1

The Laplace Transform

1.1. Introduction. The knowledge of "Integral Transforms" is an essential part of mathematical background required by scientists and engineers. This is because the transform methods provide an easy and effective means for the solutions of many problems arising in science and engineering. For example, the Laplace transformation replaces a given function F (t) by another function $f(s)$. Then Laplace transformation converts an ordinary differential equation with some given initial conditions into an algebraic equation in terms of $f(s)$ Finally, using inverse Laplace transformation we recover the original function F(t). Thus, the method of Laplace transformation is especially useful for initial value problems, as it enables us to solve the problem without the trouble of finding the general solution first and then evaluating the arbitrary constants. The use of Laplace transforms provide a powerful technique of solving differential and integral equations.

1.2. Laplace transform. Definition. **[Rohilkhand 2000, Meerut 2010]**

Given a function F(t) defined for all real $t \geq 0$, the Laplace transform of F (t) is a function of a new variable s given by

$$L\{F(t), s\} = L\{F(t)\} = f(s) = \bar{F}(s) = \int_0^\infty e^{-st} F(t)\, dt \qquad \dots (1)$$

The *Laplace transform* of F(t) is said to exist if the improper integral (1) converges for some value of s, otherwise it does not exist.

Remark 1. Some authors use new variable p in place of s. Therefore, corresponding changes may be done in proofs of articles and solutions of problems. Similarly, some authors use variable x in place of t.

Remark 2. Some authors use $f(t)$ in the place of F(t) and simultaneously use F(s) in place of $f(s)$ while defining Laplace transform.

Remark 3. L is called *Laplace transformation operator.*

Remark 4. The operation of multiplying F (t) by e^{-st} and then integrating between the limits 0 to ∞ is known as *Laplace transformation.*

Remark 5. Improper integral. Write $\int_0^\infty e^{-st} F(t)\, dt = \lim_{v \to \infty} \int_0^v e^{-st} F(t)\, dt$, where v is a positive real number. If the integral from 0 to v exists for each $v > 0$ and if the limit as $v \to \infty$ exists, then the improper integral is said to converge to that limiting value otherwise the integral is said to diverge or fail to exist. Note that with regard to upper limit, the Laplace integral $\int_0^\infty e^{-st} F(t)\, dt$ is to be understand as an improper one.

1.3. Working rule to find Laplace transform of F (t), $t \geq 0$ by using definition 1.2.

Step 1. By definition, $$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

Step 2. By definition of improper integral, we have

$$\int_0^\infty e^{-st} F(t)\, dt = \lim_{v \to \infty} \int_0^v e^{-st} F(t)\, dt$$

Step 3. Simplify e^{-st} F (*t*) and evaluate $\int_0^v e^{-st} F(t)\,dt$.

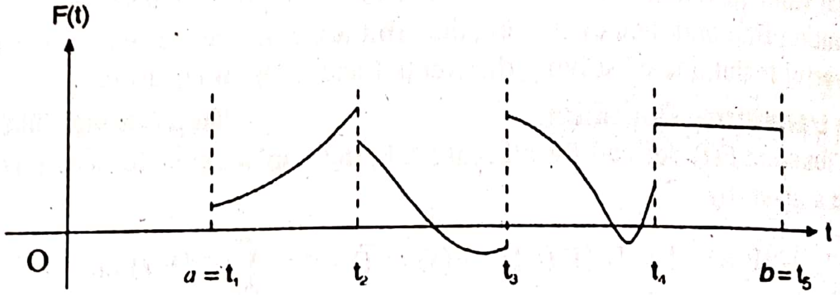
Step 4. Evaluate $\lim_{v\to\infty} \int_0^v e^{-st} F(t)\,dt$

Step 5. From steps 1 and 2, we have

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)dt = \lim_{v\to\infty} \int_0^v e^{-st} F(t)\,dt$$

1.4. Piecewise (or sectionally) continuous function. Definition.

A function F(*t*) is called piecewise continuous or sectionally continuous in the closed interval [*a*, *b*] if there exist a finite number of points $t_1, t_2, \ldots, t_n$ $(a = t_1 < t_2 < t_3 < \ldots < t_n = b)$ such that F(*t*) is continuous in each of open subinterval (t_i, t_{i+1}) and has finite right limit $F(t_i + 0)$ and finite left limit $F(t_{i+1} - 0)$. Clearly F (*t*) need not necessarily be defined at the end points of the subintervals.



An example of a function F(*t*) which is sectionally continuous is shown in the above figure by taking particular case *n* = 5. We note that F (*t*) is continuous in subinterval (t_2, t_3) and has finite right and left limits $F(t_2 + 0)$ and $F(t_3 - 0)$ respectively. We also note that F (*t*) has discontinuities at t_2, t_3 and t_4.

1.5. Functions of exponential order. Definition. [Kanpur 2002]

If there exist a positive real constant *m*, a number σ and a finite number *t* such that for all $t \geq t_0$,

$$\left|e^{-\sigma t} F(t)\right| < m \quad \text{or} \quad |F(t)| < m e^{\sigma t}$$

we say that F (*t*) is a function of exponential order σ as $t \to \infty$.

Equivalently, we also write $F(t) = O(e^{\sigma t})$, as $t \to \infty$.

1.5 A. Solved examples based on functions of exponential order

Ex.1. *Show that* $F(t) = t^2$ *is of exponential order* 3.

Sol. We have

$$\lim_{t\to\infty} e^{-\sigma t} F(t) = \lim_{t\to\infty} \frac{t^2}{e^{\sigma t}} = \lim_{t\to\infty} \frac{2t}{\sigma e^{\sigma t}}, \text{ by L' Hospital's rule}$$

$$= \lim_{t\to\infty} \frac{2}{\sigma^2 e^{\sigma t}}, \text{ by L' Hospital's rule again}$$

$$= 0, \text{ provided } \sigma > 0$$

Hence F (*t*) = t^2 is of exponential order. Again, $|t^2| = t^2 < e^{3t}$ for all $t > 0$

Therefore t^2 is a function of exponential order 3.

Ex.2. *Show that* $F(t) = t^n$ *is of exponential order as* $t \to \infty$, *n being any positive integer.* [Kanpur 2004]

Sol. Assuming that $\sigma > 0$, we have

$$\lim_{t\to\infty} e^{-\sigma t} F(t) = \lim_{t\to\infty} \frac{t^n}{e^{\sigma t}} = \lim_{t\to\infty} \frac{n\,t^{n-1}}{\sigma e^{\sigma t}}, \text{ by L'Hospital's rule} = \lim_{t\to\infty} \frac{n(n-1)t^{n-2}}{\sigma^2 e^{\sigma t}}, \text{ by L'Hospital's rule again}$$

$$= \lim_{t\to\infty} \frac{n(n-1)(n-2)\cdots 2\cdot 1\, t^{n-n}}{\sigma^n e^{\sigma t}}, \text{ after repeated use of L'Hospital's rule } (n-2) \text{ times more.}$$

$$= 0$$

Hence t^n is of exponential order σ, as $t \to \infty$.

Ex.3. *Show that the function* F $(t) = e^{t^2}$ *is not of exponential order as* $t \to \infty$.

Sol. We have, $\lim_{t\to\infty} e^{-\sigma t}\, F(t) = \lim_{t\to\infty} \left(e^{-\sigma t}\, e^{t^2}\right) = \lim_{t\to\infty} e^{t(t-\sigma)} = \infty$ for all values of σ.

Hence whatever be the value of σ, we cannot determine a real constant m such that

$$\left|e^{-\sigma t} F(t)\right| < m$$

Therefore, the given function is not of exponential order as $t \to \infty$.

1.6. Functions of class A. Definition.

A function F (t) is said to belong class A if F (t) is of some exponential order as $t \to \infty$ and is piecewise continuous over every finite interval of $t \geq 0$.

1.7. Sufficient conditions for the existence of Laplace transform [Kanpur 2014]

Theorem. *If F (t) is a function of class A, L {F (t)} exists.*

or If F (t) is of some exponential order as $t \to \infty$ *and is piecewise continuous over every finite interval of* $t \geq 0$, *then L {F (t)} exists.*

or If F(t) is a function which is piecewise continuous on every finite interval in the range $t \geq 0$ *and satisfies* $|F(t)| \leq me^{\sigma t}$ *for all* $t \geq 0$ *and for constants* σ *and* m, *then the Laplace transform of F(t) exists.* **[Osmania 2010; KU Kurukshetra 2004; Punjab 2005]**

Proof. Since $F(t)$ is of exponential order, say σ, we can find constants σ, m (> 0) and $t_0 (> 0)$ such that

$$|\,\mathrm{F}(t)\,| < m\, e^{\sigma t} \quad \text{for} \quad t \geq t_0. \qquad \ldots (1)$$

Now, $$L\,\{\mathrm{F}(t)\} = \int_0^\infty e^{-st}\,\mathrm{F}(t)\,dt = \int_0^{t_0} e^{-st}\,\mathrm{F}(t)\,dt + \int_{t_0}^\infty e^{-st}\,\mathrm{F}(t)\,dt = \mathrm{I}_1 + \mathrm{I}_2, \text{ say} \qquad \ldots (2)$$

Since F (t) is piecewise continuous on every finite interval $0 \leq t \leq t_0$, I_1 exists.

Again, $$|\,\mathrm{I}_2\,| = \left|\int_{t_0}^\infty e^{-st}\,\mathrm{F}(t)\,dt\right| \leq \int_{t_0}^\infty e^{-st}\,|\,\mathrm{F}(t)\,|\,dt < m\int_{t_0}^\infty e^{-st} e^{\sigma t}\,dt, \text{ using (1)}$$

Thus, $$|\,\mathrm{I}_2\,| < m\left[-\frac{e^{-(s-\sigma)t}}{(s-\sigma)}\right]_{t_0}^\infty = \frac{m\, e^{-(s-\sigma)t_0}}{s-\sigma}. \qquad \ldots (3)$$

Now, when $s > \sigma$, then $e^{-(s-\sigma)t_0} \to 0$ as $t \to \infty$. Hence (3) shows that $|\mathrm{I}_2|$ is finite for all $t_0 > 0$ when $s > \sigma$ and hence I_2 is also convergent. Then from (2), it follows that $L\{\mathrm{F}(t)\}$ exists for all $s > \sigma$.

Remark: The conditions stated in the above theorem are sufficient to ensure the existence of L {F (t)}. These are not necessary conditions for the existence of L{F (t)}. In other words, if the conditions of the above theorem are not satisfied, $L\{F(t)\}$ may or may not exist as shown in the following example: Consider the function $F(t) = 1/\sqrt{t}$.

As $t \to 0$, $F(t) \to \infty$. Hence $F(t)$ is not piecewise continuous on every finite interval in the range $t \geq 0$. Now, by definition, we have

$$L\{1/\sqrt{t}\} = \int_0^{\infty} e^{-st}(1/\sqrt{t})\,dt = \frac{2}{\sqrt{s}}\int_0^{\infty} e^{-x^2}dx, \text{ putting } \sqrt{st} = x \text{ and } dt/\sqrt{t} = 2dx/\sqrt{s},\ s>0$$

$$= \frac{2}{\sqrt{s}} \times \frac{\sqrt{\pi}}{2} = \left(\frac{\pi}{s}\right)^{1/2},\ s>0. \qquad \text{as} \qquad \int_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

Hence $L\{1/\sqrt{t}\}$ exists for $s > 0$ even if $1/\sqrt{t}$ is not piecewise continuous in the range $t \geq 0$.

