1}(S \cup T)$

$\Rightarrow f^{-1}(S) \cup f^{-1}(T) \subset f^{-1}(S \cup T)$ ---- (i)

Let $x \in f^{-1}(S \cup T) \Rightarrow f(x) \in S \cup T \Rightarrow f(x) \in S$ or $f(x) \in T$.

$\Rightarrow x \in f^{-1}(S)$ or $x \in f^{-1}(T)$

$\Rightarrow x \in f^{-1}(S) \cup f^{-1}(T)$

$\therefore$ $f^{-1}(S \cup T) \subset f^{-1}(S) \cup f^{-1}(T)$ ---- (ii)

From (i) & (ii) $\Rightarrow f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$

(c) $S \cap T \subset S$ and $S \cap T \subset T$

$\Rightarrow f^{-1}(S \cap T) \subset f^{-1}(S)$ and $f^{-1}(S \cap T) \subset f^{-1}(T)$.

$\Rightarrow f^{-1}(S \cap T) \subset f^{-1}(S) \cap f^{-1}(T)$ - - - (i)

Let $x \in f^{-1}(S) \cap f^{-1}(T) \Rightarrow x \in f^{-1}(S)$ and $x \in f^{-1}(T)$

$\Rightarrow f(x) \in S$ and $f(x) \in T$

$\Rightarrow f(x) \in S \cap T \Rightarrow x \in f^{-1}(S \cap T)$ - - (ii)

From (i) & (ii) $\Rightarrow$ $\underline{f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)}$.

(d) Let $x \in f^{-1}(S') \Rightarrow f(x) \in S' \Rightarrow f(x) \notin S \Rightarrow x \notin f^{-1}(S)$

$\Rightarrow x \in [f^{-1}(S)]' \Rightarrow f^{-1}(S') \subset [f^{-1}(S)]'$ - - - (i)

Let $x \in [f^{-1}(S)]' \Rightarrow x \notin f^{-1}(S) \Rightarrow f(x) \notin S \Rightarrow f(x) \in S'$

$\Rightarrow x \in f^{-1}(S') \Rightarrow [f^{-1}(S)]' \subset f^{-1}(S')$ - - - - (ii)

From (i) & (ii) $\Rightarrow$ $\underline{f^{-1}(S') = [f^{-1}(S)]'}$

Ex: Let $f: A \to B$ and $S \subset B$. Prove that

(i) $ff^{-1}(S) \subset S$ (ii) $ff^{-1}(S) = S$ if f is onto (Surjective).

(i) Let $y \in ff^{-1}(S) \Rightarrow y$ has a pre-image $x \in f^{-1}(S)$ s.t. $y = f(x)$

$\Rightarrow f(x) \in S$

$\Rightarrow y \in S$.

$\therefore ff^{-1}(S) \subset S$.

(ii) Let $y_1 \in S$ and f is onto $\Rightarrow y_1$ has a pre-image $x_1 \in f^{-1}(S) \Rightarrow f(x_1) \in ff^{-1}(S) \Rightarrow y_1 \in ff^{-1}(S)$

$\therefore S \subset ff^{-1}(S)$

From (i) & (ii): $\underline{ff^{-1}(S) = S}$, if f is onto

Ex. 12 Let $f: A \to B$ and $P \subset A$. Prove that

(i) $P \subset f^{-1}f(P)$ (ii) $P = f^{-1}f(P)$ if f be injective.

(i) Let $x \in P \Rightarrow f(x) \in f(P) \Rightarrow \exists$ a pre-image x of $f(x)$ s.t. $x \in f^{-1}f(P)$

$\therefore P \subset f^{-1}f(P)$.

(ii) Let $x_1 \in f^{-1}f(P) \Rightarrow f(x_1) \in f(P)$

Since f is injective, $f(x_1)$ has a unique pre-image x_1 in P

i.e, $x_1 \in f^{-1}f(P) \Rightarrow x_1 \in P$

$\therefore f^{-1}f(P) \subset P$, if f be injective.

From (i) & (ii), $\underline{P = f^{-1}f(P)}$, if f be injective.

Th: Let $f: A \to B$ be a function and P be a non-empty subset of A, then

(i) $[f(P)]' \subset f(P')$, if f is surjective,

(ii) $[f(P)]' = f(P')$, if f is bijective.

Proof: (i) Let $y \in [f(P)]'$, be arbitrary element.

$\Rightarrow y \notin f(P)$.

If f is surjective, then y has at least one pre-image.
But $y \notin f(P)$, so y has no pre-image in P.
Let x be a pre-image of y, then $x \in A-P$, i.e.,
$x \in P' \Rightarrow f(x) \in f(P') \Rightarrow y \in f(P')$

$\therefore y \in [f(P)]' \Rightarrow y \in f(P')$.

$\therefore [f(P)]' \subset f(P')$, if f is surjective.

(ii) Next, to prove: $f(P') \subset [f(P)]'$ if f injective.

Let $y \in f(P')$ be arbitrary element.
If f is injective, y has a unique pre-image,
say $x \in P'$. $\Rightarrow x \notin P \Rightarrow f(x) \notin f(P) \Rightarrow f(x) \in [f(P)]'$
$\Rightarrow y \in [f(P)]'$.

$\therefore y \in f(P') \Rightarrow y \in [f(P)]'$.

$\therefore f(P') \subset [f(P)]'$, if f is injective.

