V9. $(c+d)\odot\alpha = (c\odot\alpha)\oplus(d\odot\alpha)$ for all $c, d \in F$, all $\alpha \in V$;

V10. $1\odot\alpha = \alpha$, 1 being the identity element in F.

The vector space is denoted by $(V, F, +, ., \oplus, \odot)$. The elements of V are called *vectors* and the elements of F are called *scalars*. F is called the *ground field* (or the *field of scalars*) of the vector space.

Four symbols $+, ., \oplus, \odot$ denote four different compositions $+ : F \times F \to F$; $. : F \times F \to F$; $\oplus : V \times V \to V$; $\odot : F \times V \to V$. We shall dispense with $\oplus$ and use only $+$ to denote both types of addition. We shall dispense with $\odot$ and $.$ both and denote $c.d$ in F by cd and denote $c\odot\alpha$ in V by $c\alpha$.

vector space

Def. Therefore, a non-empty set V is said to form a vector space over a field F if

(i) there is a binary composition $+$ on V, called 'addition', satisfying the conditions

V1. $\alpha+\beta \in V$ for all $\alpha, \beta \in V$,

V2. $\alpha+\beta = \beta+\alpha$ for all $\alpha, \beta \in V$,

V3. $\alpha+(\beta+\gamma) = (\alpha+\beta)+\gamma$ for all $\alpha, \beta, \gamma \in V$,

V4. there exists an element θ in V such that $\alpha+\theta = \alpha$ for all α in V,

, θ is the additive identity element.

V5. for each α in V there exists an element $-\alpha$ in V such that $\alpha+(-\alpha) = \theta$;

and (ii) there is an external composition of F with V, called 'multiplication by the elements of F' satisfying the conditions

V6. $c\alpha \in V$ for all $c \in F$, all $\alpha \in V$,

V7. $c(d\alpha) = (cd)\alpha$ for all $c, d \in F$, all $\alpha \in V$,

V8. $c(\alpha+\beta) = c\alpha + c\beta$ for all $c \in F$, all $\alpha, \beta \in V$,

V9. $(c+d)\alpha = c\alpha + d\alpha$ for all $c, d \in F$, all $\alpha \in V$,

V10. $1\alpha = \alpha$, 1 being the identity element in F.

The elements of V are called *vectors* and the elements of F are called *scalars*. The external composition of F with V is also called, 'multiplication by scalars'.

In particular, V is said to be a *real vector space* (or a *complex vector space*) if the field F be $\mathbb{R}$ (or $\mathbb{C}$).

Since F is a field, it has the zero element 0 and the identity element 1. Since V is a commutative group with respect to addition, the identity element with respect to addition is the zero element in V. This is said to be the *zero vector* or the *null vector* and is denoted by θ. In order to avoid confusion, two zero elements (the scalar zero in F and the vector zero in V) will appear with different symbols. But when no such confusion occurs the symbol 0 will be used for both the zero elements.

VH-99 Ex. Show that a field F may be considered as a vector space over F if scalars multi. is defined with field multiplication.

Real vector space.

A non-empty set V is said to form a *real vector space* (or a vector space over the field $\mathbb{R}$) if

(i) there is a binary composition (+) on V, called 'addition', satisfying the conditions

V1. $\alpha+\beta \in V$ for all $\alpha, \beta \in V$;

V2. $\alpha+\beta = \beta+\alpha$ for all $\alpha, \beta \in V$;

V3. $\alpha+(\beta+\gamma) = (\alpha+\beta)+\gamma$ for all $\alpha, \beta, \gamma \in V$;

V4. there exists an element θ in V such that $\alpha+\theta = \alpha$ for all $\alpha \in V$;

V5. for each α in V there exists an element $-\alpha$ in V such that $\alpha+(-\alpha) = \theta$;

and (ii) there is an external composition of $\mathbb{R}$ with V, called 'multiplication by real numbers' satisfying the conditions

V6. $c\alpha \in V$ for all $c \in \mathbb{R}$, all $\alpha \in V$;

V7. $c(d\alpha) = (cd)\alpha$ for all $c, d \in \mathbb{R}$, all $\alpha \in V$;

V8. $c(\alpha+\beta) = c\alpha + c\beta$ for all $c \in \mathbb{R}$, all $\alpha, \beta \in V$;

V9. $(c+d)\alpha = c\alpha + d\alpha$ for all $c, d \in \mathbb{R}$, all $\alpha \in V$;

V10. $1\alpha = \alpha$, 1 being the identity element in $\mathbb{R}$.

The elements of V are called *vectors* and the elements of $\mathbb{R}$ are called *scalars*. $\mathbb{R}$ is said to be the *ground field* (or the *field of scalars*) of the vector space V.

Examples.

1. **Real vector space $\mathbb{R}^n$.** Let V be the set of all ordered n-tuples $\{(a_1, a_2, \ldots, a_n) : a_i \in \mathbb{R}\}$.

Let + be a composition on V, called 'addition', defined by
$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1+b_1, a_2+b_2, \cdots, a_n+b_n)$
and an external composition of $\mathbb{R}$ with V, called 'multiplication by real numbers' be defined by
$c(a_1, a_2, \ldots, a_n) = (ca_1, ca_2, \ldots, ca_n), c \in \mathbb{R}$.

Then the conditions V1-V10 are satisfied. Therefore V is a real vector space and it is denoted by $\mathbb{R}^n$.

