Chapter 12

Fourier Series

12.1 Introduction

The question of representing a function by a trigonometrical series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{12.1}$$

arises in many problems. For example it comes up in the solution of certain partial differential equations by the method of separation of variables. We are not going to pursue these applications in this treatise, but are mentioning them here merely to point out that the study of trigonometric series has intimate connections with many branches of mathematics.

Before beginning our discussions on the series (12.1), we observe that if the series converges, or diverges at a point $x = x_0$, then it converges or diverges at $x_0 + 2\pi$, since each term has period 2π; for periodicity implies that the partial sums at x_0 are identical with the partial sums at $x_0 + 2\pi$, or for that matter at $x_0 + 2n\pi$, where n is any integer. Thus whenever (12.1) represents a function, it represents a *periodic function* with period 2π. That means the function $f(x)$, say, must satisfy the functional equation $f(x + 2\pi) = f(x)$.

Periodic Extension.

Now if we have a function $f(x)$ defined on $-\pi < x < +\pi$, we can define the *periodic extension* of $f(x)$. This is indicated in Fig. 12.1, and is obtained by shifting the graph of $f(x)$ in $-\pi < x < \pi$ by 2π, 4π, ... to the right or to the left.

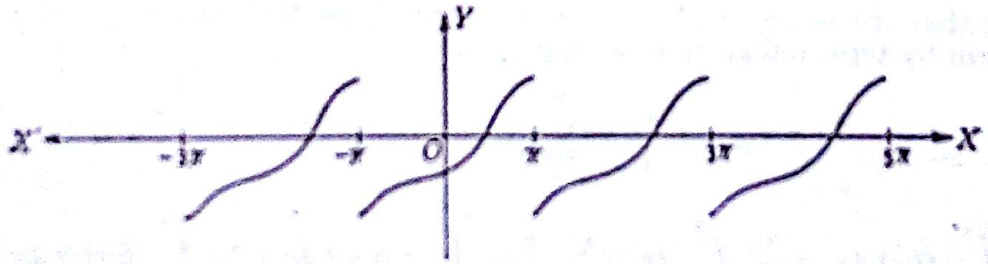

Figure 12.1

Observe that the periodicity formula provides the definition of $f(x)$ outside $-\pi < x < \pi$. Thus for x on $-3\pi < x < -\pi$, $f(x)$ is defined by $f(x) = f(x + 2\pi)$ and for x on $\pi < x < 3\pi$, $f(x)$ is defined by $f(x) = f(x - 2\pi)$ and so on. This leaves the value of $f(x)$ *undetermined* at odd multiples of π. There it can be defined *in any manner* for that will *not affect* the form of the series (12.1) associated with it.

We now put below a few results which will be much useful throughout this chapter. In these calculations m and n are *non-negative integers* not necessarily different.

(i) $\displaystyle\int_{-\pi}^{\pi} \sin nx\,dx = 0 = \int_{-\pi}^{\pi} \sin mx \cos nx\,dx$

(ii) $\displaystyle\int_{-\pi}^{\pi} \cos nx\,dx = \begin{cases} 0, & n > 0 \\ 2\pi, & n = 0 \end{cases}$

(iii) $\displaystyle\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n > 0 \\ 0, & m = n = 0 \end{cases}$

(iv) $\displaystyle\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n > 0 \\ 2\pi, & m = n = 0 \end{cases}$

These integral formulae (i) – (iv) are called *orthogonality formulae*.

12.2 Fourier Series: Determination of Fourier Constants

Assuming that $f(x)$ can be represented by a series of the form (12.1) i.e.,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{12.1}$$

and that the series (12.1) converges uniformly to $f(x)$ on $-\pi \leq x \leq \pi$, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{12.2}$$

We want to enquire, first of all, into the relations between $f(x)$ and the coefficients a_0, $\{a_n\}$ and $\{b_n\}$.

Determination of a_0, $\{a_n\}$, $\{b_n\}$.

Assuming that the series (12.1) converges uniformly to $f(x)$ on $-\pi \leq x \leq \pi$, so as to permit term by term integration, we have from

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{12.2}$$

$$\int_{-\pi}^{\pi} f(x)\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}\left(a_n \int_{-\pi}^{\pi} \cos nx\,dx + b_n \int_{-\pi}^{\pi} \sin nx\,dx\right)$$

$$= \pi a_0.$$

Thus

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx. \tag{12.3}$$

Next if we multiply (12.2) by $\cos kx$ ($k \geq 1$) it remains uniformly convergent in $-\pi \leq x \leq \pi$ and can be integrated term by term.