From (i) & (ii), we may conclude that—

$$[f(P)]' = f(P'), \text{ if } f \text{ is bijective.}$$

Ex. (12) S.K. Mapa. Let $f: A \to B$ and $P \subset A$. Prove that (i) $P \subset f^{-1}f(P)$, (ii) $P = f^{-1}f(P)$, if f is injective. give an example where $P \neq f^{-1}f(P)$.

Solution: (i) Let $x \in P$. Then x has a image, say $y = f(x)$ in $f(P)$.
$y \in f(P) \Rightarrow f(x) \in f(P) \Rightarrow x \in f^{-1}f(P)$.

$\therefore P \subset f^{-1}f(P)$.

(ii) To prove: $f^{-1}f(P) \subset P$.
Let $x \in f^{-1}f(P) \Rightarrow f(x) \in f(P) \Rightarrow x \in P$, {since f is injective, $f(x)$ has a unique pre-image x.
$\therefore f^{-1}f(P) \subset P$ and by (i) $P \subset f^{-1}f(P)$.

$\therefore \underline{P = f^{-1}f(P)}$.

Example (1) previously given is one where $P \neq f^{-1}f(P)$.

One-to-One Correspondence :-

When $f : A \to B$ be a bijection, f sets up a one-to-one correspondence between the elements of A and B.

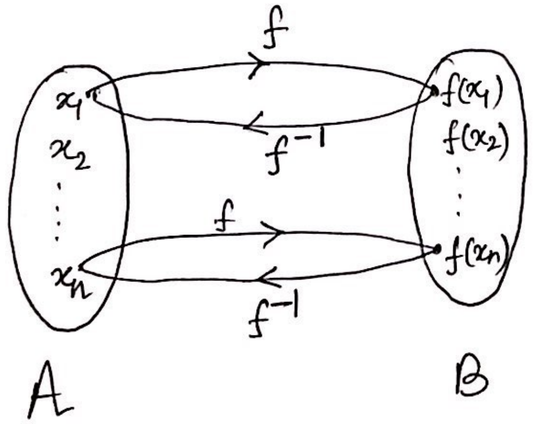
Each element $x \in A$ is put in correspondence with the single $f(x) \in B$,
and " " $y \in B$ " " " " " " " $f^{-1}(y) \in A$.

Ex: Let A and B be both finite sets of n elements and a function $f : A \to B$ is injective. Show that f is a bijection.

Let $A = \{x_1, x_2, \cdots, x_n\}$ be a set of n elements.
Then $B \supset \{f(x_1), f(x_2), \cdots, f(x_n)\}$.
Since $f : A \to B$ is injective, $f(x_1), f(x_2), \cdots, f(x_n)$ are all distinct n elements of B and $B = \{f(x_1), f(x_2), \cdots, f(x_n)\}$.
Let $y \in B$, then $y = f(x_i)$ for some $x_i \in A$, where x_i is the pre-image of y.
y is an arbitrary element of B,
$\therefore$ All the elements of B have a pre-image in A.
This proves that f is surjective.
Hence, f is a bijection. So, f sets up a one-to-one correspondence between the elements of A and B.

Each $x_i \in A$ is put in correspondence with unique $f(x_i) \in B$,
and each $f(x_i) \in B$ " " " " " " $f^{-1}\{f(x_i)\} \in A$

$\left\{ \begin{array}{l} \text{i.e., } (f^{-1} \circ f)(x_i) \in A \\ \text{or, } \quad i_A(x_i) \in A. \\ \text{or, } \quad x_i \in A. \end{array} \right\}$



Cardinality of Set :-

Definition → Let A be a non-empty set. The number of elements in A is called cardinal number of A, and is denoted by Card (A) or $|A|$.

Notes:

① The cardinality of a empty set ϕ is zero. i.e., $Card(\phi) = 0$.

② If A & B be two disjoint sets, i.e., $A \cap B = \phi$ then $|A \cap B| = |\phi| = 0$.

③ If A be a non-empty set, then $|A \cup A| = |A|$, $|A \cap A| = |A|$.

④ If A & B be two non-empty sets, then
$$|A \cup B| = |A| + |B| - |A \cap B|.$$
If $A \cap B = \phi$, then $|A \cup B| = |A| + |B|$.

⑤ Let A and B be two non-empty sets. A is said to be equipotent with B, if there exists a bijective function $f: A \to B$. We denote $A \sim B$ by A and B are equipotent.

⑥ Two equipotent sets have the same cardinal number.

⑦ The number of distinct elements of a finite set A is defined as its cardinal number.

⑧ $Card(A) = Card(B)$ if and only if $A \sim B$.

⑨ If $|A| = |B|$, A & B are said to be equivalent.

⑩ The cardinal number of an infinite set is said to be transfinite cardinal number.

Ex. 1. Let A and B be two finite sets such that $|A| = 5$ and $|B| = 9$, then find the maximum and minimum value of $|A \cup B|$

If $A \cap B = \phi$, then $|A \cup B| = |A| + |B| = 5 + 9 = 14 \leftarrow$ max.

If $A \cap B \neq \phi$, then $\max|A \cap B| = 5$, so $|A \cup B| = |A| + |B| - |A \cap B|$

and $\min|A \cup B| = |A| + |B| - \max|A \cap B| = 5 + 9 - 5 = 9$

$\therefore$ $\max|A \cup B| = 14$, $\min|A \cup B| = 9$. (Ans)

Ex. 2. If A and B be two finite sets such that $|A| = 76$, $|B| = 24$ and $|A \cap B| = 10$, then $|A \cup B| = 90$. → Show it.