$(0, 0, \ldots, 0)$ is the null vector of $\mathbb{R}^n$ and it is denoted by θ.

In a similar manner the vector spaces $\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^4, \ldots$ are defined. The set $\mathbb{R}$ itself forms a real vector space.

2. **Real vector space $\mathbb{C}$.** $\mathbb{C}$ is the set of all complex numbers $\{a+ib : a \in \mathbb{R}, b \in \mathbb{R}, i = \sqrt{(-1)}\}$.

Let + be a composition on $\mathbb{C}$, called 'addition' defined by
$$(a+ib)+(c+id) = (a+c)+i(b+d);$$

4.3. Sub-spaces.

Let V be a vector space over a field F with respect to addition (+) and multiplication by elements of F.

Let W be a non-empty subset of V. If W be stable under + and ., then the restriction of + to $W \times W$ is a mapping from $W \times W$ to W and the restriction of . to $F \times W$ is a mapping from $F \times W$ to W. The restriction of +, say $\oplus$, is a composition on W and is defined by $\alpha \oplus \beta = \alpha + \beta$ for all $\alpha, \beta \in W$. The restriction of ., say $\odot$, is an external composition of F with W and is defined by $c \odot \alpha = c.\alpha$ for all $c \in F$ and all $\alpha \in W$.

If W forms a vector space over F with respect to $\oplus$ and $\odot$, then W is said to be a *sub-vector space* or a *linear subspace* or a *subspace of* V.

Theorem 4.3.1. A non-empty subset W of a vector space V over a field F is a subspace of V if and only if

(i) $\alpha \in W,\ \beta \in W \Rightarrow \alpha + \beta \in W$; and (ii) $\alpha \in W,\ c \in F \Rightarrow c\alpha \in W$.

Proof. Let the conditions hold in W.

Let $\alpha, \beta \in W$. Since F is a field, $-1 \in F$ where 1 is the identity element in F. By (ii) $-1\beta \in W$, i.e., $-\beta \in W$.

Then by (i) $\alpha + (-\beta) \in W$, i.e., $\alpha - \beta \in W$.

Thus $\alpha, \beta \in W \Rightarrow \alpha - \beta \in W$.

This proves that W is a subgroup of the additive group V. Since V is a commutative group, W is also a commutative subgroup of V.

Therefore the conditions **V1-V5** for a vector space are satisfied in W. **V6** is satisfied in W by(ii). The conditions **V7-V10** are satisfied in W since they are hereditary properties. Thus W is by itself a vector space over F and so W is a subspace of V.

The necessity of the conditions (i) and (ii) follows from the definition of a vector space.

Note. The two conditions (i) and (ii) can also be expressed as the single condition $a\alpha + b\beta \in W$ for all $\alpha, \beta \in W$ and all $a, b \in F$.

Examples.

1. Let V be a vector space over a field F. Then V itself is a subspace of V. This subspace is called the *improper subspace* of V.

The set consisting only of the null vector θ of V forms a subspace of V. This subspace is called the *trivial subspace* of V.

2. Let S be the subset of $\mathbb{R}^3$ defined by $S = \{(x, y, z) \in \mathbb{R}^3 : y = z = 0\}$. Then S is a non-empty subset of $\mathbb{R}^3$, since $(0, 0, 0) \in S$.

Let $\alpha = (x_1, 0, 0)$, $\beta = (x_2, 0, 0) \in S$; Then x_1, x_2 are real.

Let $c \in \mathbb{R}, d \in \mathbb{R}$. Then $c\alpha + d\beta = c(x_1, 0, 0) + d(x_2, 0, 0) = (cx_1 + dx_2, 0, 0) \in S$, since $cx_1 + dx_2 \in \mathbb{R}$.

This proves that S is a subspace of $\mathbb{R}^3$.

3. Let T be the subset defined by $T = \{(x, y, z) \in \mathbb{R}^3 : x = z = 0\}$. Then T is a subspace of $\mathbb{R}^3$.

4. Let U be the subset defined by $U = \{(x, y, z) \in \mathbb{R}^3 : x = y = 0\}$. Then U is a subspace of $\mathbb{R}^3$.

5. Let S be the subset defined by $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2\}$. Then S is a non-empty subset of $\mathbb{R}^3$, since $(0, 0, 0) \in S$.

Let $\alpha = (x_1, y_1, z_1) \in S$, $\beta = (x_2, y_2, z_2) \in S$. Then $x_i, y_i, z_i \in \mathbb{R}$ and $x_1^2 + y_1^2 = z_1^2$, $x_2^2 + y_2^2 = z_2^2$.

$\alpha + \beta = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$. $\alpha + \beta$ may not belong to S because $(x_1 + x_2)^2 + (y_1 + y_2)^2$ may not be equal to $(z_1 + z_2)^2$.

For example, let $\alpha = (3, -4, 5)$, $\beta = (-3, 4, 5)$. Then $\alpha \in S$, $\beta \in S$ but $\alpha + \beta = (0, 0, 10) \notin S$.

So S is not a subspace of $\mathbb{R}^3$.

6. Let S be the subset defined by $S = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 1\}$.

S is a non-empty subset of $\mathbb{R}^3$ since $(0, 0, 1) \in S$. But S does not contain the null vector $(0, 0, 0)$. S is not a subspace of $\mathbb{R}^3$, since every subspace W of a vector space V must contain the null vector θ of V.

Theorem 4.3.2. The intersection of two subspaces of a vector space V over a field F is a subspace of V.