1.8. Linearity property of Laplace transforms. If c_1 and c_2 be constants, then

$$L\{c_1 F_1(t) + c_2 F_2(t)\} = c_1 L\{F_1(t)\} + c_2 L\{F_2(t)\}.$$

Proof. By definition, we have

$$L\{c_1 F_1(t) + c_2 F_2(t)\} = \int_0^{\infty} e^{-st}\{c_1F_1(t)+c_2F_2(t)\}dt = c_1\int_0^{\infty} e^{-st}F_1(t)\,dt + c_2\int_0^{\infty} e^{-st}F_2(t)\,dt$$

$$= c_1 L\{F_1(t)\} + c_2 L\{F_2(t)\}, \text{ by definition.}$$

1.9. Laplace transforms of some elementary functions

(i) *To find Laplace transform of $F(t) = 1$.*

[Kanpur 2003, Meerut 2004, M.S. Univ. T.N. 2007]

Sol. By definition of Laplace transform (see Art. 1.3), we have

$$L\{1\} = \int_0^{\infty} e^{-st}(1)\,dt = \lim_{v\to\infty}\int_0^{v} e^{-st}dt = \lim_{v\to\infty}\left[\frac{e^{-st}}{-s}\right]_0^{v}$$

$$= -\frac{1}{s}\lim_{v\to\infty}\left(\frac{1}{e^{sv}}-1\right) = -\frac{1}{s}(0-1) = \frac{1}{s}, \text{ provided } s>0$$

(ii) *To find Laplace transform of the function $F(t) = t^n$, n being any real number greater than -1.*

[Gorakhpur 2008; Ranchi 2010; Purvanchal 2006]

Sol. By definition, $\qquad L\{F(t)\} = \int_0^{\infty} e^{-st}F(t)\,dt.$

$$\therefore \qquad L\{t^n\} = \int_0^{\infty} e^{-st}t^n dt = \int_0^{\infty} e^{-st}t^{(n+1)-1}dt. \qquad \text{...(i)}$$

From the properties of 'Gamma function', we know that

$$\int_0^{\infty} e^{-ax}x^{m-1}dx = \frac{\Gamma(m)}{a^m}, \text{ if } a>0 \text{ and } m>0. \qquad \text{...(ii)}$$

Replacing a by s, m by $(n+1)$ and x by t in (ii), we have

$$\int_0^{\infty} e^{-st}t^{(n+1)-1}dt = \frac{\Gamma(n+1)}{s^{n+1}}, \text{ if } s>0 \text{ and } n+1>0$$

$$\therefore \quad (i) \Rightarrow \qquad \Gamma\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}, s>0 \text{ and } n>-1.$$

1.10. Table of Laplace transforms of some elementary functions. Commit to memory for direct applications. These have been evaluated in Art 1.9.

S.No.	F(t)	L {F(t)} or f(s)	
1	1	$1/s$,	$s > 0$
2	t^n, $n > -1$	$\Gamma(n+1)/s^{n+1}$,	$s > 0$
3	t^n, (n is a positive integer)	$n!/s^{n+1}$	$s > 0$
4	e^{at}	$1/(s-a)$,	$s > a$
5	$\sinh at$	$a/(s^2-a^2)$,	$s > \lvert a \rvert$
6	$\cosh at$	$s/(s^2-a^2)$,	$s > \lvert a \rvert$
7	$\sin at$	$a/(s^2+a^2)$,	$s > 0$
8	$\cos at$	$s/(s^2+a^2)$,	$s > 0$

1.10 A. Solved examples based on Art.1.8 and Art. 1.9

Ex. 1. Find the Laplace transform of the function $F(t) = (e^{at} - 1)/a$.

[Purvanchal 2003; Meerut 1994]

Sol. $L\{F(t)\} = L\{(1/a) \times (e^{at} - 1)\} = (1/a) \times L\{e^{at} - 1\} = (1/a) \times [L\{e^{at}\} - L\{1\}]$

$$= \frac{1}{a}\left(\frac{1}{s-a} - \frac{1}{s}\right) = \frac{1}{s(s-a)}, \text{ if } s > a \text{ and } s > 0.$$

Ex.2. *Find the Laplace transform of* $F(t) = (\sin t - \cos t)^2$.

[Gorakhpur 2010; Vikram 2004]

Sol. $L\{F(t)\} = L\{(\sin t - \cos t)^2\} = L\{\sin^2 t + \cos^2 t - 2 \sin t \cos t\}$

$$= L\{1 - \sin 2t\} = L\{1\} - L\{\sin 2t\} = \frac{1}{s} - \frac{2}{s^2+2^2}, \text{if } s > 0$$

$$= (s^2 - 2s + 4)/s(s^2+4), \text{ if } s > 0.$$

Ex.3. *Find the Laplace transform of* (i) $3e^{2t} + 4e^{-3t}$ (ii) $(t^2 + 1)^2$

Sol. (i) $L\{3e^{2t} + 4e^{-3t}\} = 3L\{e^{2t}\} + 4L\{e^{-3t}\} = 3 \times \frac{1}{s-2} + 4 \times \frac{1}{s+3}$, $s > 2$ and $s > -3$

$$= (7s+1)/(s^2+s-6), \text{ if } s > 2$$

(ii) $L\{(t^2+1)^2\} = L\{t^4 + 2t^2 + 1\} = L\{t^4\} + 2L\{t^2\} + L\{1\} = \frac{4!}{s^5} + 2 \times \frac{2}{s^3} + \frac{1}{s}$, as $L\{t^n\} = \frac{n!}{s^{n+1}}$

$$= \frac{24}{s^5} + \frac{4}{s^3} + \frac{1}{s} = \frac{24 + 4s^2 + s^4}{s^5}, \text{ provided } s > 0.$$

Ex.4. *Find* (i) $L\{3 \sin 2t + 5 \cos 2t\}$ **(Meerut 2013)** (ii) $L\{\sin 5t + \cos 3t\}$ **[Gulberga 2005]**
(iii) $L\{3 \cosh 5t - 4 \sinh 5t\}$ (iv) $\sin 2t \cos 3t$

Sol. (i) $L\{3 \sin 2t + 5 \cos 2t\} = 3L\{\sin 2t\} + 5L\{\cos 2t\}$

$$= 3 \times \frac{2}{s^2+2^2} + 5 \times \frac{s}{s^2+2^2}, s > 0 = \frac{6+5s}{s^2+4}, s > 0$$

(ii) $L\{\sin 5t + \cos 2t\} = L\{\sin 5t\} + L\{\cos 2t\} = 5/(s^2+25) + s/(s^2+9), s > 0$

(iii) $L\{3 \cosh 5t - 4 \sinh 5t\} = 3L\{\cosh 5t\} - 4L\{\sinh 5t\}$

$$= 3 \times \frac{s}{s^2-5^2} - 4 \times \frac{5}{s^2-5^2}, s > 5 = \frac{3s-20}{s^2-25}, s > 5$$

EXERCISE 1(*A*)

1. *Find the Laplace transform of the following functions:*

(*i*) e^{4t} **[Bangalore 2005]** (*ii*) cosh $3t$ **[Kuvempa 2005]**

(*iii*) t^7 (*iv*) $t^{2/3}$ **[Purvanchal 2004]** (*v*) $t^{3/2}$ (*vi*) $t^{5/3}$

Ans. (*i*) $1/(s-4),\ s>4$ (*ii*) $s/(s^2-9),\ s>|3|$ (*iii*) $(5040)/s^8,\ s>0$

(*iv*) $\dfrac{2\Gamma(2/3)}{3s^{5/3}},\ s>0$ (*v*) $\dfrac{3\sqrt{\pi}}{4s^{5/2}},\ s>0$ (*vi*) $\dfrac{10\Gamma(2/3)}{9s^{8/3}},\ s>0$

2. *Find the Laplace transform of the following functions*

(*i*) $F(t)=\begin{cases} e^t, & 0<t\le 1 \\ 0, & t>1 \end{cases}$ (*ii*) $F(t)=\begin{cases} t, & 0<t<4 \\ 5, & t>4 \end{cases}$ **[M.S. Univ. T.N. 2007]**

(*iii*) $F(t)=\begin{cases} \sin 3t, & 0<t<\pi \\ 0, & t>\pi \end{cases}$ **[Bangalore 2005]** (*iv*) $F(t)=\begin{cases} -1, & 0\le t\le 4 \\ 1, & t\ge 4 \end{cases}$

Ans. (*i*) $\{1-e^{-(s-1)}\}/(s-1),\ s\neq 1$ (*ii*) $\{1+(s-1)e^{-4s}\}/s^2,\ s>0$

(*iii*) $(e^{1-s}-1+s)/(s-1),\ s\neq 1$ (*iv*) $(2e^{-4s}-1)/s,\ s>0$

3. *Evaluate the following:*

(*i*) $L\{\sin 2t \sin 3t\}$ **[Kanpur 2012]** (*ii*) $L\{2 \sin t \sin 3t)$ **[Gulberga 2005]**

(*iii*) $L\{t-\sinh 2t\}$ **[CDLU 2004]** (*iv*) $L\{\sin^2 4t\}$ **[Bangalore 2004]**

(*v*) $L\{\sin^2 2t\}$ **[Karnataka 2005]** (*vi*) $\cosh^2 2t$

(*vii*) $3t^3+4t^2-5t+7$ (*viii*) $4e^{-3t}-2\sin 5t+3\cos 2t-2t^3+3t^4$

Ans. (*i*) $(12s)/(s^2+1)(s^2+25),\ s>5$ (*ii*) $(6s)/(s^2+4)(s^2+16),\ s>4$

(*iii*) $(s^2+4)/s^2(4-s^2),\ s>2$ (*iv*) $(s^2+32)/s(s^2-64),\ s>0$

(*v*) $(s^2+8)/s(s^2+16),\ s>0$ (*vi*) $(s^2-8)/s(s^2-16),\ s>4$

(*vii*) $18/s^4+8/s^3-5/s^2+7/s,\ s>0$

(*viii*) $4/(s+3)-10/(s^2+25)+3s/(s^2+4)-12/s^4+72/s^5,\ s>0$

4. *Find the Laplace transform of the following functions:*

(*i*) $\cos^3 t$ (*ii*) $\cos(at+b)$ (*iii*) $\cos^3 4t+t$

(*iv*) $1+\sqrt{t}+3/\sqrt{t},\ t>0$ (*v*) $\sin(at+b),\ t\ge 0$ (*vi*) $\sin at \sin bt$

Ans. (*i*) $\dfrac{s(s^2+7)}{(s^2+9)(s^2+1)},\ s>0$ (*ii*) $\dfrac{s\cos b-a\cos b}{s^2+a^2},\ s>0$

(*iii*) $\dfrac{s(s^2+112)}{(s^2+16)(s^2+144)}+\dfrac{1}{s^2},\ s>0$ (*iv*) $\dfrac{1}{s}+\dfrac{\sqrt{\pi}}{2s^{3/2}}+\dfrac{3\sqrt{\pi}}{s^{1/2}},\ s>0$

(*v*) $\dfrac{a\cos b+s\sin b}{s^2+a^2},\ s>0$ (*vi*) $\dfrac{2abs}{\{s^2+(a-b)^2\}\{s^2+(a+b)^2\}},\ s>0$

1.11. First shifting (or first translation) theorem. [Agra 2012]

If $L\{F(t)\}=f(s)$, *then* $L\{e^{at}F(t)\}=f(s-a)$, *where a is any real or complex constant*

