Since $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis of $V$ and $\{\beta_1, \beta_2, \ldots, \beta_m\}$ is a linearly independent set of vectors in $V$, $m \leq n$.

Since $\{\beta_1, \beta_2, \ldots, \beta_m\}$ is a basis of $V$ and $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a linearly independent set of vectors in $V$, $n \leq m$.

$m \leq n$ and $n \leq m \Rightarrow m = n$ and the theorem is done.

**Definition.** The number of vectors in a basis of a vector space $V$ is said to be the *dimension* (or *rank*) of $V$ and is denoted by dim $V$. The null space $\{\theta\}$ is said to be of dimension 0.

**Examples** (continued).

4. The demension of the vector space $\mathbb{R}^2$ is 2, since $E = \{(1,0), (0,1)\}$ is a basis.

The dimension of the vector space $\mathbb{R}^3$ is 3, since $E = \{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis.

The dimension of the vector space $\mathbb{R}^n$ is $n$, since $E = \{(1,0,0,\ldots,0), (0,1,0,\ldots,0), \ldots, (0,0,0,\ldots,1)\}$ is a basis.

5. The dimension of the vector space $\mathbb{R}_{m\times n}$ of all $m \times n$ real matrices is $mn$, since the set $\{E_{11}, E_{12}, \ldots, E_{mn}\}$, where $E_{ij}$ is an $m \times n$ matrix having 1 as the $ij$th element and 0 elsewhere, is a basis.

6. The dimension of the vector space $P_n$ of all real polynomials in $x$ of degree $< n$ together with the zero polynomial, is $n$. The set of polynomials $\{1, x, x^2, \ldots, x^{n-1}\}$ is a basis. P - 46 (6th-ch)

7. The vector space $P$ of all real polynomials is infinite dimensional.

Let $S$ be a finite subset of $P$. Since there is only a finite number of polynomials in $S$, there is a polynomial $f$ in $S$ having a maximum degree, say $m$. Therefore every polynomial in $S$ is of degree $\leq m$.

Let us consider an arbitrary polynomial $p$ in $L(S)$. The degree of $p$ cannot exceed $m$. This proves that $L(S)$ is a proper subspace of $P$ because in $P$ there are polynomials of degree $\geq m+1$. So $S$ cannot be a basis of $P$.

Since $S$ is an arbitrary finite set, $P$ cannot have a finite basis. Therefore $P$ is infinite demensional.

CW-06

**Theorem 4.5.5.** Let $V$ be a vector space of *dimension* $n$ over a field $F$. Then any linearly independent set of $n$ vectors of $V$ is a basis of $V$.

*Proof.* Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a linearly independent set of vectors in $V$. Let $\beta$ be an arbitrary vector of $V$ and $\beta \neq \alpha_i$. Since dim $V = n$, any basis of $V$ contains $n$ vectors and the set $\{\alpha_1, \alpha_2, \ldots, \alpha_n, \beta\}$ which contains $n+1$ vectors is linearly dependent.



Therefore there exist scalars $c_1, c_2, \ldots, c_n, c$, not all zero, in $F$ such that $c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n + c\beta = \theta$ ... (i)

We assert that $c \neq 0$.

Because $c = 0$ implies $c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = \theta$ where $c_1, c_2, \ldots, c_n$ are not all zero, and this implies linear dependence of the set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, a contradiction.

Since $c \neq 0, c^{-1} \in F$ and $c^{-1}c = 1$, 1 being the unity in $F$.

From (i) $\quad c\beta = -c_1\alpha_1 - c_2\alpha_2 - \cdots - c_n\alpha_n$.

Then $\beta = c^{-1}(-c_1\alpha_1 - c_2\alpha_2 - \cdots - c_n\alpha_n)$
$= d_1\alpha_1 + d_2\alpha_2 + \cdots + d_n\alpha_n$, where $d_i = -c^{-1}c_i \in F$.

This shows that $\beta$ is a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_n$.

If however, $\beta = \alpha_i$ for some $i$, then also $\beta$ is a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Thus $\beta \in L\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and so $V \subset L\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ ... (ii)

Again $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset V$ and $L\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ being the smallest subspace containing $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, $L\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset V$ ... (iii)

From (ii) and (iii), $V = L\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. This proves that the set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis of $V$.

This completes the proof.

**Theorem 4.5.6.** Let $V$ be a vector space of *dimension* $n$ over a field $F$. Then any subset of $n$ vectors of $V$ that generates $V$ is a basis of $V$.

*Proof.* Since dim $V = n$, any basis of $V$ contains $n$ vectors.

Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a generating set of $V$. We prove that it is linearly independent in $V$.

If it be linearly dependent, then by Deletion theorem there exists a proper subset $S_1$ of $S$ such that $S_1$ is linearly independent in $V$ and $S_1$ generates $V$.

Therefore $S_1$ is a basis of $V$ containing less than $n$ vectors of $V$ and this contradicts that dim $V = n$.