Thus for $k \geq 1$,

$$\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi}\cos kx\,dx + \sum_{n=1}^{\infty}\left\{a_n\int_{-\pi}^{\pi}\cos nx\cos kx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\cos kx\,dx\right\}.$$

By the results of integration given in introduction, there is only one non-vanishing term in this series, and it comes when $n = k$. Thus

$$\int_{-\pi}^{\pi} f(x)\cos kx\,dx = a_k\pi$$

whereby

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx. \tag{12.4}$$

Similarly multiplying (12.2) by $\sin kx$ $(k \geq 1)$ and integrating term by term, we obtain after the application of orthogonality formulae,

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx. \tag{12.5}$$

Turning the problem around, we can ask how to choose a_n and b_n so that for a given function $f(x)$ we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

and we see that if they can be so chosen that term-by-term integration is permissible, then they are determined by formulae (12.4) and (12.5). In general, if $f(x)$ be only integrable on $-\pi < x < \pi$, the coefficients $\{a_n\}$, $\{b_n\}$, can be computed by formulae, (12.4) and (12.5). In this case the resulting series (12.1) is called the Fourier Series of $f(x)$ and the numbers are called the Fourier Coefficients. Hence we come to a formal definition.

DEFINITION. Fourier series *corresponding* to a function $f(x)$ in an interval $-\pi \leq x \leq \pi$ under *certain conditions* (to be specified in the next article) is the trigonometric series of the form (12.1), i.e.,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) \tag{12.1}$$

where a_0, $\{a_n\}$, $\{b_n\}$ are constants determined from

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad \text{and} \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

Observation.

1. To indicate that a Fourier series arises from a function $f(x)$, we need another symbol than =, for the equality sign carries with it the connotation of convergence. The most common symbol is $\sim$, which is read 'generates'. Thus

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

where $\{a_n\}$ and $\{b_n\}$ are computed by formulae (12.4) and (12.5).

2. Note that (12.3) arises from (12.4) by putting $k = 0$. This is the reason for taking the constant term as $\frac{1}{2}a_0$ rather than a_0, for now (12.4) covers both cases.

3. Since $f(x)$ by its very definition is periodic, the integrals in (12.4) and (12.5) could equally well be taken over by any interval of length 2π, as shown in periodic extension.

4. The numbers a_0, $\{a_n\}$, $\{b_n\}$ are called Fourier constants or Fourier coefficients. Series (12.1) has already been called Fourier series. They were applied by the French mathematician J.B.J. Fourier to the study of heat conduction.

12.3 Dirichlet's Conditions

In the previous article for the expansion of $f(x)$ in a trigonometric series (12.1),

(a) we have assumed the possibility of the expansion of the function in the series,

(b) we have integrated the series term-by-term, (This would have been allowable if the convergence of the series were uniform, but this properties has not been proved and indeed is not generally applicable to the whole interval in these expansions) and

(c) the discussion does not give us any information as to the behaviour of the series at points of discontinuity, if such arise, nor does it give any suggestion as to the conditions to which $f(x)$ must be subjected if the expansion is to be valid.

The series thus formed is not necessarily convergent simply because the coefficients $\{a_n\}$ and $\{b_n\}$ are so defined, and even if the series be convergent, it may not converge to the function which generated it.

For example, let

$$f(x) = 0 \quad \text{when} \quad -\pi \le x < 0$$
$$\quad = 1 \quad \text{when} \quad 0 \le x \le \pi,$$

Then, $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} dx = 1,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi}\cos nx\,dx = 0,$$

$$b_n = \frac{1}{\pi}\int_0^{\pi}\sin nx\,dx = \frac{1-\cos n\pi}{n\pi} = \begin{cases} 2/n\pi, & n \text{ odd} \\ 0, & n \text{ even.} \end{cases}$$

Thus $$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$
$$= \frac{1}{2} + \frac{2}{\pi}\left\{\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right\}.$$

Note that at $x = 0$, the sum of the Fourier series generated by f is $\frac{1}{2}$ and hence is not equal to f at $x = 0$ which is 1.

In order to ensure that the Fourier series corresponding to $f(x)$ converges and has the sum $f(x)$, it is necessary to impose certain conditions on $f(x)$. There are several sets of *sufficient conditions* to ensure the convergence of Fourier series, the most important of which is due to Dirichlet, which we are going to discuss presently. For that let us define certain terms.

Piecewise continuous.

A function $f(x)$ is said to be *piecewise continuous* on $[a, b]$ (i) if there exists a partition P of $[a, b]$ for which f is continuous on each sub-interval, (ii) if $f(x_r + 0)$ and $f(x_r - 0)$ exist at each partition point x_r of P, and finally (iii) if $f(a+0)$ and $f(b-0)$ both exist.