Proof. Let W_1 and W_2 be two subspaces of V. $W_1 \cap W_2$ is not empty because $\theta \in W_1 \cap W_2$.

Case 1. Let $W_1 \cap W_2 = \{\theta\}$. Then $W_1 \cap W_2$ is a subspace of V.

Case 2. Let $W_1 \cap W_2 \neq \{\theta\}$ and let $\alpha_1 \in W_1 \cap W_2$, $\alpha_2 \in W_1 \cap W_2$.

Then $\alpha_1, \alpha_2 \in W_1$ and $\alpha_1, \alpha_2 \in W_2$.

Since W_1 is a subspace of V, (i) $\alpha_1 + \alpha_2 \in W_1$ and (ii) $c\alpha_1 \in W_1$, c being a scalar in F.

Since W_2 is a subspace of V, (i) $\alpha_1 + \alpha_2 \in W_2$ and (ii) $c\alpha_1 \in W_2$, c being a scalar in F.

Therefore $\alpha_1 + \alpha_2 \in W_1 \cap W_2$ and $c\alpha_1 \in W_1 \cap W_2$.

This proves that $W_1 \cap W_2$ is a subspace of V.

Note 1. Since $W_1 \cap W_2$ is the largest subset contained in both of W_1 and W_2, $W_1 \cap W_2$ is the largest subspace contained in W_1 and W_2.

Note 2. The intersection of a family of subspaces of V is a subspace of V.

Note 3. The union of two subspaces of V is not, in general, a subspace of V.

For example, let us consider two subspaces S and T of the vector space $\mathbb{R}^3$ where $S = \{(x,y,z) \in \mathbb{R}^3 : y = 0, z = 0\}$, $T = \{(x,y,z) \in \mathbb{R}^3 : x = 0, z = 0\}$.

Let $\alpha = (1,0,0) \in S$, $\beta = (0,1,0) \in T$. Then $\alpha + \beta = (1,1,0)$.

Here $\alpha \in S \cup T$, $\beta \in S \cup T$ but $\alpha + \beta \notin S$, $\alpha + \beta \notin T$ and therefore $\alpha + \beta \notin S \cup T$.

Hence $S \cup T$ is not a subspace of $\mathbb{R}^3$.

Theorem 4.3.3. If U and W be two subspaces of a vector space V over a field F, then the union $U \cup W$ is a subspace of V if and only if either $U \subset W$ or $W \subset U$.

Proof. Let $U \cup W$ be a subspace of V. We prove that either $U \subset W$ or $W \subset U$, i.e., either $U - W = \phi$ or $W - U = \phi$.

Let us assume that both $U - W \neq \phi$ and $W - U \neq \phi$. Then there exists a vector α such that $\alpha \in U$, but $\alpha \notin W$ and a vector β such that $\beta \in W$, but $\beta \notin U$.

$\alpha \in U \Rightarrow \alpha \in U \cup W$ and $\beta \in W \Rightarrow \beta \in U \cup W$.

Since $U \cup W$ is a subspace of V, $\alpha + \beta \in U \cup W$. This implies $\alpha + \beta \in U$ or $\alpha + \beta \in W$.

$\alpha + \beta \in U$ and $\alpha \in U \Rightarrow (\alpha + \beta) - \alpha \in U$, since U is a subspace
$\Rightarrow \beta \in U$, a contradiction.

$\alpha + \beta \in W$ and $\beta \in W \Rightarrow (\alpha + \beta) - \beta \in W$, since W is a subspace
$\Rightarrow \alpha \in W$, a contradiction.

Therefore $\alpha + \beta \notin U$, $\alpha + \beta \notin W$ and therefore $\alpha + \beta \notin U \cup W$.

So our assumption that both $U - W \neq \phi$ and $W - U \neq \phi$ is not tenable and therefore either $U - W = \phi$ or $W - U = \phi$, i.e., either $U \subset W$ or $W \subset U$.

Conversely, let U and W be subspaces of V such that either $U \subset W$ or $W \subset U$.

If $U \subset W$ then $U \cup W = W$ and it is a subspace of V.
If $W \subset U$ then $U \cup W = U$ and it is a subspace of V.

Therefore in any case, $U \cup W$ is a subspace of V.
This proves the theorem.

Note. A vector space V cannot be the union of two proper subspaces.

Linear sum of two subspaces.

Let U and W be two subspaces of a vector space V over a field F. Then the subset $\{u + w : u \in U, w \in W\}$ is said to be the *linear sum* of the subspaces U and W.

Let $\alpha \in U$. Then $\alpha = \alpha + \theta$, where $\alpha \in U, \theta \in W$. This shows that $\alpha \in U + W$. Therefore $U \subset U + W$.

Let $\alpha \in W$. Then $\alpha = \theta + \alpha$ where $\theta \in U, \alpha \in W$. This shows that $\alpha \in U + W$. Therefore $W \subset U + W$.

Theorem 4.3.4. Let U and W be two subspaces of a vector space V over a field F. Then the linear sum $U + W$ is a subspace of V.

Proof. Let $S = U + W = \{u + w : u \in U, w \in W\}$.
$\theta \in U, \theta \in W \Rightarrow \theta \in S$ and therefore S is non-empty.

Let $\alpha_1, \alpha_2 \in S$. Then $\alpha_1 = u_1 + w_1$ for some $u_1 \in U, w_1 \in W$;
$\alpha_2 = u_2 + w_2$ for some $u_2 \in U, w_2 \in W$.