Or *If f (s) is the Laplace transform of* $F(t)$, *then* $f(s-a)$ *is the Laplace transform of* $e^{at}F(t)$, *where a is any real or complex number.* **[Osmania 2004, Nagpur 2010, Purvanchal 2002; Gorakhpur 2008, 11]**

$$L\{F(t)\} = f(s)$$

Sol. (*a*) Given that

We have, L {(sinh *at*) F(*t*)} = L{(1/2) × ($e^{at} - e^{-at}$) F (*t*)}
= (1/2) × L{e^{at} F(*t*)} – (1/2) × L {e^{-at} F(*t*)} ...(1)
= (1/2) × {*f* (*s* – *a*) – *f* (*s* + *a*)} by first shifting theorem and (1)

Now, L{sin 3*t*} = 3/($s^2 + 3^2$) = 3/(s^2 + 9), *s* > 0

Using the result just proved by taking *a* = 2 and *f* (*s*) = 3/(s^2 + 9), we have

$$L\{\sinh 2t \sin 3t\} = \frac{1}{2}\left[\frac{3}{(s-2)^2+9} + \frac{3}{(s+2)^2+9}\right], s-2>0, \text{ and } s+2>0$$

$$= \frac{3}{2}\left[\frac{1}{s^2-4s+13} - \frac{1}{s^2+4s+13}\right] = \frac{12s}{s^4+10s^2+169}, s>2$$

(*b*) Proceed as in part (*a*) **Ans.** ($s^3 - 5s$)/($s^4 - 10s^2 + 169$), *s* > 1

Ex.11. *Prove that (i)* L {*sinh at cos at*} = *a* ($s^2 - 2a^2$)/(s^4 + 4 a^4). **[Meerut 2010]**

(*ii*) L {sinh *at* sin *at*} = ($2a^2$ *s*)/($s^4 + 4a^4$). **[Meerut 2010]**

Sol. We have L {sinh *at*} = *a*/($s^2 - a^2$) = *f*(*s*), say ...(1)

∴ L{e^{iat} sinh *at*} = *f* (*s* – *ia*), by first shifting theorem

$$= \frac{a}{(s-ia)^2-a^2}, \text{ using (1)}$$

$$= \frac{a}{(s^2-2a^2)-2\,ias} \times \frac{(s^2-2a^2)+2ias}{(s^2-2a^2)+2ias} = \frac{a(s^2-2a^2)+2ia^2s}{(s^2-2a^2)^2+4a^2s^2}.$$

$$\therefore L\{\sinh at(\cos at + i\sin at)\} = \frac{a(s^2-2a^2)+2i\,a^2s}{s^4+4a^4}, \quad \text{as } e^{i\theta} = \cos\theta + i\sin\theta$$

$$L\{\sinh at \cos at\} + i\,L\{\sinh at \sin at\} = \frac{a(s^2-2a^2)}{s^4+4a^4} + i\frac{2a^2s}{s^4+4a^4}$$

Equating real and imaginary parts, we get

$$L\{\sinh at \cos at\} = \frac{a(s^2-2a^2)}{s^4+4a^4} \quad \text{and} \quad L\{\sinh at \sin at\} = \frac{2a^2s}{s^4+4a^4}.$$

EXERCISE 1 (*B*)

Find the Laplace transform of the following functions.

1. $t^5 e^{3t}$ **[Kanpur 2004]**
2. e^{-2t} sin 4*t* **(Ravishankar 2010)**
3. e^{3t} cos 5*t*.
4. e^{-t} (3 sin 2*t* – 5 cosh 2*t*)
5. e^{-2t} (3 cos 6*t* – 5 sin 6*t*)
6. e^t {cos 2*t* + (1/2) × sin 2*t*} **[Kerala 2001]**
7. e^{2t} (t^2 + 3*t* + 4) **[Bangalore 2005]**
8. e^{-3t} sin 4*t* **[Gulberga 2005]**
9. e^{-3t} {3cos 6*t* – 5 sin 6*t*} **[Meerut 2009]**
10. e^{2t} {3sin 4*t* – 4 cos 4*t*} **[Punjab 2005]**
11. sinh *t* cos *t*.
12. e^{-4t} $t^{3/2}$
13. sinh 3*t* $\cos^2 t$ **[Kurukshetra 2004]**

14. Prove that $L\{(at^2+bt+c)e^{kt}\} = \dfrac{2a}{(s-k)^3} + \dfrac{b}{(s-k)^2} + \dfrac{c}{s-k}, s>k$ **[Bangalore 2007]**

ANSWERS

1. $120/(s-3)^6$ 2. $4/(s^2+4s+20)$ 3. $(s-3)/(s^2-6s+34)$.

4. $\dfrac{6}{s^2+2s+5}-\dfrac{5(s+1)}{s^2+2s-3}$ 5. $\dfrac{3(s-8)}{s^2+4s+40}$ 6. $\dfrac{s}{s^2-2s+5}$

7. $(4s^2-13s+12)/(s-2)^3$ 8. $4/(s^2+6s-7)$ 9. $(3s-16)/(s^2+6s+45)$

10. $(20-4s)/(s^2-4s+20)$ 11. $(s^2-2)/(s^2+4)$, $s>1$

12. $\dfrac{3\sqrt{\pi}}{4(s+4)^{5/2}}$, $s>-4$ 13. $\dfrac{1}{2}\left[\dfrac{s^2-6s+11}{(s-3)(s^2-6s+13)}+\dfrac{s^2+6s+11}{(s+3)(s^2+6s+13)}\right]$, $s>3$

1.12. The unit step function or Heaviside's unit function. Definition.

The unit step function is denoted and defined by

$$u_a(t)=\mathrm{H}(t-a)=\begin{cases}0, & \text{if } t<a\\ 1, & \text{if } t\ge a\end{cases} \qquad \ldots(1)$$

Note. $\mathrm{L}\{\mathrm{H}(t-a)\}=\int_0^{\infty}e^{-st}H(t-a)\,dt=\int_0^{a}e^{-st}H(t-a)dt+\int_a^{\infty}e^{-st}H(t-a)dt$, using (1).

$$=\int_a^{\infty}e^{-st}\cdot 1\,dt=\frac{e^{-as}}{s}.$$

1.13. Second shifting theorem or second translation theorem. [Avadh 2007; Lucknow 2009; Rohilkhand 2007; Jabalpur 2004; Kanpur 2006, Agra 2005

If $\mathrm{L}\{\mathrm{F}(t)\}=f(s)$ and $\mathrm{G}(t)=\begin{cases}F(t-a), & t>a\\ 0, & t<a\end{cases}$

then $\mathrm{L}\{\mathrm{G}(t)\}=e^{-as}f(s)$.

Proof. By definition of Laplace transform, we have

$$\mathrm{L}\{\mathrm{G}(t)\}=\int_0^{\infty}e^{-st}G(t)\,dt=\int_0^{a}e^{-st}G(t)\,dt+\int_a^{\infty}e^{-st}G(t)\,dt, \text{ where } 0<a<\infty$$

$$=\int_0^{a}e^{-st}\cdot 0\,dt+\int_a^{\infty}e^{-st}F(t-a)\,dt, \text{ putting the given values of G }(t)$$

$$=0+\int_a^{\infty}e^{-st}F(t-a)\,dt=\int_0^{\infty}e^{-s(a+u)}F(u)\,du, \text{ [Putting } t-a=u \text{ and } dt=du.$$

Also note that when $t=a$, $u=0$ and when $t=\infty$, $u=\infty$]

$$=e^{-sa}\int_0^{\infty}e^{-su}F(u)\,du=e^{-sa}\int_0^{\infty}e^{-st}F(t)dt, \text{ using a property of definite integral}$$

$=e^{-sa}\,\mathrm{L}\{\mathrm{F}(t)\}$, by definition of Laplace transform

$=e^{-sa}f(s)$, using given result $\mathrm{L}\{\mathrm{F}(t)\}=f(s)$

Another form of second shifting theorem or second translation theorem

If $\mathrm{L}\{\mathrm{F}(t)\}=f(s)$ and $a>0$, then $\mathrm{L}\{\mathrm{F}(t-a)\,\mathrm{H}(t-a)\}=e^{-as}f(s)$,

where $\mathrm{H}(t-a)=\begin{cases}0, & \text{if } t<a\\ 1, & \text{if } t>a.\end{cases}$

Proof. By definition, $\mathrm{L}\{\mathrm{F}(t-a)\,\mathrm{H}(t-a)\}=\int_0^{\infty}e^{-st}F(t-a)\,H(t-a)dt$

$$L\{G(t)\} = \int_0^\infty e^{-st} G(t)\, dt = \int_0^{\pi/3} e^{-st} G(t)\, dt + \int_{\pi/3}^\infty e^{-st} G(t)\, dt$$

$$= \int_0^{\pi/3} e^{-st} \cdot 0\, dt + \int_{\pi/3}^\infty e^{-st} \sin\left(t - \frac{\pi}{3}\right) dt,$$ using given values of G (t)

$$= 0 + \int_{\pi/3}^\infty e^{-st} \sin\left(t - \frac{\pi}{3}\right) dt = \int_0^\infty e^{-s(u+\pi/3)} \sin u\, du$$ [putting $t - (\pi/3) = u$ and $dt = du$. Also note that when $t = \pi/3$, $u = 0$ and when $t = \infty$, $u = \infty$]

$$= e^{-s\pi/3} \int_0^\infty e^{-su} \sin u\, du = e^{-s\pi/3} \int_0^\infty e^{-st} \sin t\, dt$$

$$= e^{-s\pi/3} L\{\sin t\},$$ by definition of Laplace transform

Thus, $$L\{G(t)\} = e^{-s\pi/3} \times \{1/(s^2+1)\}, s > 0$$

EXERCISE 2(C)

Ex.1. *Find L {G (t)}, where* $G(t) = \begin{cases} \sin(t - 2\pi/3), & t > 2\pi/3 \\ 0, & t < 2\pi/3. \end{cases}$ **[Osmania 2004]**

Sol. Proceed as in Ex.2. Here F (t) = sin t and $a = 2\pi/3$. **Ans.** $e^{-2\pi s/3} \times \{1/(s^2+1)\}, s > 0$.

Ex. 2. *Find L{G (t)}, where* $G(t) = \begin{cases} \cos(t - 2\pi/3), & t > 2\pi/3 \\ 0, & t < 2\pi/3. \end{cases}$ **Ans.** $\dfrac{se^{-2\pi s/3}}{s^2+1}, s > 0$

[Meerut 2008]

Ex. 3. *Find L* $\{e^{t-1} H(t-1)\}$. **[Purvanchal 2004]**

[Sol. Let F$(t) = e^t$. Then, $L\{F(t)\} = 1/(s-1) = f(s)$, say.

Using second shifting theorem, we have

$$L\{e^{t-1} H(t-1)\} = e^{-s} f(s) = e^{-s}/(s-1)$$

$$[\because L\{F(t)\} = f(s) \Rightarrow L\{F(t-1) H(t-a)\} = e^{-as} f(s)]$$

Ex. 4. *Find the Laplace transform of the following functions:*

(*i*) $(t-1) H(t-1)$ (*ii*) $\cos t\ H(t-\pi)$

Ans. (*i*) $e^{-s}/s^2, s > 0$ (*ii*) $-(se^{-\pi s})/(s^2+1), s > 0$

1.14. Change of scale property.

[Avadh 2006; Rohilkhand 2008, 11, Avadh 2006; Agra 2007; Nagpur 2005, Osmania 2004, Lucknow 2007, Kanpur 2007, 10]

If $L\{F(t)\} = f(s)$, then $L\{F(at)\} = (1/a) \times f(s/a)$.