In dealing with a function which is piecewise continuous on $[-\pi, +\pi]$ we will assume that the function is extended by periodicity. Also we will normalise the value of the function at the partition points of P by

$$f(x_r) = \frac{1}{2}\{f(x_r+0) + f(x_r-0)\}.$$

By periodicity this implies

$$f(\pi) = f(-\pi) = \frac{1}{2}\{f(-\pi+0) + f(\pi-0)\}.$$

At all other points x, f is continuous, so that

$$f(x) = \frac{1}{2}\{f(x+0) + f(x-0)\}$$

holds for all points.

Piecewise monotone.

A function f is called ***piecewise monotone*** on $[a, b]$, if there exists a partition P of $[a, b]$ such that f is monotone on each of the sub-intervals.

Thus if f be piecewise monotone and bounded on $[a, b]$, it can have discontinuities of the first kind only.

The possibility of the expansion of a function $f(x)$ on $-\pi \leq x \leq \pi$ in the corresponding Fourier series depends upon some integrals by means of which Dirichlet gave the first rigorous proof that for a large class of functions the Fourier series converges to $f(x)$. These integrals again exist under certain conditions, called Dirichlet's conditions.

Dirichlet's conditions.

A function $f(x)$ will be said to satisfy Dirichlet's conditions on an interval $-\pi \leq x \leq \pi$ in which it is defined when it is subjected to one of the two following conditions:

(i) $f(x)$ is bounded periodic with period 2π and integrable on $-\pi \leq x \leq \pi$ and the interval can be broken up into a finite number of open partial intervals in each of which $f(x)$ is monotonic; [or in a simpler form: $f(x)$ is bounded periodic with period 2π and integrable on $[-\pi, \pi]$, and piecewise monotonic on $[-\pi, \pi]$];

(ii) $f(x)$ has a finite number of points of infinite discontinuity in the interval. When arbitrary small neighbourhoods of these points are excluded, $f(x)$ is bounded in the remainder of the interval, and this can be broken up into a finite number of open partial intervals, in each of which $f(x)$ is monotonic.

Further the improper integral $\int_{-\pi}^{\pi} f(x)\,dx$ is to be absolutely convergent.

Convergence.

When $f(x)$ satisfies Dirichlet's conditions on $-\pi \leq x \leq \pi$, the Fourier series corresponding to $f(x)$ converges to $f(x)$ at any point x on $-\pi < x < \pi$ when $f(x)$ is continuous and converges to $\frac{1}{2}\{f(x+0) + f(x-0)\}$ when there is an ordinary discontinuity at the point. In particular at $x = \pi$ and $x = -\pi$ it converges to $\frac{1}{2}\{f(-\pi+0) + f(\pi-0)\}$ when $f(-\pi+0)$ and $f(\pi-0)$ exist.

We merely content ourselves by stating the conditions, proofs of which are beyond the scope of this book.

Observation.

If the functions of bounded variation be included in the class of functions available for discussion, $f(x)$ may be said to satisfy Dirichlet's conditions (i) when it is of bounded variation in the whole interval, or (ii) when it has a finite number of points of infinite discontinuity in the interval and it is of bounded variation in the remainder of that interval, when the arbitrary small neighbourhoods of these points have been excluded; provided that the improper integral $\int_{-\pi}^{\pi} f(x)\,dx$ be absolutely convergent.

12.4 Odd and Even Functions

When we compute the coefficients of a Fourier series from its defining function, it is useful to recall the following facts:

$f(x)$ is even implies $f(-x) = f(x)$

$f(x)$ is odd implies $f(-x) = -f(x)$

$f(x)$ is even implies $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

$f(x)$ is odd implies $\int_{-a}^{a} f(x)\,dx = 0.$

Thus when $f(x)$ *is odd*, $f(x)\cos nx$ is odd whereas $f(x)\sin nx$ is even whereby

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$$

and
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

which represent a sine series $\sum_{n=1}^{\infty} b_n \sin nx$ only.

Again *if* $f(x)$ *be even*, $f(x)\cos nx$ is even and $f(x)\sin nx$ is odd. Hence

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx \quad \text{and} \quad b_n = 0$$

indicating a cosine series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ only.

12.5 Change of Scale

In $-l \le y \le l$ ..

We can have Fourier series over other intervals than $\{-\pi \le x \le \pi\}$. Let $\phi(y)$ be integrable over $\{-l \le y \le l\}$. If we substitute $y = lx/\pi$ and let $f(x) = \phi(lx/\pi)$, we obtain Fourier series corresponding to $f(x)$ as

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

or corresponding to $\phi(y)$ as

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi y}{l} + b_n \sin\frac{n\pi y}{l}\right)$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{l}\int_{-l}^{l}\phi(y)\cos\frac{n\pi y}{l}\,dy$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{l}\int_{-l}^{l}\phi(y)\sin\frac{n\pi y}{l}\,dy.$$

Evidently no new problems arise in discussing convergence. We have simply made a change of scale taking $\{-\pi \le x \le \pi\}$ into $\{-l \le y \le l\}$. It is clear that there are sine and cosine series for this new interval.

