Chapter 6

Quadratic Forms

6.1 Definition

An expression of the form $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j$, where the coefficients a_{ij}'s are real and $a_{ij} = a_{ji}$, is said to be a real quadratic form in n variables $x_k, 1 \le k \le n$.

Example: $x^2 - 2xy + 3y^2$ is a real quadratic form in two variables x, y.

If we take $X = \begin{bmatrix} x \\ y \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then

$$AX = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{bmatrix}$$

and

$$X^TAX = [x \quad y]\begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{bmatrix} = x(a_{11}x + a_{12}y) + y(a_{21}x + a_{22}y)$$

$$= a_{11}x^2 + (a_{12} + a_{21})xy + a_{22}y^2.$$

In matrix notation, a quadratic form in n variables can be written as $Q = X^TAX$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & & & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} \end{pmatrix} \text{ and } X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ .with } a_{ij} = a_{ji}$$

Thus every real quadratic form in n variables is associated with a real symmetric square matrix of order n which is called matrix of the quadratic form.

Example: (i) The matrix associated with the real quadratic form $3x^2 + 4xy - 2y^2$in two variables x, y is $\begin{bmatrix} 3 & 2 \\ 2 & -2 \end{bmatrix}$.

(ii) The associated matrix of the real quadratic form $2xy - 4yz$ in three variables x, y, z is

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & -2 & 0 \end{bmatrix}$$

6.2 Classes of Quadratic forms

From the expression of quadratic form it is clear that $Q(x_1, x_2, \dots, x_n)$ is zero when $X = 0$. But if $X \neq 0$, then Q assumes different real values for different non- zero values of X as can be seen from below:

A real quadratic form $Q = X^TAX$ is said to be

(i) *positive definite* if $Q > 0$ for all $X \neq 0$,

(ii) *positive semi- definite* if $Q \ge 0$ for all X and $Q = 0$ for some $X \neq 0$,

(iii) *negative definite* if $Q < 0$ for all $X \neq 0$,

(iv) *negative semi- definite* if $Q \le 0$ for all X and $Q = 0$ for some $X \neq 0$,

(v) *indefinite,* if $Q \ge 0$ for some non- zero values X and $Q \le 0$ for other non-zero values of X.

The associated real symmetric matrix A of a quadratic form Q is said to be positive definite, positive semi-definite, negative definite, negative definite, indefinite according as quadratic form is positive definite, positive semi-definite, negative definite, negative definite, indefinite.

6.3 Reduction to Canonical form

Consider the real quadratic form $Q = X^TAX$, where A is real symmetric matrix. By the transformation $X = PY$, where P is a non-singular matrix, the above quadratic form transforms to $Q = Y^T(P^TAP)Y$. Since, $(P^TAP)^T = (AP)^T(P^T)^T = P^TA^TP = P^TAP$, Q is a quadratic form in Y. Thus Q is transformed to another quadratic form Q' under the transformation $X = PY$. The associated matrix of Q' is congruent to that of Q as both of these are congruent to A.

If A be a real symmetric matrix of rank $r(\le n)$, then there exists a non-singular matrix P such that P^TAP becomes a diagonal matrix of the form

$$D = \begin{pmatrix} I_m & O & O \\ O & -I_{r-m} & O \\ O & O & O \end{pmatrix}$$

of rank r , where $0 \le m < r$.

So, by suitable transformation $X = PY$, where P is non-singular matrix, the real quadratic form $Q = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j$ can be transformed to another form $\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}y_iy_j$ in which $b_{ij} = 0$ for $i \neq j$ and $b_{ii} = 1$ for $i \le m, b_{ii} = -1$ for $m < i \le r$ and $b_{ii} = 0$ for $r < i < n$. This form of the quadratic is called normal or diagonal form of Q and the integer m is invariant.

The rank and signature of a real quadratic form is defined to be those of the associated real symmetric matrix.

Further it can be shown that a real quadratic form $Q = X^TAX$ can be reduced to it to the following normal form

$$Q = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigen values of the associated matrix of the quadratic form. Depending on the eigen values of the associated matrix of the real quadratic form $Q = X^TAX$, it is classified as follows:

(i) *positive definite* iff all the eigen values of A are positive,

(ii) *positive semi- definite* iff all the eigen values of A are non- negative and at least one eigen value is zero,

(iii) *negative definite* iff all the eigen values of A are negative,

(iv) *negative semi- definite* iff all the eigen values of A are non- positive and at least one eigen value is zero,

(v) *indefinite*, A has both negative and positive eigen values.