Proof. By definition, $$\int_0^\infty e^{-st} F(t)\, dt = L\{F(t)\} = f(s) \quad \text{... (1)}$$

Now, $$L\{F(at)\} = \int_0^\infty e^{-st} F(at)\, dt = \frac{1}{a}\int_0^\infty e^{-su/a} F(u)\, du,$$ putting $at = u$ and $a\, dt = du$

$$= \frac{1}{a}\int_0^\infty e^{-(s/a)t} F(t)\, dt = \frac{1}{a} f\left(\frac{s}{a}\right),$$ using (1)

1.14 A SOLVED EXAMPLES BASED ON ART. 1.14.

Ex.1. *If* $L\{F(t)\} = (s^2 - s + 1)/(2s+1)^2 (s-1)$ *prove that*

$$L\{F(2t)\} = (s^2 - 2s + 4)/4(s+1)^2(s-2).$$

Remark 3. If F (t) is discontinuous at $t = a_1, a_2, \ldots, a_n$, we break up the integral into ... from 0 to ∞ as shown below:

$$L\{F'(t)\} = \int_0^{a_1} e^{-st} F'(t)\,dt + \int_{a_1}^{a_2} e^{-st} F'(t)\,dt + \ldots + \int_{a_n}^{\infty} e^{-st} F'(t)\,dt$$

and then proceed as explained in remark 2.

Theorem II. *Let F (t) and F′ (t) be continuous functions for all t ≥ 0 and F″ be of exponential order σ as t → ∞ and if F″ (t) is of class A, then Laplace transform of F″ (t) exists when s > σ, and is given by* L {F″ (t)} = s^2 L {F (t)} – s F (0) – F′ (0). [Madras 20..]

Proof. Let G (t) = F′ (t)

so that G′ (t) = F″ (t) and G (0) = F′ (0).

Applying theorem I to function G (t), we get

L {G′ (t)} = s L {G (t)} – G (0) or L {F″ (t)} = s L {F′ (t)} – F′ (0), using (1) and (2).

But by theorem I, L {F′ (t)} = s L {F (t)} – F (0).

Using (4), (3) reduces to L {F″ (t)} = s [s L {F (t)} – F (0)] – F′ (0)

or L {F″ (t)} = s^2 L {F (t)} – s F (0) – F′ (0).

Theorem III. *Let F (t), F′ (t) and be F″ (t) be continuous for all t ≥ 0 and be of exponential order σ as t → ∞ and if F‴ (t) is of class A then Laplace transform of F‴ (t) exists when s > ... and is given by*

$$L\{F'''(t)\} = s^3 L\{F(t)\} - s^2 F(0) - s F'(0) - F''(0).$$ [Mysore 20..]

Proof Let G (t) = F″ (t)

so that G′ (t) = F‴ (t) and G (0) = F″ (0).

Applying theorem I to function G (t), we get L {G′ (t)} = s L {G (t)} – G...

or L { F‴ (t)} = sL {F″(t)}– F″ (0), using (1) and (2)

But, from theorem II, we have L {F″ (t)} = s^2 L {F (t)} – s F (0) – F′ (0)

Using (4), (3) reduces to L {F‴(t)} = s [s^2 L {F (t)} – s F (0) – F′ (0)] – F...

or L { F‴ (t)} = s^3 L {F (t)} – s^2 F (0) – s F′ (0) – F″ (0).

Theorem IV. Laplace transform of the n^th order derivative. General case.

Let F (t) and its derivative F′ (t), F″ (t) . . . . . ., $F^{(n-1)}$ (t) be continuous for all t ≥ 0 and be of exponential order σ as t → ∞ and if $F^{(n)}$ (t) is of class A, then Laplace transform of $F^{(n)}$ exists when s > σ, and is given by

$$L\{F^{n}(t)\} = s^n L\{F(t)\} - s^{n-1} F(0) - s^{n-2} F'(0) - \ldots - F^{(n-1)}(0),$$

where $F^{(n)}(t) = d^n F(t)/dt^n$ etc.

Proof. We shall use the principle of mathematical induction to prove

$$L\{F^{(n)}(t)\} = s^n L\{F(t)\} - s^{n-1} F(0) - s^{n-2} F'(0) - \ldots - F^{(n-1)}(0) \quad (1)$$

For $n = 1$, (1) reduces to L {F′ (t)} = s L {F (t)} – F (0), which is true by theorem ...

Hence (1) is true for $n = 1$.

Assume that the result (1) is true for $n = k$, so that

$$L\{F^{(k)}(t)\} = s^k L\{F(t)\} - s^{k-1} F(0) - s^{k-2} F'(0) - \ldots - F^{(k-1)}(0)$$

We shall now prove the result (1) for $n = k + 1$ *i.e.*, we shall prove that

$$L\{F^{(k+1)}(t)\} = s^{k+1} L\{F(t)\} - s^k F(0) - s^{k-1} F'(0) - \ldots - F^{(k)}(0)$$

Let G (t) = $F^{(k)}(t)$

so that G′ (t) = $F^{(k+1)}(t)$ and G (0) = $F^{(k)}(0)$

Applying theorem I to function G (t), we get

$$L\{G'(t)\} = s\,L\{G(t)\} - G(0)$$

or $L\{F^{(k+1)}(t)\} = s\,L\{F^{(k)}(t)\} - F^{(k)}(0)$, using (4) and (5)

$$= s\,[s^k L\{F(t)\} - s^{k-1} F(0) - s^{k-2} F'(0) - \ldots - F^{(k-1)}(0)] - F^{(k)}(0).$$

$$\therefore L\{F^{(k+1)}(t)\} = s^{k+1} L\{F(t)\} - s^k F(0) - s^{k-1} F'(0) - \ldots - F^{(k)}(0).$$

This shows that the required result (1) is true for $n = k + 1$ whenever it is true for $n = k$. Hence, by the principle of mathematical induction, the required result (1) is true for all positive integers.

1.15A. SOLVED EXAMPLES BASED ON ART. 1.15

Ex.1. *If* $L\left\{2\sqrt{\left(\frac{t}{\pi}\right)}\right\} = \frac{1}{s^{3/2}}$, *show that* $\frac{1}{s^{1/2}} = L\left\{\frac{1}{\sqrt{(\pi t)}}\right\}$. **[Meerut 2000]**

Sol. Let $F(t) = 2\sqrt{(t/\pi)}$. Then $F(0) = 2\sqrt{(0/\pi)} = 0$. ... (1)

Again, $$F'(t) = \frac{d}{dt}\left[\frac{2}{\sqrt{\pi}} t^{1/2}\right] = \frac{2}{\sqrt{\pi}} \times \frac{1}{2} t^{-1/2} = \frac{1}{\sqrt{(\pi t)}}. \quad \text{... (2)}$$

Now, we know that (refer theorem I, Art. 1.15) $L\{F'(t)\} = s\,L\{F(t)\} - F(0)$. ... (3)

Substituting values of $F'(t)$, $F(t)$ and $F(0)$ in (3), we get

$$L\left\{\frac{1}{\sqrt{(\pi t)}}\right\} = s\,L\left\{2\sqrt{\left(\frac{t}{\pi}\right)}\right\} - 0 = s \times \frac{1}{s^{3/2}} = \frac{1}{s^{1/2}},$$ as given that $L\left\{2\sqrt{\left(\frac{t}{\pi}\right)}\right\} = \frac{1}{s^{3/2}}$

Ex.2. *If* $L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}$, *show that* $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\left(\frac{\pi}{s}\right)} e^{-1/4s}$. **[Meerut 2001, 05]**

Sol. Let $F(t) = \sin\sqrt{t}$ so that $F(0) = 0$. ... (1)

Again, $$F'(t) = \frac{d}{dt} \sin t^{1/2} = \cos t^{1/2} \times \frac{1}{2} t^{-1/2} = \frac{\cos\sqrt{t}}{2\sqrt{t}} \quad \text{... (2)}$$

Now, we know that $L\{F'(t)\} = s\,L\{F(t)\} - F(0)$. ... (3)

Substituting values of $F'(t)$, $F(t)$ and $F(0)$ in (3), we get

$$L\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = s\,L\{\sin\sqrt{t}\} - 0 = s \times \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s},$$ as given that $L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}$

or $$\frac{1}{2} L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}}{2s^{1/2}} e^{-1/4s}$$ or $$L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\left(\frac{\pi}{s}\right)} e^{-1/4s}.$$

Proof. By definition, $f(s)=\int_0^\infty e^{-st}F(t)\,dt.$ ...(1)

Since F (t) belongs to class A, Leibnitz's rule *for differentiation under the sign of integral is justified and so from (1), we get

$$\frac{d}{ds}f(s)=\int_0^\infty \frac{d}{ds}\left\{e^{-st}F(t)\right\}dt=\int_0^\infty (-t)e^{-st}F(t)\,dt=-\int_0^\infty e^{-st}\{tF(t)\}\,dt.$$

$\therefore$ $f'(s)=-\text{L}\{t\,\text{F}(t)\}$ or L $\{t\,\text{F}(t)\}=-f'(s).$

Theorem II: (General case). *If F (t) is a function of class A and if* L $\{\text{F}(t)\}=f(s)$,

then $\text{L}\{t^n\text{F}(t)\}=(-1)^n\dfrac{d^nf(s)}{ds^n}$, $n=1,2,3,\ldots.$ **[Agra 2011]**

[Gorakhpur 2011; Meerut 2002; Kanpur 2008; Lucknow 2009, 11; Nagpur 2005]

Proof. We use the principle of mathematical induction to prove that

$$\text{L}\{t^n\,\text{F}(t)\}=(-1)^n\frac{d^nf(s)}{ds^n}. \quad ...(1)$$

For $n=1$, (1) reduces to $\quad \text{L}\{t\,\text{F}(t)\}=-\dfrac{d}{ds}f(s)=-f'(s),$

which is true by theorem I. Hence (1) is true for $n=1$

Assume that the result (1) is true for $n=k$ so that

$$\text{L}\{t^k\,\text{F}(t)\}=(-1)^k\frac{d^kf(s)}{ds^k}. \quad ...(2)$$

or

$$\int_0^\infty e^{-st}\{t^kF(t)\}dt=(-1)^k\frac{d^kf(s)}{ds^k}. \quad ...(3)$$

Differentiating both sides of (3) w.r.t. 's' and applying the Leibnitz's rule* for differentiation under the integral sign, we get

$$\int_0^\infty \frac{\partial}{\partial s}\{e^{-st}t^kF(t)\}dt=(-1)^k\frac{d^{k+1}f(s)}{ds^{k+1}} \quad \text{or} \quad \int_0^\infty (-t)e^{-st}t^kF(t)\,dt=(-1)^k\frac{d^{k+1}f(s)}{ds^{k+1}}$$

or

$$\int_0^\infty e^{-st}\{t^{k+1}F(t)\}\,dt=(-1)^{k+1}\frac{d^{k+1}f(s)}{ds^{k+1}}$$

or

$$\text{L}\{t^{k+1}\,\text{F}(t)\}=(-1)^{k+1}\frac{d^{k+1}f(s)}{ds^{k+1}}, \text{ using definition of Laplace transform} \quad ...(4)$$

(2) and (4) show that the required result (1) is true for $n=k+1$ whenever it is true for $n=k$. Hence, by the principle of mathematical induction, the required result (1) is true for all positive integers.