Summary.

Thus, if f be bounded periodic with period $2l$ and integrable on $[-l, l]$ and piecewise monotone on $[-l, l]$,

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right)$$

where

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx,\quad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

Convergence.

When $f(x)$ satisfies Dirichlet's conditions on $-l \le x \le l$, the Fourier series corresponding to $f(x)$ converges to $f(x)$ at any point x on $-l < x < l$ when $f(x)$ is continuous and converges to $\frac{1}{2}\{f(x+0)+f(x-0)\}$ when there is an ordinary discontinuity at the point. In particular at $x = l$ and at $x = -l$, it converges to $\frac{1}{2}\{f(-l+0)+f(l-0)\}$.

In $0 \le x \le 2\pi$.

Substituting $y = x + \pi$ and letting $f(x) = \phi(x+\pi) = \phi(y)$, Fourier series corresponding to $\phi(y)$ on $\{0 \le x \le 2\pi\}$ will be

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n\cos n(y-\pi) + b_n\sin n(y-\pi)\} = \frac{a_0}{2} + \sum_{n=1}^{\infty}(-1)^n\{a_n\cos ny + b_n\sin ny\}$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi}\phi(y)\cos n(y-\pi)\,dy = \frac{(-1)^n}{\pi}\int_0^{2\pi}\phi(y)\cos ny\,dy$$

and

$$b_n = \frac{1}{\pi}\int_0^{2\pi}\phi(y)\sin n(y-\pi)\,dy = \frac{(-1)^n}{\pi}\int_0^{2\pi}\phi(y)\sin ny\,dy$$

or, *in other words*, Fourier series will be

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos ny + b_n\sin ny)$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi}\phi(y)\cos ny\,dy \quad\text{and}\quad b_n = \frac{1}{\pi}\int_0^{2\pi}\phi(y)\sin ny\,dy.$$

Summary.

Thus if f be bounded periodic with period 2π and integrable on $[0, 2\pi]$ and piecewise monotone on $[0, 2\pi]$,

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx, \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx.$$

Convergence.

When $f(x)$ satisfies Dirichlet's conditions on $0 \le x \le 2\pi$, the Fourier series corresponding to $f(x)$ converges to $f(x)$ at any point x on $0 < x < 2\pi$ when $f(x)$ is continuous and converges to $\frac{1}{2}\{f(x+0)+f(x-0)\}$ when there is an ordinary discontinuity at the point. In particular, at $x = 0$ and at $x = 2\pi$, it converges to $\frac{1}{2}\{f(0+0) + f(2\pi - 0)\}$.

12.6 Sine and Cosine Series: The Interval 0 to π Only

(A) Cosine series.

Let $f(x)$ satisfy Dirichlet's conditions on $0 \le x \le \pi$. That means if f be bounded, integrable and piecewise monotone on $[0, \pi]$, then

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \qquad \text{and} \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

is called Fourier cosine series corresponding to $f(x)$ on the interval. The series is equal to $\frac{1}{2}\{f(x+0) + f(x-0)\}$ at every x on $0 < x < \pi$ where $f(x+0)$ and $f(x-0)$ exist; and is equal to $f(0+0)$ *at* $x = 0$ and equal to $f(\pi - 0)$ at $x = \pi$, provided both $f(0+0)$ and $f(\pi - 0)$ exist.

If moreover $f(x)$ be continuous on the interval, the cosine series represents $f(x)$ on the closed interval $0 \le x \le \pi$.

Defining $f(x)$ on $-\pi \le x < 0$ by the equation $f(-x) = f(x)$, i.e., taking $f(x)$ as an even function we can at once prove the results.

(B) Sine series.

Defining $f(x)$ on $-\pi \le x < 0$ by the equation $f(-x) = -f(x)$, i.e., taking $f(x)$ as an odd function, we can show that:

If $f(x)$ satisfies Dirichlet's conditions on $0 \le x \le \pi$, that is if f be bounded, integrable and piecewise monotone on $[0, \pi]$, then

$$\sum_{n=1}^{\infty} b_n \sin nx \qquad \text{where} \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

represents $f(x)$ in Fourier sine series on the interval. The series is equal to $\frac{1}{2}\{f(x+0)+f(x-0)\}$ at every point x on $0<x<\pi$ when $f(x+0)$ and $f(x-0)$ exist; and when $x=0$ and $x=\pi$, the sum is zero.

Observation.

It may be noticed that when $f(x)$ is continuous at the end points $x=0$ and $x=\pi$, the cosine series gives the value of the function at these points, the sine series, however, only gives the value of $f(x)$ at these points only if $f(x)$ is zero there.

12.7 Illustrative Examples

Ex. 1. Expand $f(x)=x$ in Fourier series on the interval $-\pi\le x\le\pi$.