$\alpha_1 + \alpha_2 = (u_1 + u_2) + (w_1 + w_2) \in S$, since $u_1 + u_2 \in U$, $w_1 + w_2 \in W$.

Let c be a scalar in F.

Then $c\alpha_1 = c(u_1 + w_1) = cu_1 + cw_1 \in S$, since $cu_1 \in U$, $cw_1 \in W$.

Therefore $\alpha_1 \in S$, $\alpha_2 \in S \Rightarrow \alpha_1 + \alpha_2 \in S$; and $c \in F$, $\alpha_1 \in S \Rightarrow c\alpha_1 \in S$.

This proves that S is a subspace of V, i.e., $U + W$ is a subspace of V.

Theorem 4.3.5. The subspace $U + W$ is the smallest subspace of V containing the subspaces U and W.

Proof. Let P be any subspace of V containing the subspaces U and W.
Let α be an element of $U + W$.
Then $\alpha = u_1 + w_1$ for some $u_1 \in U$, $w_1 \in W$.

Since $U \subset P, u_1 \in P$ and since $W \subset P$, $w_1 \in P$.
Since P is a subspace of V and $u_1, w_1 \in P$, $u_1 + w_1 \in P$, i.e., $\alpha \in P$.

Thus $\alpha \in U + W \Rightarrow \alpha \in P$ and therefore $U + W \subset P$.

This proves that $U + W$ is the smallest subspace containing U and V.

Some important subspaces of a vector space.

1. Let V be a vector space over a field F and let $\alpha \in V$. Then the set $W = \{c\alpha : c \in F\}$ forms a subspace of V

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Case 1. Let $\alpha = \theta$. Then $W = \{\theta\}$ and W is a subspace of V.

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Case 2. Let $\alpha \neq \theta$.

W is a non-empty subset of V, since ~~$\alpha \in W$~~. $\theta (= 0 \cdot \alpha) \in W$

Let $\alpha_1, \alpha_2 \in W$. Then $\alpha_1 = c_1\alpha, \alpha_2 = c_2\alpha$ for some scalars c_1, c_2 in F.

$\alpha_1 + \alpha_2 = c_1\alpha + c_2\alpha = (c_1 + c_2)\alpha \in W$, since $c_1 + c_2 \in F$.

Therefore $\alpha_1 \in W, \alpha_2 \in W \Rightarrow \alpha_1 + \alpha_2 \in W$ ... (i)

Let p be a scalar in F.

Then $p\alpha_1 = p(c_1\alpha) = (pc_1)\alpha \in W$, since $pc_1 \in F$.

Therefore $p \in F, \alpha_1 \in W \Rightarrow p\alpha_1 \in W$ ... (ii)

From (i) and (ii) it follows that W is subspace of V.

Note. This subspace is said to be *generated by* the vector α and α is said to be a *generator* of the subspace W.

(2) Let V be a vector space over a field F and let $\alpha, \beta \in V$. Then the set $W = \{c\alpha + d\beta : c \in F, d \in F\}$ forms a subspace of V.

W is a non-empty subset of V, since $\theta (= 0\alpha + 0\beta) \in W$.

Let $\alpha_1 = c_1\alpha + d_1\beta \in W, \alpha_2 = c_2\alpha + d_2\beta \in W$, where $c_1, d_1, c_2, d_2 \in F$.

$\alpha_1 + \alpha_2 = (c_1\alpha + d_1\beta) + (c_2\alpha + d_2\beta) = (c_1 + c_2)\alpha + (d_1 + d_2)\beta \in W$ since $c_1 + c_2 \in F, d_1 + d_2 \in F$.

Therefore $\alpha_1 \in W, \alpha_2 \in W \Rightarrow \alpha_1 + \alpha_2 \in W$ ... (i)

Let p be a scalar in F.

Then $p\alpha_1 = p(c_1\alpha + d_1\beta)$
$= (pc_1)\alpha + (pd_1)\beta \in W$, since $pc_1 \in F, pd_1 \in F$.

Therefore $p \in F, \alpha_1 \in W \Rightarrow p\alpha_1 \in W$ ... (ii)

From (i) and (ii) it follows that W is a subspace of V.

Note. This subspace is said to be generated by the vectors α, β. The set $\{\alpha, \beta\}$ is said to be a *generating set* of the subspace W.

(3) Let V be a vector space over a field F and let $\alpha_1, \alpha_2, \ldots, \alpha_r \in V$. Then the set $W = \{c_1\alpha_1 + c_2\alpha_2 + \cdots + c_r\alpha_r : c_i \in F\}$ forms a subspace of V.

The set of vectors $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ is said to be a *generating set* of the subspace W.

Definition. Let V be a vector space over a field F. Let $\alpha_1, \alpha_2, \ldots, \alpha_r \in V$. A vector β in V is said to be a *linear combination* of the vectors $\alpha_1, \alpha_2, \ldots, \alpha_r$ if β can be expressed as
$\beta = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_r\alpha_r$ for some scalars $c_1, c_2, \ldots, c_r$ in F.

Example.

Let V be a real vector space and $\alpha, \beta, \gamma \in V$. Then $\alpha + \beta + \gamma$, $\alpha + 2\beta + 3\gamma, 0\alpha + \beta + 0\gamma, 0\alpha + 0\beta + \gamma, 0\alpha + 0\beta + 0\gamma$ are linear combinations of α, β, γ.