***Leibnitz's rule for differentiation under the integral sign.** Let a and b be constants.

Then $\quad f(s)=\int_a^b \text{F}(s,t)dt \Rightarrow \dfrac{d}{ds}f(s)=\int_a^b \dfrac{\partial}{\partial s}\text{F}(s,t)dt.$

S.No.	Operation	$F(t)$	$L\{F(t)\}$ or $f(s)$ or $\bar{F}(s)$
1.	Linearity property	$c_1F_1(t) + c_2F_2(t)$	$c_1L\{F_1(t)\} + c_2L\{F_2(t)\}$
2.	First translation or shifting theorem	$e^{at}F(t)$	$f(s-a)$
3.	Second translation or shifting theorem	$G(t) = \begin{cases} F(t-a), t > 0 \\ 0 \quad t < a \end{cases}$ or $F(t-a)\,H(t-a)$	$e^{-as}f(s)$
4.	Change of scale property	$F(at)$	$(1/a) \times f(s/a)$
5.	Differentiation theorem	$F'(t)$	$s\,L\{F(t)\} - F(0)$
		$F''(t)$	$s^2L\{F(t)\} - sF(0) - F'(0)$
		$F'''(t)$	$s^3L\{F(t)\} - s^2F(0) - sF'(0) - F''(0)$
6.	Multiplication theorem	$t\,F(t)$	$-f'(s)$
		$t^n F(t)$	$(-1)^n \frac{d^n}{ds^n} f(s)$
7.	Division theorem	$\frac{1}{t}F(t)$	$\int_s^\infty f(x)\,dx$
8.	Integral theorem	$\int_0^t F(x)\,dx$	$\frac{1}{s}f(s)$

2

The Inverse Laplace Transform

2.1. Introduction.

We have already said that the purpose of studying Laplace transform is to solve differential and integral equations. In next chapters, 3, 4 and 5, we shall observe that during the process of solving differential and integral equations we have to determine a function whose Laplace transform is already known. This is the reverse process of determining the Laplace transform. In this chapter, we propose to give methods of finding the function whose Laplace transform is already known.

2.2. Inverse Laplace Transform. [Osmania 2004, Purvanchal 1996]

Definition. If $f(s)$ be the Laplace transform of a function $F(t)$, *i.e.*, if $L\{F(t)\} = f(s)$, then $F(t)$ is called the *Inverse Laplace Transform* of the function $f(s)$ and is written as $F(t) = L^{-1}\{f(s)\}$, where L^{-1} is called the *Inverse Laplace Transformation operator.*

Remark. In view of the above definition, we note that $L\{F(t)\} = f(s) \Leftrightarrow L^{-1}\{f(s)\} = F(t)$.

For example,
$$L\{e^{at}\} = \frac{1}{s-a} \Leftrightarrow L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

2.3. Null Function. [Osmania 2004]

Definition. If $N(t)$ is a function of t such that $\int_0^t N(t)\,dt = 0$, for all $t > 0$, then $N(t)$ is called a *Null Function.*

Example *The function*
$$N(t) = \begin{cases} 1, & t = 1/2 \\ -1, & t = 1 \\ 0, & \text{otherwise} \end{cases}$$

is a null function.

Also, we can show that $L\{N(t)\} = 0$

2.3A. Uniqueness of inverse Laplace Transforms. Lerch's Threorem

Since the Laplace Transform of a null function $N(t)$ is zero, hence
$$L\{f(t)\} = f(s) \quad \Rightarrow \quad L\{F(t) + N(t)\} = f(s),$$
showing that we can have two different functions with the same Laplace transform.

As an illustration, consider two functions

$$F_1(t) = e^{-3t} \quad \text{and} \quad F_2(t) = \begin{cases} 0, & t = 1 \\ e^{-3t}, & \text{otherwise} \end{cases}$$

Then we can easily show that both functions have the same Laplace transform, *i.e.* $1/(s+3)$.

Thus, if we allow null functions, then the inverse Laplace transform is not unique. Now null functions do not, in general, arise in problems of physical interest. Hence if null functions is not allowed, then the inverse Laplace transform of a function is unique. This result is stated in the following theorem.

Lerch's theorem. *If we restrict ourselves to functions F (t) which are sectionally continuous in every finite interval $0 \le t \le t_0$ and of exponential order for $t > t_0$, then the inverse of Laplace transform of f (s), i.e., $L^{-1}\{f(s)\} = F(t)$, is unique.*

While evaluating inverse Laplace transform of any function, we shall always assume such uniqueness unless otherwise stated.

2.4. Inverse Laplace Transform of Some Elementary Functions

The student in adviced to study Art. 1.9 of Chapter 1 before studying this article.

(i) To find inverse Laplace transform of 1/s, s > 0.

Proof. Since $L\{1\} = 1/s,\ s > 0$, so by definition $L^{-1}\{1/s\} = 1,\ s > 0.$

(ii) To find inverse Laplace transform of $1/s^{n+1}$, n being any real number such that n > –1.

Proof. Since $L\{t^n\} = \Gamma(n+1)/s^{n+1},\ s > 0,\ n > -1$, so by definition

$$L^{-1}\left\{\frac{\Gamma(n+1)}{s^{n+1}}\right\} = t^n \quad \text{or} \quad L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{\Gamma(n+1)},\ n > -1, s > 0.$$ **[Guwahati 2007]**

(iii) To Find inverse Laplace tranform of $1/s^{n+1}$, n being a positive integer.

Proof. Since $L\{t^n\} = n!/s^{n+1},\ s>0$, so by definition

$$L^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n \quad \text{or} \quad L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!},\ s > 0.$$

(iv) To find inverse Laplace transform of 1/(s – a), s > a. **[Andhra 1990]**

Proof. Since $L\{e^{at}\} = 1/(s-a),\ s > a$, so by definition

$$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at},\ s > a$$

(v) To find inverse Laplace transform of $1/(s^2 - a^2)$, s > |a|.

Proof. Since $L\{\sinh at\} = a/(s^2 - a^2),\ s > |a|$, so by definition

$$L^{-1}\left\{\frac{a}{s^2 - a^2}\right\} = \sinh at \quad \text{or} \quad L^{-1}\left\{\frac{1}{s^2 - a^2}\right\} = \frac{\sinh at}{a},\ s > |a|$$

(vi) To find inverse Laplace transform of $s/(s^2 - a^2)$, s > |a|.

Since $L\{\cosh at\} = s/(s^2 - a^2),\ s > |a|$, so by definition

$$L^{-1}\left\{\frac{s}{s^2 - a^2}\right\} = \cosh at,\ s > |a|$$ **(Agra 2010)**

(vii) To find inverse Laplace transform of $1/(s^2 + a^2)$, s > 0. **[Meerut 2002]**

Proof. Since $L\{\sin at\} = a/(s^2 + a^2),\ s > 0$, so by definition

$$L^{-1}\left\{\frac{a}{s^2 + a^2}\right\} = \sin at \quad \text{or} \quad L^{-1}\left\{\frac{1}{s^2 + a^2}\right\} = \frac{\sin at}{a},\ s > 0$$

(viii) To find inverse Laplace transform of $s/(s^2 + a^2)$, s > 0.

Proof. Since $L\{\cos at\} = s/(s^2 + a^2),\ s > 0$, so by definition

$$L^{-1}\left\{\frac{s}{s^2 + a^2}\right\} = \cos at,\ s > 0$$

Remarks. Results of this article should be commited to memory. With these results, the inverse Laplace transform of all functions can be evaluated by using some general theorems, which will be presented at proper places in this chapter. For ready reference the results of this article are presented in the table given on next page.

$$= \frac{3}{5}\cos t + \frac{1}{5}\sin t - \frac{3}{5}e^{-t}\mathrm{L}^{-1}\left\{\frac{s-(1/3)}{s^2+1}\right\}, \text{ by first shifting theorem}$$

$$= \frac{3}{5}\cos t + \frac{1}{5}\sin t - \frac{3}{5}e^{-t}\left[\mathrm{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{1}{3}\mathrm{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\right]$$

$= (3/5) \times \cos t + (1/5) \times \sin t - (3/5) \times e^{-t}\, \{\cos t - (1/3) \times \sin t\}$

$= (3/5) \times (1 - e^{-t}) \cos t + (1/5) \times (1 + e^{-t}) \sin t$

EXERCISE 2 (D)

Evaluate the inverse Laplace transform of the following:

1. $1/(s-4)^3$ **[Gulberga 2005]**
2. $(s+1)/(s^2+6s+25)$ **[Osmania 2010]**
3. $(s+2)/(s^2-2s+5)$ **[Kanpur 2008]**
4. $(4s+5)/\{(s-4)^4\,(s+3)\}$ **[Bilaspur 2004]**
5. $(s+5)/\{(s+2)\,(s^2+4)\}$
6. $1/(2s+3)^{1/3}$ **[Meerut 1998]**
7. $s/(s+3)^{7/3}$ **[Kanpur 2010]**
8. $1/(s-4)^5 + 5/\{(s-2)^2+5^2\}$
9. $1/(s+4)^{3/2}$ **[Sagar 2004]**
10. $(s+2)/(s^2-4s+13)$ **[Ravishanker 2004]**
11. $1/(s^2-6s+10)$ **[Ravishanker 2004]**
12. $9/(s^2+s+1/2)$
13. $\dfrac{3s+1}{(s+1)^4}$ **[Ranchi 2010]**
14. $\dfrac{1}{s^2+4s+13} - \dfrac{s+4}{s^2+8s+97} + \dfrac{s+2}{s^2-4s+29}$

15. $(2s)/\{(s-1)(s+2)^3\}$ **[Gulberga 2005]** 16. $1/\{(s-1)^2(s^2+1)\}$ **[Purvanchal 2005]**

17. $(3s-2)/(s^2-4s+20)$ **[Meerut 2011, Agra 2014]**

ANSWERS

1. $(1/2)\times t^2 e^{4t}$
2. $e^{-3t}\{\cos 4t-(1/2)\times \sin 4t\}$
3. $e^t[\cos 2t+(3/2)\times \sin 2t]$
4. $3te^{4t}+(1/7)\times(e^{4t}-e^{-3t})$
5. $(1/8)\times(3e^{-2t}-3\cos 2t+7\sin 2t)$
6. $(1/2\pi t)^{1/2}e^{-3t/2}$
7. $[4e^{-3t}t^{3/2}(5-6t)]/15\sqrt{\pi}$
8. $(1/24)\times t^4 e^{4t}+e^{2t}\sin 5t$
9. $2(t/\pi)^{1/2}e^{-4t}$
10. $e^{2t}\cos 3t+(4/5)\times e^{2t}\sin 3t$
11. $e^{3t}\sin t$
12. $18e^{-t/2}\sin(t/2)$
13. $(1/6)\times e^{-t}(9t^2-2t^3)$
14. $(1/3)\times e^{-2t}\sin 3t-e^{-4t}\cos 9t+e^{2t}\cos 5t+(4/5)\times e^{2t}\sin 5t$
15. $(2/27)\times e^{-2t}(18t^2-3t-1)+(2/27)\times e^t$
16. $2\cos t+(1/2)\times e^t(t-1)$
17. $e^{2t}(3\cos 4t+\sin 4t)$

2.9. Second shifting theorem or second translation theorem

[Lucknow 2000; Purvanchal 1995; Kanpur 1994, 96; Meerut 1999]

$$\int_0 \frac{}{a-b}\left[e^{(a-b)t}-1\right]=\frac{1}{a-b}(e^{at}-e^{bt})$$

2.16. The Convolution Theorem (or Convolution property)

[Agra 2011, 12 ; Kanpur 2009]