Solution. Observe that $f(x) = x$ is bounded and integrable on $-\pi \le x \le \pi$, since it is continuous there. Further $f'(x)=1>0$ indicates that $f(x)$ is monotone increasing on the entire interval. Thus $f(x)$ satisfies Dirichlet's conditions on $[-\pi,\pi]$. Hence the Fourier series corresponding to $f(x)=x$ is

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

where

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x\cos nx\,dx=0,\qquad a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}x\,dx=0$$

since $x\cos nx$ and x are odd functions, and

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin nx\,dx=\frac{2}{\pi}\int_{0}^{\pi}x\sin nx\,dx$$

since $x\sin nx$ is even. Thus

$$b_n=\frac{2}{\pi}\left[-x\frac{\cos nx}{n}\right]_0^{\pi}+\frac{2}{\pi}\int_0^{\pi}\frac{\cos nx}{n}dx=-\frac{2}{n}\cos n\pi=\begin{cases}-2/n, & n\text{ even}\\ 2/n, & n\text{ odd.}\end{cases}$$

Hence $f(x)=x$ generates Fourier series in the form

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\sum_{n=1}^{\infty}b_n\sin nx$$

$$=b_1\sin x+b_2\sin 2x+b_3\sin 3x+b_4\sin 4x+\cdots$$

$$=2\left\{\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\frac{\sin 4x}{4}+\cdots\right\} \qquad (12.6)$$

Observations.

(1) See that $f(x)=x$ is also a continuous function on the interval. Hence (12.6) is equal to $f(x)$ at any point on $-\pi<x<\pi$. That means on $-\pi<x<\pi$,

$$x=2\left\{\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\frac{\sin 4x}{4}+\cdots\right\}$$

Thus when $x = \frac{\pi}{2}$,

$$\frac{\pi}{2} = 2\left\{1 - \frac{1}{3} + \frac{1}{5} - \cdots\right\} \quad \text{whereby} \quad 1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4}.$$

(2) Again (12.6) converges to 0 at $x = \pi$ and at $x = -\pi$, since $f(\pi - 0) = \pi$ and $f(-\pi + 0) = -\pi$.

(3) For other values of x it converges to the periodic extension.

(4) $f(x) = x$ being an odd function has generated a sine series only as expected.

Ex. 2. Find a series of cosines of multiples of x which will represent $f(x) = x$ on the closed interval $0 \le x \le \pi$.

Solution. $f(x) = x$ satisfies Dirichlet's conditions on $0 \le x \le \pi$ by the arguments given in Ex. 1. Hence it can be expanded in Fourier cosine series in the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{where} \quad a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx\,dx = \frac{2}{n^2\pi}(\cos n\pi - 1) = \begin{cases} 0, & n \text{ even} \\ -4/\pi n^2, & n \text{ odd.} \end{cases}$$

Also $f(x) = x$ is continuous on $0 \le x \le \pi$ and hence on $0 \le x \le \pi$,

$$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \frac{\pi}{2} - \frac{4}{\pi}\left\{\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right\}$$

Deduction.

At $x = 0$, $0 = \frac{\pi}{2} - \frac{4}{\pi}\left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right\}$ or, $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

Ex. 3. Find a series of sines of multiples of x which will represent x on $0 \le x < \pi$.

Solution. Let $f(x) = x$ on $0 \le x < \pi$. We may define $f(x)$ at the end point π arbitrarily (See introduction). But for convenience let us define $f(x) = x$ on $0 \le x \le \pi$. Then as in Ex. 1, we can verify that $f(x) = x$ satisfies Dirichlet's conditions on $[0, \pi]$. Hence Fourier sine series corresponding to $f(x) = x$ is

$$\sum_{n=1}^{\infty} b_n \sin nx$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx\,dx = \begin{cases} -2/n, & n \text{ even} \\ 2/n, & n \text{ odd.} \end{cases}$$

and therefore for $0 < x < \pi$, since $f(x) = x$ is continuous there,

$$x = 2\left\{\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right\}.$$

Observe that $f(x) = x$ being continuous gives the value of $f(x)$ at $x = 0$ only since $f(x)$ is zero there, but not at $x = \pi$.