Theorem 6.3.1: *The nature of a real quadratic form $Q = X^TAX$ regarding positive definiteness, negative definiteness remains invariant under the transformation $X = PY$, where P is non-singular.*

Proof: By the transformation $X = PY$, where P is a non-singular matrix, the quadratic form $Q = X^TAX$ transforms to $Q = Y^T(P^TAP)Y$. From $X = PY$, we have $Y = P^{-1}X$. Now, let Q be positive definite, then

$$Q > 0 \text{ for all } X \neq 0$$

$$\Rightarrow Q > 0 \text{ for all } PY \neq 0$$

$$\Rightarrow Q' > 0 \text{ for all } PY \neq 0$$

$$\Rightarrow Q' > 0 \text{ for all } Y \neq 0$$

Thus Q' is positive definite.

Again, if Q be semi-definite

$$Q \geq 0 \text{ for all } X \text{ and } Q = 0 \text{ for some } X \neq 0$$

$$\Rightarrow Q \geq 0 \text{ for all } PY \text{ and } Q = 0 \text{ for some } PY \neq 0$$

$$\Rightarrow Q' \geq 0 \text{ for all } PY \text{ and } Q' = 0 \text{ for some } PY \neq 0$$

$$\Rightarrow Q' \geq 0 \text{ for all } Y \text{ and } Q' = 0 \text{ for some } Y \neq 0$$

Thus $Q''$is positive semi-definite.

Theorem 6.3.2: A real quadratic form $Q = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j$ of rank r and index m is

(i) positive definite, if $r = n, m = r$;

(ii) positive semi-definite, if $r < n, m = r$;

(iii) negative definite, if $r = n, m = 0$;

(iv) negative semi- definite, if $r < n, m = 0$;

(v) indefinite, if $r \leq n, 0 < m < r$.

Proof: By a suitable transformation $X = PY$, where P is non-singular matrix, the real quadratic form $Q = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j$can be transformed to another form $D = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}y_iy_j$ in which $b_{ij} = 0$ for $i \neq j$ and $b_{ii} = 1$ for $i \leq m$, $b_{ii} = -1$ for $m < i \leq r$ and $b_{ii} = 0$ for $r < i < n$.

(i) If $r = n, m = r$, then $D = \sum_{i=1}^{n} y_i^2$. Hence D is positive definite and therefore Q is so.

(ii) If $r < n, m = r$, then $D = \sum_{i=1}^{r} y_i^2$, $r < n$. Hence D is positive semi-definite and therefore Q is so.

(iii) If $r = n, m = 0$, then $D = -\sum_{i=1}^{n} y_i^2$. Hence D is negative definite and therefore Q is so.

(iv) If $r < n, m = 0$, then $D = -\sum_{i=1}^{r} y_i^2$, $r < n$. Hence D is negative semi-definite and therefore Q is so.

(v) If $r \leq n, 0 < m < r$, then $D = \sum_{i=1}^{m} y_i^2 - \sum_{i=m+1}^{r} y_i^2$, $r \leq n$. Hence D is indefinite and therefore Q is so.

Theorem 6.3.3: *A real symmetric matrix is positive definite iff all its eigen values are positive.*

Proof: Let A be a real symmetric matrix of order n then it has n real eigen values $c_1, c_2, \ldots, c_n$ and there exists an orthogonal matrix P such that $P^{-1}AP$, i.e., P^TAP is a diagonal matrix. Since A and $P^{-1}AP$ has same eigen values,

$$P^{-1}AP = \begin{pmatrix} c_1 & 0 & 0 & \cdots & 0 \\ 0 & c_2 & 0 & \cdots & 0 \\ 0 & 0 & c_3 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & c_n \end{pmatrix}.$$

If A be positive definite, then A is congruent to I_n and also P^TAP is congruent to A , hence P^TAP is congruent to I_n. So $c_i > 0$, $i = 1,2, \ldots, n$.