Theorem 4.3.6. Let V be a vector space over a field F and let S be a non-empty finite subset of V. Then the set W of all linear combinations of the vectors in S forms a subspace of V and this is the smallest subspace containing the set S.

Proof. Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Then $W = \{c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n : c_1, c_2, \ldots, c_n \in F\}$.

W is a non-empty subset of V, since $\alpha_1 \in W$.

Let $\alpha = r_1\alpha_1 + r_2\alpha_2 + \cdots + r_n\alpha_n \in W$, $\beta = s_1\alpha_1 + s_2\alpha_2 + \cdots + s_n\alpha_n \in W$. Then $r_1, r_2, \ldots, r_n \in F$, $s_1, s_2, \ldots, s_n \in F$.

$\alpha + \beta = (r_1\alpha_1 + r_2\alpha_2 + \cdots + r_n\alpha_n) + (s_1\alpha_1 + s_2\alpha_2 + \cdots + s_n\alpha_n)$
$= (r_1 + s_1)\alpha_1 + (r_2 + s_2)\alpha_2 + \cdots + (r_n + s_n)\alpha_n \in W$, since $r_i + s_i \in F$.

Therefore $\alpha \in W, \beta \in W \Rightarrow \alpha + \beta \in W$ ... (i)

Let $p \in F$.

Then $p\alpha = p(r_1\alpha_1 + r_2\alpha_2 + \cdots + r_n\alpha_n)$
$= (pr_1)\alpha_1 + (pr_2)\alpha_2 + \cdots + (pr_n)\alpha_n \in W$, since $pr_i \in F$.

Therefore $p \in F, \alpha \in W \Rightarrow p\alpha \in W$ ... (ii)

From (i) and (ii) it follows that W is a subspace of V.

Let P be any subspace of V containing the set S.

Let $\xi \in W$. Then $\xi = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n$ for some scalars $x_i \in F$.

Since P is a subspace of V containing α_i and $x_i \in F$, it follows that $x_i\alpha_i \in P$ for $i = 1, 2, \ldots, n$.

Since P is a subspace and $x_1\alpha_1, x_2\alpha_2, \ldots, x_n\alpha_n \in P$, it follows that $x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n \in P$, i.e., $\xi \in P$.

Thus $\xi \in W \Rightarrow \xi \in P$ and therefore $W \subset P$.

This proves that W is the smallest subspace containing S.

Note. The smallest subspace containing S is the intersection of all subspaces containing S.

Definition. The smallest subspace containing a finite set S of vectors of a vector space V is said to be the *linear span* of S and is denoted by $L(S)$. $L(S)$ is said to be generated (or spanned) by the set S and S is said to be the *generating set* (or *spanning set*) of $L(S)$.

Note. If S be a non-empty finite set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, then $L(S)$ is the set of all linear combinations of the vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$.

If $S = \phi$, then $L(S) = \{\theta\}$, since $\{\theta\}$ is the intersection of all subspaces containing ϕ.

Theorem 4.3.7. If S and T be two non-empty finite subsets of a vector space V over a field F and $S \subset T$, then $L(S) \subset L(T)$.

Proof. Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and let $\xi \in L(S)$.

Then $\xi = r_1\alpha_1 + r_2\alpha_2 + \cdots + r_n\alpha_n$ for some scalars $r_i \in F$.

$\alpha_i \in S \Rightarrow \alpha_i \in T$ and therefore $\alpha_i \in L(T)$.

Since $L(T)$ is a subspace of $V, r_i \in F, \alpha_i \in L(T) \Rightarrow r_i\alpha_i \in L(T)$.

Similarly, $r_2\alpha_2 \in L(T), \ldots, r_n\alpha_n \in L(T)$. $i=1,2,\ldots,n$

Since $L(T)$ is a subspace, it also follows that

$r_1\alpha_1 + r_2\alpha_2 + \cdots + r_n\alpha_n \in L(T)$, i.e., $\xi \in L(T)$.

Thus $\xi \in L(S) \Rightarrow \xi \in L(T)$ and therefore $L(S) \subset L(T)$.

Theorem 4.3.8. If S and T be two non-empty finite subsets of a vector space V over a field F and each element of T is a linear combination of the vectors of S, then $L(T) \subset L(S)$.

Proof. Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, $T = \{\beta_1, \beta_2, \ldots, \beta_m\}$ and let

$\beta_i = c_{i1}\alpha_1 + c_{i2}\alpha_2 + \cdots + c_{in}\alpha_n$ for some $c_{ij} \in F$, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$.

Let ξ be an element of $L(T)$.

Then $\xi = p_1\beta_1 + p_2\beta_2 + \cdots + p_m\beta_m$ for some scalars $p_i \in F$.

$\xi = p_1(c_{11}\alpha_1 + c_{12}\alpha_2 + \cdots + c_{1n}\alpha_n) + p_2(c_{21}\alpha_1 + c_{22}\alpha_2 + \cdots + c_{2n}\alpha_n) + \cdots + p_m(c_{m1}\alpha_1 + c_{m2}\alpha_2 + \cdots + c_{mn}\alpha_n)$

$= d_1\alpha_1 + d_2\alpha_2 + \cdots + d_n\alpha_n \in L(S)$, since $d_i = p_1c_{1i} + p_2c_{2i} + \cdots + p_mc_{mi} \in F; i = 1, 2, \ldots, n$.