Hence

$$x = 2\left\{\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots\right\} \quad \text{for} \quad 0 \le x < \pi.$$

Ex. 4. Develop $f(x)$ in Fourier series on $-\pi < x < \pi$ if

$$f(x) = 0, \quad \text{for} \quad -\pi < x < 0$$
$$= \pi, \quad \text{for} \quad 0 < x < \pi.$$

Solution. See that $f(x)$ is not defined at $x = 0, \pi, -\pi$ where it can be defined in any manner. For convenience let us take $f(x) = 0$ at $x = -\pi$, 0 and $f(x) = \pi$ at $x = \pi$. Thus $f(x)$ being continuous on $-\pi \le x \le \pi$ except at $x = 0$ where there is an ordinary discontinuity, is bounded and integrable there. Further $f(x)$ is monotone on each of the open intervals $-\pi < x < 0$ and $0 < x < \pi$. Thus $f(x)$ satisfies Dirichlet's condition on $[-\pi, +\pi]$. Now

$$a_0 = \frac{1}{\pi}\left\{\int_{-\pi}^{0} 0 \cdot dx + \int_{0}^{\pi} \pi\, dx\right\} = \pi$$

$$a_n = \frac{1}{\pi}\left\{\int_{-\pi}^{0} 0 \cdot \cos nx\, dx + \int_{0}^{\pi} \pi \cos nx\, dx\right\} = 0$$

$$b_n = \frac{1}{\pi}\left\{\int_{-\pi}^{0} 0 \cdot \sin nx\, dx + \int_{0}^{\pi} \pi \sin nx\, dx\right\}$$

$$= \frac{1}{n}(1 - \cos n\pi) = \begin{cases} 0, & n \text{ even} \\ 2/n, & n \text{ odd.} \end{cases}$$

The Fourier series corresponding to $f(x)$ on $-\pi < x < \pi$ is then

$$\frac{\pi}{2} + 2\left\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right\}.$$

Ex. 5. Expand in Fourier series $x + x^2$ on $-\pi < x < \pi$ and deduce that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots.$$

Solution. Let $f(x) = x + x^2$ on $-\pi < x < \pi$. We may define $f(x)$ at the end points $x = \pm\pi$ arbitrarily. But for convenience we take $f(x) = x + x^2$ on $-\pi \le x \le \pi$. Now $f(x)$ is bounded and integrable on $[-\pi, +\pi]$ because of its continuity on the same closed interval. Further $f'(x) = 1 + 2x$, so that $f'(x) > 0$ for $x > -\frac{1}{2}$ and $f'(x) < 0$ for $x < -\frac{1}{2}$. Thus $f(x)$ is monotone decreasing on $-\pi \le x < -\frac{1}{2}$ and monotone increasing on $-\frac{1}{2} < x \le \pi$ whereby $f(x)$ is piecewise monotone on $-\pi \le x \le \pi$. Hence $f(x)$ satisfies Dirichlet's conditions on $[-\pi, +\pi]$.

Now

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)\, dx = \frac{2}{3}\pi^2.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)\cos nx\, dx = \frac{4}{n^2}\cos n\pi = \begin{cases} 4/n^2, & n \text{ even} \\ -4/n^2, & n \text{ odd.} \end{cases}$$

Similarly $b_n = -2/n$ when n is even and $b_n = 2/n$ when n is odd.

Thus,

$$x + x^2 \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= \frac{\pi^2}{3} - 4\left\{\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right\} + 2\left\{\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right\}.$$

And $x+x^2$ is a continuous function; hence on $-\pi < x < \pi$,

$$x+x^2 = \frac{\pi^2}{3} - 4\left\{\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \dots\right\} + 2\left\{\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots\right\}.$$

Next at $x = \pm\pi$, the sum of the series

$$= \frac{1}{2}\{f(-\pi+0) + f(\pi-0)\} = \frac{1}{2}\{-\pi+\pi^2+\pi+\pi^2\} = \pi^2$$

whereby

$$\pi^2 = \frac{1}{3}\pi^2 - 4\left\{-\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \dots\right\} \quad \text{or,} \quad \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{1}{6}\pi^2.$$

Ex. 6. **Obtain the Fourier series expansion of the function $f(x) = x\sin x$ on $[-\pi, \pi]$. Hence deduce that**

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \dots$$

[C.H. 1998]

Solution. First verify that $f(x) = x\sin x$ satisfies Dirichlet's conditions on $[-\pi, +\pi]$. Also see that $x\sin x$ is an even function of x indicating that it represents a cosine series of the form $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$. And

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\,dx = \frac{1}{\pi}\{[-x\cos x]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \cos x\,dx\}$$

$$= \frac{1}{\pi}\{-\pi\cos\pi - \pi\cos\pi\} = 2.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos nx\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\{\sin(n+1)x - \sin(n-1)x\}dx$$

When $n \neq 1$,

$$a_n = \frac{1}{2\pi}\left\{\left[x\frac{\cos(n-1)x}{n-1} - x\frac{\cos(n+1)x}{n+1}\right]_{-\pi}^{\pi}\right\}$$

$$+ \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right\}dx$$

$$= \frac{1}{2\pi}\left\{2\pi\cdot\frac{\cos(n-1)\pi}{n-1} - 2\pi\frac{\cos(n+1)\pi}{n+1}\right\}$$

$$= (-1)^{n-1}\left\{\frac{1}{n-1} - \frac{1}{n+1}\right\}$$

$$= (-1)^{n-1}\cdot\frac{2}{n^2-1} = -(-1)^n\cdot\frac{2}{n^2-1}, \quad \text{since } \cos n\pi = (-1)^n.$$