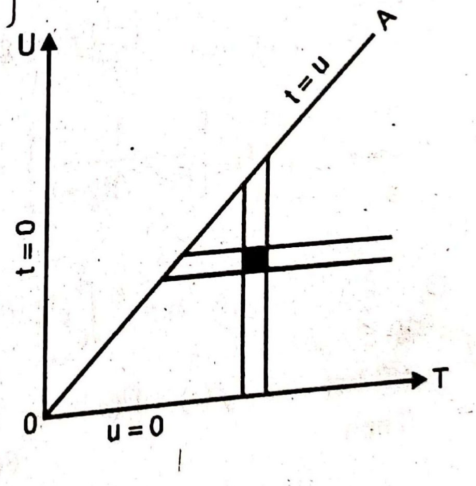
If $L^{-1}\{f(s)\} = F(t)$ and $L^{-1}\{g(s)\} = G(t)$, then $L^{-1}\{f(s)\,g(s)\} = \int_0^t F(u)G(t-u)\,du = F * G$

[Guwahati 2000, Osmania 2004; Meerut 2011, 12; Nagpur 2005, 10; Purvanchal 2002; Rohilkhand 2007, 14; Avadh 2006, 09; Lucknow 2010, 11]

Proof. Let
$$H(t) = \int_{u=0}^{t} F(u)G(t-u)\,du = F*G \qquad \text{... (1)}$$

Now,
$$L\{H(t)\} = \int_{t=0}^{\infty} e^{-st} H(t)\,dt, \text{ by definition of Laplace transform}$$

or
$$L\{H(t)\} = \int_{t=0}^{\infty} e^{-st}\left\{\int_{u=0}^{t} F(u)G(t-u)\,du\right\}dt, \text{ using (1)} \qquad \text{... (2)}$$



In (2), the region of integration in the double integral is the infinite area below the line OA (with equation $u = t$) and above the line OT (with equation $u = 0$). Here OT and OU are rectangular axes, t and u being measured along OT and OU respectively. In the double integral the strip is taken parallel to OU and if the order of integration is changed, the strip is taken parallel to OT and hence the new limits of t are from u to ∞ and those of u are from 0 to ∞. Hence changing the order of integration, (2) reduces to

$$L\{H(t)\} = \int_{u=0}^{\infty} F(u)\left\{\int_{t=u}^{\infty} e^{-st} G(t-u)\,dt\right\}du = \int_{u=0}^{\infty} e^{-su} F(u)\left\{\int_{t=u}^{\infty} e^{-s(t-u)} G(t-u)dt\right\}du$$

$$= \int_{u=0}^{\infty} e^{-su} F(u)\left\{\int_{v=0}^{\infty} e^{-sv} G(v)dv\right\}du, \text{ putting } t-u=v \text{ so that } dt = dv$$

$$= \int_{u=0}^{\infty} e^{-su} F(u)\, g(s)du = g(s)\int_{u=0}^{\infty} e^{-su} F(u)du, \text{ by definition}$$

$\therefore$ $L\{H(t)\} = g(s) f(s)$ or $L^{-1}\{f(s)\, g(s)\} = H(t)$

or $$L^{-1}\{f(s)\, g(s)\} = \int_0^t F(u) G(t-u)du = F*G, \text{ by (1)}$$

Remark 1. The convolution theorem can be re-written as:

$$L\{\int_0^t F(u)G(t-u)du\} = L\{F(t)*G(t)\} = L\{F(t)\}\times L\{G(t)\}$$ [CDLU 2004]

Remark 2. While using the convolution theorem, we use one of the following two forms. The particular choice will depend on the problem in hand keeping in mind the later work of integration involved in the later part of the solution.

$$L^{-1}\{f(s)\, g(s)\} = \int_0^t F(u)\, G(t-u)\, du \quad \text{or} \quad L^{-1}\{f(s)\, g(s)\} = \int_0^t G(u)\, F(t-u)\, du$$

2.16 A. Solved examples based on Art. 2.16.

Ex.1. *Use the convolution theorem to evaluate:*

(i) $L^{-1}\left\{\dfrac{1}{(s+a)(s+b)}\right\}$ (ii) $L^{-1}\left\{\dfrac{1}{(s+1)(s-1)}\right\}$

(iii) $L^{-1}\left\{\dfrac{1}{(s-1)(s+2)}\right\}$ [Bilaspur 2004, Meerut 2004, Kanpur 2004]

Sol. (i) Let $f(s) = 1/(s+a)$ and $g(s) = 1/(s+b)$ ...(1)

Then $F(t) = L^{-1}\{f(s)\} = L^{-1}\{1/(s+a)\} = e^{-at}$ ...(2)

and $G(t) = L^{-1}\{g(s)\} = L^{-1}\{1/(s+b)\} = e^{-bt}$ ...(3)

Now, using the convolution theorem, we have

$$L^{-1}\{f(s)\, g(s)\} = \int_0^t F(u)\, G(t-u)\, du$$

or $$L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\} = \int_0^t e^{-au} e^{-b(t-u)} du, \text{ using (1), (2) and (3)}$$

$$= e^{-bt}\int_0^t e^{(b-a)u} du = e^{-bt}\left[\frac{e^{(b-a)u}}{b-a}\right]_0^t = \frac{e^{-bt}}{b-a}\left[e^{(b-a)t}-1\right] = \frac{1}{b-a}\left(e^{-at}-e^{-bt}\right)$$

(ii) Let $f(s) = 1/(s+1)$ and $g(s) = 1/(s-1)$ ...(1)

Then $F(t) = L^{-1}\{f(s)\} = L^{-1}\{1/(s+1)\} = e^{-t}$ ...(2)

and $$G(t) = L^{-1}\{g(s)\} = L^{-1}\{1/(s-1)\} = e^t. \quad ..(3)$$

Now, using the convolution theorem, we have

$$L^{-1}\{f(s)\,g(s)\} = \int_0^t F(u)\,G(t-u)\,du$$

or $$L^{-1}\left\{\frac{1}{(s+1)(s-1)}\right\} = \int_0^t e^{-u} e^{t-u}\,du = e^t \int_0^t e^{-2u}\,du, \text{ by (1), (2) and (3)}$$

$$= e^t\left[-\frac{1}{2}e^{-2u}\right]_0^t = -\frac{1}{2}e^t\left(e^{-2t}-1\right) = \frac{1}{2}(e^t - e^{-t})$$

(*iii*) Proceed as in part (*ii*). **Ans.** $(e^t - e^{-2t})/3$.

Ex.2(*a*). *Evaluate* $L^{-1}\{1/s^2(s^2-a^2)\}$ **[Purvanchal 1996, Meerut 1993]**

(*b*) $L^{-1}\{1/s^2(s^2+a^2)\}$ **[MDU Rohtak 2005]**

Sol(*a*). Let $f(s) = 1/s^2$ and $g(s) = 1/(s^2-a^2)$ ... (1)

Then $F(t) = L^{-1}\{f(s)\} = L^{-1}\{1/s^2\} = t/1! = t$ ... (2)

and $G(t) = L^{-1}\{g(s)\} = L^{-1}\{1/(s^2-a^2)\} = (1/a) \times \sinh at$ ... (3)

Now, using the convolution theorem, we have

$$L^{-1}\{f(s)\,g(s)\} = \int_0^t F(u)\,G(t-u)\,du$$

or $$L^{-1}\left\{\frac{1}{s^2(s^2-a^2)}\right\} = \int_0^t u \times \frac{1}{a} \times \sinh a(t-u)\,du, \text{ by (1), (2) and (3)}$$

$$= \frac{1}{a}\left\{\left[u\frac{\cosh a(t-u)}{-a}\right]_0^t - \int_0^t 1 \times \frac{\cosh a(t-u)}{-a}\,du\right\}$$

$$= \frac{1}{a}\left[-\frac{t}{a}\cosh 0 + 0 + \frac{1}{a}\int_0^t \cosh a(t-u)\,du\right] = -\frac{t}{a^2} + \frac{1}{a^2}\left[\frac{\sinh a(t-u)}{-a}\right]_0^t$$

$$= -(t/a^2) - (1/a^3) \times (0 - \sinh at) = (\sinh at - at)/a^3$$

(*b*) Proceed like part (*a*). **Ans.** $(at - \sin at)/a^3$

Ex.3. *Use the Convolution theorem to evaluate:*

(*i*) $L^{-1}\left\{\dfrac{1}{(s+1)(s^2+1)}\right\}$ **[Osmania 2010]** (*ii*) $L^{-1}\left\{\dfrac{1}{(s+2)(s^2+4)}\right\}$

(*iii*) $L^{-1}\left\{\dfrac{1}{(s-2)(s^2+1)}\right\}$

Sol. (*i*) Let $f(s) = 1/(s+1)$ and $g(s) = 1/(s^2+1)$ ...(1)

$\therefore F(t) = L^{-1}\{f(s)\} = L^{-1}\{1/(s+1)\} = e^{-t}$, $G(t) = L^{-1}\{g(s)\} = L^{-1}\{1/(s^2+1)\} = \sin t$ ...(2)

Now, using the convolution theorem, we have

$$L^{-1}\{f(s)\,g(s)\} = \int_0^t G(u)\,F(t-u)\,du$$

3

Applications to Ordinary Differential Equations

3.1. Solution of ordinary differential equation with constant coefficients.

Suppose we wish to solve the nth order ordinary differential equation with constant coefficients

$$d^n y/dt^n + a_1 (d^{n-1} y/d t^{n-1}) + a_2 (d^{n-2} y/d t^{n-2}) + \ldots\ldots + a_n y = F(t), \quad \ldots (1)$$

where $a_1, a_2, \ldots\ldots, a_n$ are constants, subject to the initial conditions

$$y(0) = k_0, \quad y'(0) = k_1, \quad y''(0) = k_2, \ldots\ldots, \quad y^{(n-1)}(0) = k_{n-1}, \quad \ldots (2)$$

where $k_0, k_1, \ldots\ldots, k_{n-1}$ are constants.

On taking the Laplace transform of both sides of (1) and using (2), we obtain an algebraic equation (which is known as "subsidiary equation") for determination of L $\{y(t)\}$. The required solution is then obtained by finding the inverse Laplace transform of L $\{y(t)\}$. Since the formulas of Art 1.15, Chapter 1 depend on the initial conditions of the differential equation, these conditions are automatically contained in the final solution of the given differential equation when the inverse is found.

Very important note. Students are adviced to commit to memory all rules of finding Laplace transforms and inverse Laplace transforms discussed in chapters 1 and 2.

Notations. In what follows, we shall adopt the following notations:

(*i*) $dy/dt = Dy = y'(t) = y^{(1)}(t), \quad d^2y/dt^2 = D^2 y = y''(t) = y^{(2)}(t), \ldots$

$d^n y/dt^n = D^n y = y^{(n)}(t)$ etc.

(*ii*) At $t = 0$, we have $y(0) = y_0, \quad y'(0) = y_1, \quad y''(0) = y_2, \ldots\ldots y^{(n)}(0) = y_n$.

In some problems, the dependent variable may be x or z etc in place of y in (1). Then we make the corresponding changes in the notations which have just been discussed.

The whole procedure of solution will become clear by reading the following solved examples.

3.1A. Solved examples based on Art. 3.1

Ex. 1(a) *Solve* $(D^2 + 4)y = t$ *given that* $y(0) = y'(0) = 0$ *and* $D \equiv d/dt$.