Thus $\xi \in L(T) \Rightarrow \xi \in L(S)$ and therefore $L(T) \subset L(S)$.

Note. The theorem says that if $T \subset L(S)$ then $L(T) \subset L(S)$.

Worked Examples.

1. In $\mathbb{R}^3$, $\alpha = (4,3,5)$, $\beta = (0,1,3)$, $\gamma = (2,1,1)$, $\delta = (4,2,2)$.

Examine if (i) α is a linear combination of β and γ,

(ii) β is a linear combination of γ and δ.

(i) Let $\alpha = c\beta + d\gamma$, where $c, d \in \mathbb{R}$.

Then $(4,3,5) = c(0,1,3) + d(2,1,1) = (0 + 2d, c + d, 3c + d)$.

Therefore $2d = 4, c + d = 3, 3c + d = 5$ giving $c = 1, d = 2$.

Hence $\alpha = \beta + 2\gamma$ and α is a linear combination of β and γ.

(ii) Let $\beta = c\gamma + d\delta$ where $c, d \in \mathbb{R}$.

Then $(0,1,3) = c(2,1,1) + d(4,2,2) = (2c + 4d, c + 2d, c + 2d)$.

Therefore $2c + 4d = 0, c + 2d = 1, c + 2d = 3$.

The equations are inconsistent. Therefore β cannot be expressed as $c\gamma + d\delta$ for real c, d. Hence β is not a linear combination of γ and δ.

2. Determine the subspace of $\mathbb{R}^3$ spanned by the vectors $\alpha = (1,2,3)$, $\beta = (3,1,0)$. Examine if

(i) $\gamma = (2,1,3)$ is in the subspace,

(ii) $\delta = (-1,3,6)$ is in the subspace.

$L\{\alpha, \beta\}$ is the set of vectors $\{c\alpha + d\beta : c \in \mathbb{R}, d \in \mathbb{R}\}$.

$c\alpha + d\beta = c(1,2,3) + d(3,1,0) = (c + 3d, 2c + d, 3c)$.

(i) If $\gamma \in L\{\alpha, \beta\}$ then there must be real numbers c, d such that $(2,1,3) = (c + 3d, 2c + d, 3c)$.

Therefore $c + 3d = 2, 2c + d = 1$ and $3c = 3$.

These equations are inconsistent and so γ is not in $L\{\alpha, \beta\}$.

(ii) If $\delta \in L\{\alpha, \beta\}$, then there must be real numbers c, d such that $(-1,3,6) = (c + 3d, 2c + d, 3c)$

Therefore $c + 3d = -1, 2c + d = 3$ and $3c = 6$, giving $c = 2, d = -1$.

Therefore $\delta = 2(1,2,3) - 1(3,1,0)$, showing that $\delta \in L\{\alpha, \beta\}$.

3. Let $S = \{\alpha, \beta, \gamma\}$, $T = \{\alpha, \beta, \alpha + \beta, \beta + \gamma\}$ be subsets of a real vector space V. Show that $L(S) = L(T)$.

S and T are finite subsets of V and each element of T is a linear combination of the vectors of S and therefore $L(T) \subset L(S)$.

Again $\alpha = \alpha + 0\beta + 0(\alpha + \beta) + 0(\beta + \gamma)$,

$\beta = 0\alpha + \beta + 0(\alpha + \beta) + 0(\beta + \gamma)$

and $\gamma = 0\alpha - \beta + 0(\alpha + \beta) + (\beta + \gamma)$.

This shows that each element of S is a linear combination of the vectors of T, and therefore $L(S) \subset L(T)$.

It follows that $L(S) = L(T)$.

Definition. Linear span of a set

Let S be a non-empty finite subset of a vector space V over a field F. The set of all *finite* linear combinations of the elements of S is said to be the *linear span* of S and it is denoted by $L(S)$. and S is said to be generating set (or spanning set) of L(S). as L(S) is spanned by the vectors of S.

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Exercises 6

1. Prove that in a real vector space V

   (i) $c(\alpha - \beta) = c\alpha - c\beta$, where $c \in \mathbb{R}$; $\alpha, \beta \in V$;

   (ii) $(c - d)\alpha = c\alpha - d\alpha$, where $c, d \in \mathbb{R}$; $\alpha \in V$;

   (iii) $(-c)\alpha = -(c\alpha) = c(-\alpha)$, where $c, d \in \mathbb{R}$; $\alpha, \beta \in V$.

2. Examine if the set S is a subspace of $\mathbb{R}^3$.

   (i) $S = \{(x, y, z) \in \mathbb{R}^3 : x = 0\}$;

   (ii) $S = \{(x, y, z) \in \mathbb{R}^3 : x = 1\}$;

   (iii) $S = \{(x, y, z) \in \mathbb{R}^3 : xy = z\}$;

   (iv) $S = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$;

   (v) $S = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 1\}$;

   (vi) $S = \{(x, y, z) \in \mathbb{R}^3 : x + 2y - z = 0, 2x - y + z = 0\}$; — vn10

   (vii) $S = \{(x, y, z) \in \mathbb{R}^3 : x + 2y - z = 1, 2x - y + z = 2\}$.

3. If $\alpha = (1, 1, 2), \beta = (0, 2, 1), \gamma = (2, 2, 4)$, determine whether

   (i) α is a linear combination of β and γ,

(ii) β is a linear combination of γ and α,

(iii) γ is a linear combination of α and β.