Conversely, let $c_i > 0$, $i = 1,2, \ldots, n$, then the matrix

$$\begin{pmatrix} c_1 & 0 & 0 & \cdots & 0 \\ 0 & c_2 & 0 & \cdots & 0 \\ 0 & 0 & c_3 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & c_n \end{pmatrix}$$

is positive definite. But A is congruent to P^TAP. Hence, A is positive definite.

Worked Out Examples

1. **Write down the associated matrix of the following quadratic forms:**

   (i) $3x^2 + 2y^2 - 6xy$ in two variables x, y.

   (ii) $3x^2 + 2y^2 - 6xy - 4yz$ in three variables x, y, z.

□ (i) In the given quadratic form $a_{11} = 3, a_{12} + a_{21} = 2a_{12} = 2a_{21} = -6, a_{22} = 2$So the associated matrix is $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 2 \end{bmatrix}$.

(ii) In the given quadratic form

$$a_{11} = 3, a_{22} = 2, a_{33} = 0, a_{12} + a_{21} = 2a_{12} = 2a_{21} = -6,, a_{13} + a_{31} = 2a_{13} = 2a_{31} = 0,$$
$$a_{32} + a_{23} = 2a_{32} = 2a_{23} = -4,$$

So the associated matrix is $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 3 & -3 & 0 \\ -3 & 2 & -2 \\ 0 & -2 & 0 \end{bmatrix}$.

2. **Write down the quadratic forms corresponding to the following associated matrices:**

(i) $\begin{bmatrix} 1 & -3 & 2 \\ -3 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$, (ii) $\begin{bmatrix} 1 & -3 & 0 \\ -3 & 2 & -1 \\ 0 & -1 & 0 \end{bmatrix}$.

□ (i) Given $a_{11} = a_{22} = a_{33} = 1, a_{12} = a_{21} = -3, a_{13} = a_{31} = 2, a_{23} = a_{32} = -2$. Hence, the quadratic form is $x^2 + y^2 + z^2 - 6xy + 4xz - 4yz$.

(ii) Given $a_{11} = 1, a_{22} = 2, a_{33} = 0, a_{12} = a_{21} = -3, a_{13} = a_{31} = 0, a_{23} = a_{32} = -1$. Hence, the quadratic form is $x^2 + 2y^2 - 6xy - 2yz$

3. **Show that the quadratic forms are positive definite**

   (i) $2x^2 + 4y^2 + z^2 - 4xy + 2xz - 2yz$

   (ii) $5x^2 + y^2 + 10z^2 - 10xz - 4yz$

□ Here $Q(x, y, z) = 2x^2 + 4y^2 + z^2 - 4xy + 2xz - 2yz = (x - y + z)^2 + (x - y)^2 + 2y^2 > 0$ for all (x, y, z) and $Q = 0$ for $x = y = z = 0$. Hence Q is positive definite.

(ii) Here $Q(x, y, z) = 5x^2 + y^2 + 10z^2 - 10xz - 4yz = (y - 2z)^2 + 5(x - z)^2 + z^2 > 0$ for all (x, y, z) and $Q = 0$ for $x = y = z = 0$. Hence Q is positive definite.

4. **Prove that the real quadratic $ax^2 + bxy + cy^2$ in two variables x, y, is positive definite if $a > 0$ and $b^2 < 4ac, a, b, c \neq 0$.**

□ Here

$$Q(x, y) = ax^2 + bxy + cy^2$$
$$= a\left[x^2 + 2x\frac{b}{2a}y + \left(\frac{b}{2a}y\right)^2\right] + cy^2 - a\left(\frac{b}{2a}y\right)^2$$
$$= a\left(x + \frac{b}{2a}y\right)^2 + \frac{1}{4a}(4ac - b^2)y^2 > 0$$

for all (x, y) if $a > 0$ and $b^2 < 4ac,\ a, b, c \neq 0$and $Q = 0$ if $x = y = 0$. Hence Q is positive definite if $a > 0$ and $b^2 < 4ac, a, b, c \neq 0$.

5. **Examine whether the quadratic form $3x^2 + 5y^2 + 3z^2 + 2xy + 2xz + 2yz$ is positive definite or not.**

□ Characteristic equation of the associated matrix of the quadratic form is given by

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 5-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0$$
$$\text{or, } \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

and its solutions are $\lambda = 2, 3, 6$.