[M.S.Univ. T.N. 2007]

Sol. Given $$y''(t) + 4y(t) = t \quad \ldots (1)$$

with initial conditions: $y(0) = 0$ and $y'(0) = 0$ ... (2)

Taking Laplace transform of both sides of (1), we have

$L\{y''(t)\} + 4L\{y(t)\} = L\{t\}$ or $s^2 L\{y\} - sy(0) - y'(0) + 4L\{y\} = 1/s^2$

or $(s^2 + 4) L\{y\} = 1/s^2$ or $L\{y\} = 1/\{s^2(s^2+4)\}$, using (2) ...(3)

Taking inverse Laplace transform on both sides of (3), we get

$$y = L^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} = \frac{1}{4} L^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+4}\right\} = \frac{1}{4}\left(t - \frac{\sin 2t}{2}\right).$$

Ex. 1(*b*) *Using Laplace transform method solve* $(D + D^2)\,x = 2$, *if* $x = 3$ *and* $x' = 1$ *when* $t = 0$.

[Rohilkhand 2011; Kanpur 2007]

Sol. Re-writing, the given equation gives $x'(t) + x''(t) = 2$ ...(1)

Given boundary conditions are $x(0) = 3$ and $x'(0) = 1$ ...(2)

Taking Laplace transform of both sides of (1), we get

$L\{x'(t)\} + L\{x''(t)\} = 2L\{1\}$ or $sL\{x\} - x(0) + s^2L\{x\} - sx(0) - x'(0) = 2/s$

or $sL\{x\} - 3 + s^2\,L\{x\} - 3s - 1 = 2/s$, using (2)

or $(s + s^2)L\{x\} = \dfrac{2}{s} + 4 + 3s$ or $L\{x\} = \dfrac{3s^2 + 4s + 2}{s^2(s+1)}$ ...(3)

Let $\dfrac{3s^2 + 4s + 2}{s^2(s+1)} = \dfrac{A}{s+1} + \dfrac{B}{s} + \dfrac{C}{s^2}$ ...(4)

(4) $\Rightarrow$ $3s^2 + 4s + 2 = As^2 + Bs\,(s + 1) + C(s + 1)$ ...(5)

Equating coeffients of s, s^2 and constant terms on both sides of (5), we have

$A + B = 3$, $B + C = 4$ and $C = 2$ $\Rightarrow$ $A = 1$, $B = 2$, $C = 2$

$\therefore$ From (3), $L\{x\} = 1/(s + 1) + 2/s + 2/s^2$ ...(6)

Taking inverse Laplace transform of both sides of (6), we get

$$x = L^{-1}\{1/(s + 1)\} + 2L^{-1}\{1/s\} + 2L^{-1}\{1/s^2\} = e^{-t} + 2 + 2t$$

Ex.1(c). *Solve* $(D^2 - 1)\,y = a \cosh nt$, *if* $y = Dy = 0$ *when* $t = 0$.

Sol. Re-writing the given equation and conditions, we have

$$y'' - y = a \cosh nt, \quad ...(1)$$

with initial conditions: $y(0) = 0$ and $y'(0) = 0$. ...(2)

Taking Laplace transform of both sides of (1), we get

$$L\{y''\} - L\{y\} = L\{a \cosh nt\}$$

or $s^2\,L\{y\} - s\,y(0) - y'(0) - L\{y\} = as/(s^2 - n^2)$

or $(s^2 - 1)\,L\{y\} - s \times 0 - 0 = as/(s^2 - n^2)$, using (2)

or $$L\{y\} = \frac{as}{(s^2-1)(s^2-n^2)} = \frac{as}{n^2-1}\left\{\frac{(s^2-1)-(s^2-n^2)}{(s^2-1)(s^2-n^2)}\right\}$$

or $$L\{y\} = \frac{as}{n^2-1}\left[\frac{1}{s^2-n^2} - \frac{1}{s^2-1}\right] = \frac{a}{n^2-1}\left[\frac{s}{s^2-n^2} - \frac{s}{s^2-1}\right]$$

Taking inverse Laplace transform of both sides, we have

$$y = \frac{a}{n^2-1}\left[L^{-1}\left\{\frac{s}{s^2-n^2}\right\} - L^{-1}\left\{\frac{s}{s^2-1}\right\}\right] = \frac{a}{n^2-1}(\cosh nt - \cosh t).$$

Ex.2. *Solve* $(D^2 + 1)y = 6\cos 2t$, *if* $y = 3$, $Dy = 1$, *when* $t = 0$.

[GATE 2011; KU Kurukshetra 2005, Meerut 2012, Purvanchal 2002, 07]
Kanpur 2012; Gorakhpur 2009; Rohilkhand 2009]

Sol. Re-writing the given equation and conditions, we have

$$y'' + y = 6\cos 2t, \quad ...(1)$$

with initial conditions: $y(0) = 3$, and $y'(0) = 1$. ...(2)

Taking Laplace transform of both sides of (1), we get

$L\{y''\} + L\{y\} = 6\,L\{\cos 2t\}$ or $s^2\,L\{y\} - s\,y(0) - y'(0) + L\{y\} = 6s/(s^2 + 2^2)$

or $$(s^2+1)\,L\{y\} - 3s - 1 = 6s/(s^2+4), \text{ using (2)}$$

or $$L\{y\} = \frac{3s}{s^2+1} + \frac{1}{s^2+1} + \frac{6s}{(s^2+1)(s^2+4)} = \frac{3s}{s^2+1} + \frac{1}{s^2+1} + 2s\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)$$

$\therefore$ $$L\{y\} = 5\times\frac{s}{s^2+1} + \frac{1}{s^2+1} - 2\times\frac{s}{s^2+2^2}. \quad \text{... (3)}$$

Taking inverse Laplace transform of both sides of (3), we get

$$y = 5\cos t + \sin t - 2\cos 2t.$$

Ex.3. *Solve* $(D^2 + 4D + 8)\,y = 0,\ y(0) = 2,\ y'(0) = 2.$ **[Rohilkhand 1997]**

Sol. Re-writing the given equation and conditions, we have

$$y'' + 4y' + 8y = 0, \quad \text{... (1)}$$

with initial conditions: $y(0) = 2$ and $y'(0) = 2$. ... (2)

Taking Laplace transform of both sides of (1), we get $L\{y''\} + 4L\{y'\} + 8L\{y\} = 0$

or $$s^2 L\{y\} - s\,y(0) - y'(0) + 4[s\,L\{y\} - y(0)] + 8L\{y\} = 0$$

or $$(s^2 + 4s + 8)\,L\{y\} - 2s - 2 - (4\times 2) = 0, \text{ using (2)}$$

or $$L\{y\} = (2s+10)/(s^2+4s+8) \quad \text{... (3)}$$

Taking inverse Laplace transform of both sides of (3), we get

$$y = L^{-1}\left\{\frac{2(s+2)+6}{(s+2)^2+4}\right\} = e^{-2t}L^{-1}\left\{\frac{2s+6}{s^2+4}\right\}, \text{ using the first shifting theorem,}$$

$$= e^{-2t}\left[2L^{-1}\left\{\frac{s}{s^2+2^2}\right\} + 6L^{-1}\left\{\frac{1}{s^2+2^2}\right\}\right]$$

$$= e^{-2t}\{2\cos 2t + 6\times(1/2)\times\sin 2t\} = e^{-2t}(2\cos 2t + 3\sin 2t).$$

Ex.4. *Find the general solution of* $(D^2 + k^2)\,y = 0$ *and verify it.*

Sol. Given that $$y'' + k^2 y = 0 \quad \text{... (1)}$$

Taking Laplace transform of both sides of (1), $L\{y''\} + k^2 L\{y\} = 0$

or $$s^2 L\{y\} - sy(0) - y'(0) + k^2 L\{y\} = 0$$

or $$(s^2+k^2)\,L\{y\} - As - B = 0, \text{ where } y(0) = A \text{ and } y'(0) = B, \text{ say}$$

or $$L\{y\} = \frac{As+B}{s^2+k^2} = A\times\frac{s}{s^2+k^2} + B\times\frac{1}{s^2+k^2} \quad \text{... (2)}$$

Taking inverse Laplace transform of both sides of (3), we get

$$y = A\cos kt + (B/k)\times\sin kt = A\cos kt + C\sin kt, \quad \text{... (3)}$$

where $C = B/k$. (3) is the general solution of given equation and A and C are arbitrary constants.

Verification: Differentiating (3) twice w.r.t. 't', we have

$y' = -Ak\sin kt + Ck\cos kt$ and $y'' = -Ak^2\cos kt - Ck^2\sin kt$ ... (4)

Substituting the values of y and y'' from (3) and (4) in (1) we get

$-Ak^2\cos kt - Ck^2\sin kt + k^2(A\cos kt + C\sin kt) = 0$ or $0=0$,

showing that (3) is a solution of (1).

Ex.5 (a). *Solve* $(D+2)^2 x = 4e^{-2t},\ x(0) = -1$ *and* $x'(0) = 4$ **[Agra 2008; Meerut 2001]**

Or Solve $x''(t) + 4x'(t) + 4x(t) = 4e^{-2t},\ x(0) = -1,\ x'(0) = 4$

Sol. Given $(D+2)^2 x = 4e^{-2t}$ or $(D^2 + 4D + 4)\,x = 4e^{-2t}$.

or $$x'' + 4x' + 4x = 4e^{-2t}, \quad \text{... (1)}$$

with initial conditions: $x(0) = -1$ and $x'(0) = 4$. ... (2)

... $x(0) = A$ and $x'(0) = B$.

EXERCISE 3 (A)

Using Laplace transform method, solve the following equations:

1. $(D + 1)\,y = 0$, $t > 0$, given that $y = y_0$, when $t = 0$.
2. $(D + 1)\,y = 1$, given that $y = 2$ when $t = 0$.
3. $(D^2 + 1)\,y = 0$, if $y = 1$, $dy/dt = 0$, when $t = 0$. **[Osmania 2010, Ravishankar 2010]**
4. $(D^2 + 9)\,y = 18t$, if $y(0) = 0$, $y(\pi/2) = 0$. **[Osmania 2004]**
5. $(D^2 + 3D + 2)\,y = 0$, $y = y_0$ and $Dy = y_1$ at $t = 0$. **[Rohilkhand 1994]**
6. $(D^2 + 6D + 25)\,y = 208\,e^{3t}$, $t > 0$, if $y = 1$, $Dy = 0$ when $t = 0$.
7. $(D^3 + 1)\,y = 1$, $t > 0$ if $y = Dy = D^2 y = 0$ when $t = 0$. **[Ravishankar 2004]**
8. $(D^3 + D)\,y = e^{2t}$, $y(0) = y'(0) = y''(0) = 0$.
9. $(D^3 - D)\,y = 2\cos t$, $y = 3$, $Dy = 2$, $D^2 y = 1$ when $t = 0$.
10. $(D^2 + 4D + 4)\,x = \sin wt$, $t > 0$ with x_0 and x_1 for values of x and Dx when $t = 0$.
11. $(D^3 - D^2 + 4D - 4)\,y = 68\,e^t \sin 2t$, $y = 1$, $Dy = -19$, $D^2 y = -37$, *at* $t = 0$.
12. $y''''(t) + y'''(t) = \cos t$, $y(0) = y'(0) = y'''(0) = 0$, $y''(0)$ arbitrary. **[Agra 2005]**
13. $y_2 + 4y_1 + 5y = (\cos t - \sin t)\,e^{-2t}$, $y(0) = 1$, $y'(0) = -3$. **[Osmania 2010; Avadh 2008]**
14. $(D^2 - 3D + 2)y = 1 - e^{2t}$, $y(0) = 1$, $y'(0) = 0$. **[Guwahati 2007]**
15. $d^2x/dt^2 + x = t$, $x(0) = 1$, $x'(0) = -2$ **[Kanpur 2006]**
16. $y'' + 25y = 10\cos 5t$, $y(0) = 2$, $y'(0) = 0$ **[Purvanchal 2004]**
17. $y'' + 5y' + 6y = 5e^{2t}$, $y(0) = 2$, $y'(0) = 1$ **[MDU Rohtak 2005]**
18. $y'' + 2y' - 3y = \sin t$, $y(0) = 0$, $y'(0) = 0$ **[Kanpur 2005]**
19. $2y'' + 5y' + 2y = e^{-2t}$, $y(0) = y'(0) = 1$ **[Kuvempa 2005]**
20. $x''(t) + 4x'(t) + 4x(t) = 4e^{-2t}$, $x(0) = -1$, $x'(0) = 4$ **[Bangalore 2005]**
21. $y''(t) - 2y'(t) + y(t) = e^t$, $y(0) = 2$, $y'(0) = -1$

22. $y''(t) + y(t) = 0,\ y(0) = 1,\ y'(0) = 1$

23. $y'''(t) + y''(t) - 4y'(t) - 4y(t) = F(t)$, if $y(0) = y''(0) = 0$ $y'(0) = 2$. [Mysore 2004]

24. $y''(t) - k^2 y(t) = F(t)$, if $y(0) = 0 = y'(0)$, $k \neq 0$.