4. In $\mathbb{R}^2$, $\alpha = (3, 1), \beta = (2, -1)$. Determine $L\{\alpha, \beta\}$ and show that $L\{\alpha, \beta\} =$ $\mathbb{R}^2$. p-646 133

5. In $\mathbb{R}^3$, $\alpha = (1, 3, 0), \beta = (2, 1, -2)$. Determine $L\{\alpha, \beta\}$. Examine if $\gamma =$ $(-1, 3, 2), \delta = (4, 7, -2)$ are in $L\{\alpha, \beta\}$.

6. Let $\alpha_1, \alpha_2, \alpha_3$ are vectors in a real vector space V such that $\alpha_1 + \alpha_2 + \alpha_3 = \theta$. Prove that $L\{\alpha_1, \alpha_2\} = L\{\alpha_2, \alpha_3\} = L\{\alpha_3, \alpha_1\}$.

7. Let $S = \{\alpha, \beta, \gamma\}, P = \{\alpha, \alpha + \beta, \alpha + \beta + \gamma\}, T = \{\alpha + \beta, \beta + \gamma, \gamma + \alpha\}$ be subsets in a real vector space V. Prove that $L(S) = L(T) = L(P)$.

8. If $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are vectors in a real vector space V such that $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 = \theta$, where c_1, c_2, c_3, c_4 are real numbers with $c_1c_4 \neq 0$. Prove that $L\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = L\{\alpha_2, \alpha_3, \alpha_4\} = L\{\alpha_1, \alpha_2, \alpha_3\}$.

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9. Prove that the set $C[a, b]$ of all real valued continuous functions defined on the closed interval $[a, b]$ forms a real vector space if

(i) addition is defined by $(f + g)(x) = f(x) + g(x), f, g \in C[a, b]$,

(ii) multiplication by a real number r is defined by $(r.f)(x) = rf(x), f \in$ $C[a, b]$.

10. Prove that the subset $D[a, b]$ of all real valued differentiable functions defined on $[a, b]$ is a subspace of $C[a, b]$.

11. Examine if the set S is a subspace of the vector space $\mathbb{R}_{2\times2}$, where

(i) $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}_{2\times2} : a + b = 0 \right\}$;

(ii) $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}_{2\times2} : a + b + c + d = 0 \right\}$;

(iii) $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}_{2\times2} : \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0 \right\}$;

(iv) S is the set of all 2×2 real diagonal matrices;

(v) S is the set of all 2×2 real symmetric matrices;

(vi) S is the set of all 2×2 real skew symmetric matrices;

(vii) S is the set of all 2×2 real upper triangular matrices;

(viii) S is the set of all 2×2 real lower triangular matrices.

12. Show that the set S is a subspace of the vector space $C[0, 1]$, where (i) $S = \{f \in C[0, 1] : f(0) = 0\}$; (ii) $S = \{f \in C[0, 1] : f(0) = 0, f(1) = 0\}$.

13. If a vector space V is the set of real valued continuous functions over $\mathbb{R}$, then show that the set W of solutions of $2\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 2y = 0$ is a subspace of V.

4.4. Linear dependence and Linear independence.

A *finite set* of vectors $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of a vector space V over a field F is said to be *linearly dependent* in V if there exist scalars $c_1, c_2, \ldots, c_n$, not all zero, in F such that

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = \theta \quad \cdots \quad \text{(i)}$$

The set is said to be *linearly independent* in V if the equality (i) is satisfied only when $c_1 = c_2 = \cdots = c_n = 0$.

Def:- An *arbitrary set* S of vectors of a vector space V over a field F is said to be linearly dependent in V if there exists a *finite* subset of S which is linearly dependent in V. Otherwise S is L.I.

Def: A set of vectors which is not linearly dependent is said to be linearly independent.

Although the definition of linear dependence or linear independence refers to a set of vectors, we shall also state that the individual vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly dependent or independent.

Theorem 4.4.1. A superset of a linearly dependent set of vectors in a vector space V over a field F is linearly dependent.

Proof. **Case (i).** Let S be a linearly dependent set of vectors containing a finite number of elements $\alpha_1, \alpha_2, \ldots, \alpha_n$. Let T be a superset of S. Now S being a finite subset of T and being linearly dependent, T is linearly dependent, by definition.

Case (ii). Let S be a linearly dependent set of vectors containing an infinite number of elements and T be a superset of S. Since S is linearly dependent, there exists a finite subset P of S such that P is linearly dependent. Now P being a linearly dependent finite subset of T, T is linearly dependent, by definition.
This completes the proof.

Theorem 4.4.2. A subset of a linearly independent set of vectors in a vector space V over a field F is linearly independent.

Proof. Let S be a linearly independent set of vectors and P be a subset of S. If P be linearly dependent then S being a superset of the linearly dependent set P, must be linearly dependent, by Theorem 4.4.1. But it is not so. This proves that P is linearly independent.

Note. The set ϕ is linearly independent.

Theorem 4.4.3. A set of vectors containing the null vector θ in a vector space V over a field F is linearly dependent.

Proof. Let $S = \{\theta\}$. The set S is linearly dependent since the relation $c\theta = \theta$ holds for a non-zero scalar c.

Let T be an arbitrary set of vectors containing the null vector θ. Then T being a superset of S, is linearly dependent by Theorem 4.4.1.

Theorem 4.4.4. The set consisting of a single non-zero vector α in a vector space V over a field F is linearly independent.