When $n = 1$, $a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos x\,dx = -\frac{1}{2}.$

Hence $f(x) \sim 1 - \frac{1}{2}\cos x - 2\sum_{n=2}^{\infty}\frac{(-1)^n\cos nx}{n^2-1}.$

And $f(x) = x\sin x$ is continuous, hence on $-\pi \le x \le \pi$

$$x\sin x = 1 - \frac{1}{2}\cos x - 2\sum_{n=2}^{\infty}\frac{(-1)^n \cos nx}{n^2-1}.$$

Put $x = \frac{\pi}{2}$,

$$\frac{\pi}{2} = 1 + 2\left\{\frac{1}{2^2-1} - \frac{1}{4^2-1} + \frac{1}{6^2-1} - \cdots\right\}.$$

and the result follows.

Ex. 7. Find the Fourier series of the periodic function f with period 2π, where

$$f(x) = 0, \quad -\pi < x < a; \quad f(x) = 1, \quad a \le x \le b; \quad f(x) = 0, \quad b < x < \pi.$$

Find the sum of the series for $x = 4\pi + a$ and deduce that

$$\sum_{n=1}^{\infty}\frac{\sin n(b-a)}{n} = \frac{\pi - b + a}{2}.$$ [C.H. 2000]

Solution. Argue as in Ex. 4 to show that f satisfies Dirichlet's conditions.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_a^b dx = \frac{b-a}{\pi},$$

$$a_n = \frac{1}{\pi}\int_a^b \cos nx\, dx = \frac{1}{n\pi}(\sin bn - \sin an),$$

$$b_n = \frac{1}{\pi}\int_a^b \sin nx\, dx = \frac{1}{n\pi}(\cos an - \cos bn)$$

$$\therefore \quad f(x) \sim \frac{b-a}{2\pi} + \frac{1}{\pi}\left\{\sum_{n=1}^{\infty}\frac{1}{n}(\sin bn - \sin an)\cos nx + \sum_{n=1}^{\infty}\frac{1}{n}(\cos an - \cos bn)\sin nx\right\}$$

Since f satisfies Dirichlet's conditions on convergence, we have

$$f(x) = \frac{b-a}{2\pi} + \frac{1}{\pi}\left\{\sum_{n=1}^{\infty}\frac{1}{n}(\sin bn - \sin an)\cos nx + \sum_{n=1}^{\infty}\frac{1}{n}(\cos an - \cos bn)\sin nx\right\}$$

At $x = 4\pi + a$, that is at $x = a$ (period 2π), the series converges to $\frac{1}{2}\{f(a-0) + f(a+0)\}$ $= \frac{1}{2}(0+1) = \frac{1}{2}$, a being an ordinary discontinuity.

$$\therefore \quad \frac{1}{2} = \frac{b-a}{2\pi} + \frac{1}{\pi}\left\{\sum_{n=1}^{\infty}\frac{1}{n}(\sin bn - \sin an)\cos an + \sum_{n=1}^{\infty}\frac{1}{n}(\cos an - \cos bn)\sin an\right\}$$

$$\text{Thus} \quad \frac{1}{2} - \frac{b-a}{2\pi} = \frac{\pi - b + a}{2\pi} = \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{\sin bn\cos an - \cos bn\sin an}{n}\right).$$

Ex. 8. Compute $f(x)$ in Fourier series on the interval $-2 < x < 2$ if

$$f(x) = 0, \quad \text{for } -2 < x < 0$$
$$\quad = 1, \quad \text{for } \; 0 < x < 2.$$

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Solution. Argue as in Ex 4, and show that $f(x)$ satisfies Dirichlet's conditions. Now,

$$a_0 = \frac{1}{2}\left\{\int_{-2}^{0} 0\cdot dx + \int_0^2 1\cdot dx\right\} = 1$$

$$a_n = \frac{1}{2}\left\{\int_{-2}^{0} 0\cdot \cos\left(\frac{n\pi}{2}x\right) dx + \int_0^2 1\cdot \cos\left(\frac{n\pi}{2}x\right) dx\right\} = 0$$

$$b_n = \frac{1}{2}\left\{\int_{-2}^{0} 0\cdot \sin\left(\frac{n\pi}{2}x\right) dx + \int_0^2 1\cdot \sin\left(\frac{n\pi}{2}x\right) dx\right\}$$

$$= \frac{1}{n\pi}(1-\cos n\pi) = \begin{cases} 0, & n \text{ even} \\ 2/n\pi, & n \text{ odd.}\end{cases}$$

Thus on $-2 < x < 2$, Fourier series corresponding to $f(x)$ is

$$\frac{1}{2} + \frac{2}{\pi}\left\{\sin\frac{\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \frac{1}{5}\sin\frac{5\pi x}{2} + \cdots\right\}.$$

Examples XII

1. Show that on $-\pi \le x \le \pi$, if $f(x)$ be even, $b_n = 0$; and that if $f(x)$ be odd, $a_n = 0$.

2. (i) Obtain Fourier series corresponding to $f(x) = x$ on $[-\pi, +\pi]$ and show that

$$x = 2\left\{\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right\}$$

and hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$.