25. Find the general solution of $d^2y/dt^2 - 2k\,(dy/dt) + k^2 y = F(t)$.

26. $d^2y/dt^2 + a^2 y = F(t),\ y(0) = 1,\ y'(0) = -2$ [Delhi Physics (H) 2005]

ANSWERS

1. $y = y_0\, e^{-t}$.
2. $y = 1 + e^{-t}$.
3. $y = \cos t$.
4. $y = 2t + \pi \sin 3t$
5. $y = (2y_0 + y_1)\, e^{-t} - (y_0 + y_1)\, e^{-2t}$.
6. $y = 4\, e^{3t} - (3/4) \times e^{-3t}\,(4 \cos 4t + 7 \sin 4t)$
7. $y = 1 - (1/3) \times e^{-t} - (2/3) \times e^{t/2} \cos(t\sqrt{3}/2)$.
8. $y = -(1/2) + (1/10) \times e^{2t} + (1/5) \times (2\cos t - \sin t)$
9. $y = \cosh t + 3 \sinh t - \sin t + 2$.

10. $y = e^{-2t}\left\{x_0(1-2t) + (x_1 + 4x_0) + \dfrac{\omega t}{4+\omega^2} + \dfrac{4\omega}{(4+\omega^2)^2}\right\} - \dfrac{4\omega}{(4+\omega^2)^2}\cos \omega t + \dfrac{4-\omega^2}{(4+\omega^2)^2}\sin \omega t$

11. $y = (1/5) \times (e^t + 14 \cos 2t - 3 \sin 2t) - 2\, e^t (\cos 2t + 4 \sin 2t)$.
12. $y = t - 1 + (1/2) \times At^2 + (1/2) \times (e^{-t} + \cos t - \sin t)$, if $y''(0) = A$.
13. $y = e^{-2t}\,[\cos t - (3/2) \times \sin t + (1/2) \times t \sin t + (1/2) \times t \cos t]$.
14. $y(t) = (1/2) \times (1 + e^{2t}) - te^{2t}$
15. $x(t) = \cos t - 3 \sin t + t$
16. $y(t) = (2 + t) \cos 5t - (1/5) \times \sin 5t$
17. $y(t) = (1/4) \times (23e^{-2t} + e^{2t}) - 4e^{-t}$
18. $y(t) = (1/40) \times (e^{3t} - 5e^{-t}) + (\cos t - 2 \sin t)/2$
19. $y(t) = (20/9) \times e^{-t/2} + (1/9) \times (3t - 11)\, e^{-2t}$
20. $y(t) = (2t - 1 + 2t^2)\, e^{-2t}$
21. $y(t) = (2 - 3t + t^2/2)\, e^t$
22. $y(t) = e^t \cos t$
23. $\sin t + \cos t$

24. $y = \sinh 2t + \dfrac{1}{12}\displaystyle\int_0^t (e^{2u} + 3e^{-2u} - 4e^{-u})F(t-u)\, du$

25. $y = \dfrac{1}{2k}\displaystyle\int_0^t (e^{ku} - e^{-ku})F(t-u)\, du$

26. $y = \{(B - Ak)t + A\}e^{kt} + e^{kt}\displaystyle\int_0^t e^{-ku}(t-u)F(u)\, du$

3.2. Solution of ordinary differential equations with variable coefficients

Suppose the given differential equation contain a term of the form

$t^m y^{(n)}(t)$, *i.e.*, $t^m \dfrac{d^n y(t)}{dt^n}$ the Laplace transform of which is $(-1)^m \dfrac{d^m}{ds^m} L\{y^{(n)}(t)\}$.

The method is illustrated in the following examples

3.2A. Solved examples based on Art 3.2.

Ex.1. *Solve* $[t D^2 + (1 - 2t) D - 2]\, y = 0,\ y(0) = 1,\ y'(0) = 2.$ [Meerut 2000]

Sol. Given
$$t y'' + y' - 2t y' - 2y = 0, \quad \dots (1)$$
with initial conditions: $y(0) = 1$ and $y'(0) = 2$. ... (2)

Taking Laplace transform of both sides of (1), $L\{t y''\} + L\{y'\} - 2L\{ty'\} - 2L\{y\} = 0$

or
$$(-1)^1 \frac{d}{ds} L\{y''\} + L\{y'\} - 2(-1)^1 \frac{d}{ds} L\{y'\} - 2L\{y\} = 0$$

or
$$-\frac{d}{ds}[s^2 L\{y\} - sy(0) - y'(0)] + s L\{y\} - y(0) + 2\frac{d}{ds}[s L\{y\} - y(0)] - 2L\{y\} = 0$$

$$-\frac{d}{ds}[s^2\bar{y}-s-2]+s\bar{y}-1+2\frac{d}{ds}[s\bar{y}-1]-2\bar{y}=0, \text{ by (2)} \quad ...(3)$$

or

where we have written $\quad L\{y\} = \bar{y}\ (s). \quad ...(4)$

From (3), we get $\quad -\left[s^2\frac{d\bar{y}}{ds}+2s\bar{y}-1\right]+s\bar{y}-1+2\left[s\frac{d\bar{y}}{ds}+\bar{y}\right]-2\bar{y}=0$

or $\quad (s^2-2s)\frac{d\bar{y}}{ds}=s\bar{y} \quad$ or $\quad \frac{d\bar{y}}{\bar{y}}+\frac{ds}{s-2}=0.$

Integrating, $\quad \log\bar{y}+\log(s-2)=\log C \quad$ or $\quad \bar{y}=C/(s-2).$

$\therefore \quad L\{y\}=C/(s-2)$, using (4). ... (5)

Taking inverse transform of both sides of (5), we have

$$y(t)=C\,L^{-1}\{1/(s-2)\}=Ce^{2t}. \quad ... (6)$$

Putting $t=0$ in (6), we get $y(0)=C$ so that $C=1$, using (2).

Hence, from (6), $y=e^{2t}$, which is the required solution

Ex.2. *Solve $ty''+y'+4ty=0$, if $y(0)=3$, $y'(0)=0$.*

Sol. Given $\quad ty''+y'+4ty=0 \quad ...(1)$

with initial conditions: $\quad y(0)=3 \quad$ and $\quad y'(0)=0. \quad ...(2)$

Taking Laplace transform of both sides of (1), $\quad L\{ty''\}+L\{y'\}+4L\{ty\}=0$

or $$(-1)^1\frac{d}{ds}L\{y''\}+L\{y'\}+4(-1)^1\frac{d}{ds}L\{y\}=0$$

or $$-\frac{d}{ds}\left[s^2L\{y\}-sy(0)-y'(0)\right]+sL\{y\}-y(0)-4\frac{d}{ds}L\{y\}=0$$

or $$-\frac{d}{ds}[s^2L\{y\}-3s]+sL\{y\}-3-4\frac{d}{ds}L\{y\}=0, \text{ using (2)} \quad ...(3)$$

Let $\quad L\{y\}=\bar{y}\ (s). \quad ...(4)$

Then, (3) $\Rightarrow \quad -\frac{d}{ds}(s^2\bar{y}-3s)+s\bar{y}-3-4\frac{d\bar{y}}{ds}=0$

or $$-\left[s^2\frac{d\bar{y}}{ds}+2s\bar{y}-3\right]+s\bar{y}-3-4\frac{d\bar{y}}{ds}=0$$

or $\quad (s^2+4)\frac{d\bar{y}}{ds}+s\bar{y}=0 \quad$ or $\quad \frac{d\bar{y}}{\bar{y}}+\frac{1}{2}\times\frac{2s}{s^2+4}ds=0.$

Integrating, $\quad \log\bar{y}+(1/2)\log(s^2+4)=\log c \quad$ or $\quad \bar{y}=c/(s^2+4)^{1/2}$

$\therefore \quad L\{y\}=c/(s^2+4)^{1/2}$, using (4). ...(5)

(5) $\Rightarrow \quad y(t)=L^{-1}\left\{\frac{c}{(s^2+4)^{1/2}}\right\}=c\,L^{-1}\left\{\frac{1}{(s^2+2^2)^{1/2}}\right\}=c\,J_0(2t) \quad ...(6)$

[$\because$ from Ex 1 (*i*), of Art. 1.22B, Chapter 1, $L\{J_0(at)\}=1/(s^2+a^2)^{1/2}$]

Putting $t=0$ in (6), we get $y(0)=C\,J_0(0)$ or $3=C$, using (2) and noting that $J_0(0)=1$.

Hence from (6), $\quad y=3\,J_0(2t)$, which is the required solution

$$\therefore \quad \bar{y} = \frac{c}{s^2(s+1)} + \frac{5}{s+1} = C\left(\frac{1}{s+1} - \frac{1}{s} + \frac{1}{s^2}\right) + \frac{5}{s+1}$$

[On resolving into partial ...

Taking inverse Laplace transform of both sides of (6) we get

$$y(t) = c\,(e^{-t} - 1 + t) + 5\,e^{-t}.$$

Since given that $y(\infty) = 0$, (7) holds only if $C = 0$.

Then from (7), $y = 5\,e^{-t}$, which is the required solution.

EXERCISE 3 (*B*)

Solve the following differential equations:

1. $t\,y''(t) + y'(t) + t\,y(t) = 0,\ y(0) = 1,\ y'(0) = 0.$
2. $t\,y''(t) + (2t+3)\,y'(t) + (t+3)\,y(t) = a\,e^{-t}$ [Agra ...] [Luckn...]
3. $y''(t) + at\,y'(t) - 2ay(t) = 1,\ y(0) = y'(0) = 0,\ a > 0$

ANSWERS

1. $y = J_0(t)$ **2.** $y = [A + (1/3) \times at]\,e^{-t}$

3.3. Solution of simultaneous ordinary differential equations

Simultaneous ordinary differential equations involve more than one dependent vari... Laplace transform is needed for each variable. The procedure is to solve the simultaneous ... ... invert to recover each de...