The proof follows from the Theorem 4.2.1.(iv).

Worked Examples.

1. Examine if the set of vectors $\{(2,1,1)(1,2,2),(1,1,1)\}$ is linearly dependent in $\mathbb{R}^3$.

Let $\alpha = (2,1,1)$, $\beta = (1,2,2)$, $\gamma = (1,1,1)$.

Let us consider the relation $c_1\alpha + c_2\beta + c_3\gamma = \theta$, where c_1, c_2, c_3 are real numbers.

Then $c_1(2,1,1) + c_2(1,2,2) + c_3(1,1,1) = (0,0,0)$.

Therefore $2c_1 + c_2 + c_3 = 0$, $c_1 + 2c_2 + c_3 = 0$, $c_1 + 2c_2 + c_3 = 0$,

or equivalently, $2c_1 + c_2 + c_3 = 0$, $c_1 + 2c_2 + c_3 = 0$.

The solution is $c_1 = -k$, $c_2 = -k$, $c_3 = 3k$, where k is a real number.

Since k is arbitrary, there exist c_1, c_2, c_3, not all zero, such that $c_1\alpha + c_2\beta + c_3\gamma = \theta$. For example, $c_1 = 1, c_2 = 1, c_3 = -3$.

Therefore the set of vectors is linearly dependent.

2. Prove that the set of vectors $\{(1,2,2),(2,1,2),(2,2,1)\}$ is linearly independent in $\mathbb{R}^3$.

Let $\alpha = (1,2,2)$, $\beta = (2,1,2)$, $\gamma = (2,2,1)$.

Let us consider the relation $c_1\alpha + c_2\beta + c_3\gamma = \theta$, where c_1, c_2, c_3 are real numbers.

Then $c_1(1,2,2) + c_2(2,1,2) + c_3(2,2,1) = (0,0,0)$.

Therefore $c_1 + 2c_2 + 2c_3 = 0$, $2c_1 + c_2 + 2c_3 = 0$, $2c_1 + 2c_2 + c_3 = 0$.

This is a homogeneous system of three equations in c_1, c_2, c_3.

The co-efficient determinant of the system is $\begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = 5 \neq 0$.

By Cramer's rule, there exists a unique solution for c_1, c_2, c_3 and the solution is $c_1 = 0, c_2 = 0, c_3 = 0$.

for unique solⁿ apply it must

This proves that the set of vectors is linearly independent.

Theorem 4.4.5. If the set of vectors $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ in a vector space V over a field F be linearly dependent, then at least one of the vectors of the set can be expressed as a linear combination of the remaining others.

Conversely, if one of the vectors of the set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a linear combination of the remaining others, the set is linearly dependent.

Proof. Since the set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is linearly dependent, there exist scalars $c_1, c_2, \ldots, c_n$ in F, not all zero, such that

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = \theta.$$

Let $c_j \neq 0$. Then $c_j^{-1} \in F$ and $c_j^{-1}c_j = 1$, 1 being the identity element in F.

Now $c_j\alpha_j = -c_1\alpha_1 - c_2\alpha_2 - \cdots - c_{j-1}\alpha_{j-1} - c_{j+1}\alpha_{j+1} - \cdots - c_n\alpha_n$.

Therefore $\alpha_j = c_j^{-1}[-c_1\alpha_1 - c_2\alpha_2 - \cdots - c_{j-1}\alpha_{j-1} - c_{j+1}\alpha_{j+1} - \cdots - c_n\alpha_n]$
$= d_1\alpha_1 + d_2\alpha_2 + \cdots + d_{j-1}\alpha_{j-1} + d_{j+1}\alpha_{j+1} + \cdots + d_n\alpha_n$,
where $d_i = -c_j^{-1}c_i \in F$, $i = 1, 2, \ldots, j-1, j+1, \ldots, n$.

This shows that α_j is a linear combination of the vectors $\alpha_1, \alpha_2, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_n$.

Conversely, let one of the vectors, say α_j, is a linear combination of the other vectors of the set.

Then $\alpha_j = r_1\alpha_1 + r_2\alpha_2 + \cdots + r_{j-1}\alpha_{j-1} + r_{j+1}\alpha_{j+1} + \cdots + r_n\alpha_n$, for some scalars $r_i \in F$, $i = 1, 2, \ldots, j-1, j+1, \ldots, n$.

Therefore $r_1\alpha_1 + r_2\alpha_2 + \cdots + r_{j-1}\alpha_{j-1} + (-1)\alpha_j + r_{j+1}\alpha_{j+1} + \cdots + r_n\alpha_n = \theta$.

Since the above equality holds for scalars $r_1, r_2, \ldots, r_{j-1}, -1, r_{j+1}, \ldots, r_n$ in F and since one of them at least is non-zero, the set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is linearly dependent.

Corollary. Two vectors α, β in a vector space V over a field F are linearly dependent if at least one of them is a scalar multiple of the other.

Examples.

1. The set of vectors $S = \{\alpha, 2\alpha, \beta\}$ of a real vector space V is linearly dependent, since $2\alpha \in S$ and $2\alpha (= 2\alpha + 0\beta)$ is a linear combination of the remaining vectors of S.

2. The set of vectors $S = \{\alpha, \beta, \gamma, \beta + \gamma\}$ of a real vector space V is linearly dependent, since $\beta + \gamma \in S$ and $\beta + \gamma (= 0\alpha + \beta + \gamma)$ is a linear combination of the remaining vectors of S.