So, the eigen values of the matrix A are 2,3,6. Since the eigen values of the associated matrix are all positive, the quadratic is positive definite.

6. **Reduce the quadratic form $5x^2 + y^2 + 10z^2 - 4yz - 10xz$ to the normal form. Find its rank, index, signature and show that it is positive definite.**

□ The associated symmetric matrix is $\begin{pmatrix} 5 & 0 & -5 \\ 0 & 1 & -2 \\ -5 & -2 & 10 \end{pmatrix}$. Now we reduce it to the normal form as follows:

$$\begin{pmatrix} 5 & 0 & -5 \\ 0 & 1 & -2 \\ -5 & -2 & 10 \end{pmatrix}$$
$$\xrightarrow{R_{31}(1)} \begin{pmatrix} 5 & 0 & -5 \\ 0 & 1 & -2 \\ 0 & -2 & 5 \end{pmatrix}$$
$$\xrightarrow{C_{31}(1)} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 5 \end{pmatrix}$$
$$\xrightarrow{C_{32}(2)} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$
$$\xrightarrow{R_{32}(2)} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\xrightarrow{R_1\left(\frac{1}{\sqrt{5}}\right)} \begin{pmatrix} \sqrt{5} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\xrightarrow{C_1\left(\frac{1}{\sqrt{5}}\right)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Here, $m = 3, m - r = 0, 2m - r = 3$. So, rank = 3, index = 3 and signature 3. So, the quadratic form is positive definite. From the normal form of the associated matrix, we get the normal form of the given quadratic form is

$$x^2 + y^2 + z^2.$$

7. Reduce the quadratic form $x^2 + 2y^2 + 2z^2 + 2xy + 2xz$ to the normal form. Find its rank, index, signature and show that it is positive semi-definite.

□ The associated symmetric matrix is $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$. Now we reduce it to the normal form as follows:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

$$\xrightarrow{R_{31}(-1), R_{21}(-1)} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\xrightarrow{C_{31}(-1), C_{21}(-1)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\xrightarrow{R_{32}(1)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\xrightarrow{C_{32}(1)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Here, $m = 2, m - r = 0, 2m - r = 2$. So, rank $= 2 < 3$ (number of variables), index = 2 and signature 2. So, the quadratic form is positive semi-definite. From the normal form of the associated matrix, we get the normal form of the given quadratic form is

$$x^2 + y^2.$$

8. Reduce the quadratic form $x^2 + 2y^2 + z^2 + 4xy$ to the normal form. Find its rank, index, signature and show that it is indefinite.

□ The associated symmetric matrix is $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Now we reduce it to the normal form as follows:

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_{21}(-2)} \begin{pmatrix} 1 & 2 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{C_{21}(-2)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2\left(\frac{1}{\sqrt{2}}\right)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{C_2\left(\frac{1}{\sqrt{2}}\right)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_{23}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\xrightarrow{C_{23}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Here, $m = 2, m - r = -1, 2m - r = 1$. So, rank = 3 = number of variables, index = 2 < 3 (number of variables) and signature 1. So, the quadratic form is indefinite. From the normal form of the associated matrix, we get the normal form of the given quadratic form is

$$x^2 + y^2 - z^2.$$

9. Show that the quadratic form $x_1x_2 + x_2x_3 + x_1x_3$ can be reduced to the normal form $y_1^2 - y_2^2 - y_3^2$ by means of the transformation

$$x_1 = y_1 - y_2 - y_3, x_2 = y_1 + y_2 - y_3, x_3 = y_3.$$

□ The associated symmetric matrix is $A = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$.

Now we diagonalize A by congruent operations as follows:

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \frac{1}{2} & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ by } R_{12}(1), C_{12}(1)$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -\frac{1}{4} & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} A \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ by } R_{21}\left(-\frac{1}{2}\right), C_{21}\left(-\frac{1}{2}\right)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -1 & -1 & 1 \end{pmatrix} A \begin{pmatrix} 1 & -\frac{1}{2} & -1 \\ 1 & \frac{1}{2} & -1 \\ 0 & 0 & 1 \end{pmatrix}, \text{ by } R_{31}(-1), C_{31}(-1)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} A \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \text{ by } R_2(2), C_2(2)$$

Which is of the form $D = P^T AP$. Thus under the transformation $X = PY$, i.e.,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

i.e.. $x_1 = y_1 - y_2 - y_3, x_2 = y_1 + y_2 - y_3, x_3 = y_3,$

the given quadratic form reduces to the normal form $y_1^2 - y_2^2 - y_3^2$.