(ii) Show that Fourier series corresponding to x^2 on $-\pi \le x \le \pi$ is

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$$

and hence deduce that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{1}{6}\pi^2;\quad 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{1}{12}\pi^2;$$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{1}{8}\pi^2.$$

(iii) Prove that $f(x) = x + x^2$ on $-\pi < x < \pi$ has the Fourier series

$$x + x^2 = \frac{\pi^2}{3} - 4\left\{\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right\}$$

$$+2\left\{\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right\}.$$

Deduce that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{1}{6}\pi^2.$$

(iv) If $f(x) = -x$, for $-\pi < x < 0$
$= 0$, for $0 < x < \pi$

then show that Fourier series corresponding to $f(x)$ on $-\pi < x < \pi$ is

$$\frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} + \sum_{n=1}^{\infty}\frac{(-1)^n \sin nx}{n}.$$

(v) State Dirichlet's conditions for convergence of a Fourier series. Prove that if the periodic function (with period 2π),

$$\begin{aligned} f(x) &= -1 \quad \text{for} \quad -\pi < x < 0 \\ &= 0 \quad \text{for} \quad x = 0 \\ &= +1 \quad \text{for} \quad 0 < x < \pi, \end{aligned}$$

then $f(x) = \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$. **[C.H. 1996]**

Deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$.

What is the value of the series for $x = \pm\pi$ and $x = 0$?

(vi) Prove that the even function $f(x) = |x|$ on $-\pi < x < \pi$ has a cosine series in Fourier's form as

$$|x| \sim \frac{\pi}{2} - \frac{4}{\pi}\left\{\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right\}.$$

Apply Dirichlet's conditions of convergence to show that the series converges to $|x|$ throughout $-\pi \le x \le \pi$.

Also show that $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{1}{8}\pi^2$. **[C.H. 1994]**

(vii) Let

$$\begin{aligned} f(x) &= x, & 0 \le x \le \tfrac{\pi}{2}, \\ &= \pi - x, & \tfrac{\pi}{2} \le x \le \pi, \\ &= -f(-x), & -\pi \le x \le 0. \end{aligned}$$

Verify that f satisfies Dirichlet's condition on $[-\pi, \pi]$. Obtain the Fourier series for f in $[-\pi, \pi]$. **[C.H. 1988]**

[**Ans.** $\sum_{n=1}^{\infty} \frac{4}{\pi n^2} \sin \frac{n\pi}{2} \sin nx$]

3. Obtain the Fourier series corresponding to the following functions on $[-\pi, \pi]$:

(i)

$$\begin{aligned} f(x) &= 0, \text{ when } -\pi < x < 0 \\ &= \pi, \text{ when } 0 < x < \pi \end{aligned}$$

[**Ans.** $\frac{\pi}{2} + 2(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots)$]

(ii)

$$\begin{aligned} f(x) &= -\tfrac{\pi}{4}, \text{ when } -\pi \le x \le 0 \\ &= \tfrac{\pi}{4}, \text{ when } 0 < x \le \pi \end{aligned}$$

[**Ans.** $\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots$]

(iii)

$$\begin{aligned} f(x) &= 0 \text{ for } -\pi < x < 0 \\ &= \tfrac{1}{2} \text{ at } x = 0 \\ &= 1 \text{ for } 0 < x < \pi \end{aligned}$$

[**Ans.** $\frac{1}{2} + \frac{2}{\pi}(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots)$]

(iv)

$$\begin{aligned} f(x) &= x, \text{ for } -\pi \le x \le 0 \\ &= 2x, \text{ for } 0 \le x \le \pi \end{aligned}$$

[C.H. 1985]

(v)

$$\begin{aligned} f(x) &= 0, \text{ for } -\pi \le x \le 0 \\ &= \sin x, \text{ for } 0 < x \le \pi \end{aligned}$$

[**Ans.** $\frac{1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2-1}\cos 2nx + \frac{1}{2}\sin x$]

(vi)

$$\begin{aligned} f(x) &= 0, \text{ when } -\pi < x < 0 \\ &= 1, \text{ when } 0 \le x \le \pi. \end{aligned}$$

[C.H. 1993]

[**Ans.** $\frac{1}{2} + \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{1}{2n+1}\sin(2n+1)x$]