Exercise

1. Write down the associated matrix of the following quadratic forms:

(i) $x^2 + 2y^2 - 4xy$ in two variables x, y.

(ii) $x^2 + 3z^2 - 4xy - 4xz$ in three variables x, y, z.

(iii) $x^2 + y^2 + 3z^2 + 2xy - 2xz + 2yz$ in three variables x, y, z.

2. Write down the quadratic forms corresponding to the following associated matrices:

(i) $\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}$, (ii) $\begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix}$, (iii) $\begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -1 \\ -2 & -1 & 0 \end{bmatrix}$

3. Examine whether the following quadratic forms are positive definite or not.

(i) $x^2 + 4y^2 + z^2 + 2xy + 2yz$

(ii) $3x^2 + y^2 + 10z^2 + 2xy - 8xz$

(iii) $x^2 + y^2 - z^2 + 2xy + 2xz - 2yz$

(iv) $8x^2 + 7y^2 + 3z^2 - 12xy + 4xz - 8yz$

(v) $x^2 + y^2 + z^2 + yz$

(vi) $6x^2 + y^2 + 18z^2 - 4yz - 12xz,$

(vii) $x^2 + y^2 + 2yz.$

4. Show that the following forms are positive semi-definite

(i) $x^2 + 2y^2 + 2z^2 + 2xy + 2xz,$

(ii) $x^2 + 5y^2 + 2z^2 - 4xy - 6zy + 2xz,$

(iii) $2x^2 + 2y^2 + 5z^2 - 4xy - 2xz + 2yz,$

(iv) $x^2 + y^2 + z^2 - xy - xz - yz.$

5. Show that the following forms are indefinite

(i) $x^2 + 2y^2 + z^2 + 4xy,$

(ii) $x^2 + y^2 + 2xy + 2zy,$

(iii) $x^2 + y^2 + z^2 + 3yz,$

(iv) $x^2 + y^2 - z^2 + 2xy + 2zx - 2yz.$

6. Reduce the quadratic forms to their normal forms

(i) $x^2 + 2y^2 + 2z^2 + 2xy + 2xz,$

(ii) $x^2 + 5y^2 + 2z^2 - 4xy - 6zy + 2xz,$

(iii) $x^2 + y^2 + z^2 + 3yz,$

(iv) $x^2 + y^2 - z^2 + 2xy + 2zx - 2yz,$

(v) $xy + yz + xz,$

(vi) $xy - 3yz + xz.$

7. Find the rank, index and signature of the following quadratic forms

(i) $x^2 + 2y^2 + 2z^2 + 2xy + 2xz,$

(ii) $x^2 + 5y^2 + 2z^2 - 4xy - 6zy + 2xz,$

(iii) $x^2 + y^2 + z^2 + 3yz,$

(iv) $x^2 + y^2 - z^2 + 2xy + 2zx - 2yz.$

Answers

1. (i) $\begin{pmatrix} 1 & -2 \\ -2 & 2 \end{pmatrix}$, (ii) $\begin{pmatrix} 1 & -2 & -2 \\ -2 & 0 & 0 \\ -2 & 0 & 3 \end{pmatrix}$, (iii) $\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 3 \end{pmatrix}$,

2. (i) $x^2 + y^2 + z^2 - 2xy + 4xz - 2yz,$

(ii) $x^2 + 3z^2 - 4xy - 2yz,$

(iii) $x^2 + y^2 + 2xy - 4xz - 2yz,$

3. (i) Positive definite, (ii) Positive definite,, (iii) Indefinite, (iv) Positive semi definite,(v) Positive definite,(vi) Positive definite, (vii) Indefinite,

6. (i) $x^2 + y^2$, (ii) $x^2 + y^2$, (iii) $x^2 + y^2 - z^2$, (iv) $x^2 + y^2 - z^2$, (v) $x^2 - y^2 - z^2$, (vi) $x^2 - y^2 + z^2$,

7. (i) 2,2,2, (ii) 2,2,2, (iii) 2,3,1, (iv) 2,3,1.