Therefore $S$ is a linearly independent set in $V$. Thus the set $S$ is a linearly independent generating set of $V$ and therefore it is a basis of $V$.

This completes the proof.

**Worked Examples** (continued).

2. Find a basis for the vector space $\mathbb{R}^3$ that contains the vectors (1, 2, 0) and (1, 3, 1).

$\mathbb{R}^3$ is a vector space of dimension 3. The standard basis for $\mathbb{R}^3$ is $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ where $\epsilon_1 = (1,0,0), \epsilon_2 = (0,1,0), \epsilon_3 = (0,0,1)$.

Let $\alpha = (1,2,0), \beta = (1,3,1)$. Then $\alpha = 1\epsilon_1 + 2\epsilon_2 + 0\epsilon_3$.

Since the coefficient of $\epsilon_1$ in the representation of $\alpha$ is non-zero, by Replacement theorem $\alpha$ can replace $\epsilon_1$ in the basis $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ and $\{\alpha, \epsilon_2, \epsilon$ can be a new basis for $\mathbb{R}^3$.

Let $\beta = c_1\alpha + c_2\epsilon_2 + c_3\epsilon_3$.
Then $(1,3,1) = c_1(1,2,0) + c_2(0,1,0) + c_3(0,0,1)$.
Therefore $c_1 = 1, 2c_1 + c_2 = 3, c_3 = 1$.
We have $c_1 = 1, c_2 = 1, c_3 = 1$ and $\beta = \alpha + \epsilon_2 + \epsilon_3$.

Since the coefficient of $\epsilon_2$ is non-zero, by Replacement theorem $\beta$ ca replace $\epsilon_2$ in the basis $\{\alpha, \epsilon_2, \epsilon_3\}$ and $\{\alpha, \beta, \epsilon_3\}$ can be a new basis f $\mathbb{R}^3$.

**Note.** The replacement can be done in more than one ways and th different bases for $\mathbb{R}^3$ can be obtained.

3. Prove that the set $S = \{(2,1,1), (1,2,1), (1,1,2)\}$ is a basis of $\mathbb{R}^3$.

Let $\alpha_1 = (2,1,1), \alpha_2 = (1,2,1), \alpha_3 = (1,1,2)$.

Let us consider the relation $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = \theta$, where $c_1, c_2, c_3 \in$
Then $c_1(2,1,1) + c_2(1,2,1) + c_3(1,1,2) = (0,0,0)$.

This gives $2c_1 + c_2 + c_3 = 0, \quad c_1 + 2c_2 + c_3 = 0, \quad c_1 + c_2 + 2c_3 =$

This is a homogeneous system of equations in $c_1, c_2, c_3$. Here th coefficient determinant $= -4 \neq 0$.

By Cramer's rule, there exists a unique solution and the solution $c_1 = 0, c_2 = 0, c_3 = 0$. This proves that the set $S$ is linearly independe

Since $\mathbb{R}^3$ is a vector space of dimension 3 and $S$ is a linearly indepe dent set containing 3 vectors of $\mathbb{R}^3$, $S$ is a basis of $\mathbb{R}^3$.

4. Let $V$ be a real vector space with $\{\alpha, \beta, \gamma\}$ as a basis. Prove that th set $\{\alpha+\beta+\gamma, \beta+\gamma, \gamma\}$ is also a basis of $V$.

Let $\alpha + \beta + \gamma = \alpha_1, \beta + \gamma = \alpha_2, \gamma = \alpha_3$.
Let us consider the relation $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = \theta$, where $c_1, c_2, c_3 \in$
Then $c_1(\alpha+\beta+\gamma) + c_2(\beta+\gamma) + c_2\gamma = \theta$
or, $c_1\alpha + (c_1+c_2)\beta + (c_1+c_2+c_3)\gamma = \theta$.
This implies $c_1 = 0, c_1 + c_2 = 0, c_1 + c_2 + c_3 = 0$, since the set $\{\alpha, \beta,$ is linearly independent.



The solution is $c_1 = 0, c_2 = 0, c_3 = 0$ and this proves that the set $\{\alpha_1, \alpha_2, \alpha_3\}$ is linearly independent.

$V$ is a vector space of dimension 3 and $\{\alpha_1, \alpha_2, \alpha_3\}$ is a linearly independent set containing 3 vectors of $V$. Therefore $\{\alpha_1, \alpha_2, \alpha_3\}$ is a basis of $V$.

5. $V$ is the vector space of all $2 \times 2$ real matrices. Prove that the set $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ is a basis of $V$.

Let $\alpha_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Let us consider the relation $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 = O$, where $c_1, c_2, c_3, c_4 \in \mathbb{R}$.

Then $c_1 + c_2 + c_3 + c_4 = 0, c_2 + c_3 + c_4 = 0, c_3 + c_4 = 0, c_4 = 0$.
This gives $c_1 = 0, \quad c_2 = 0, \quad c_3 = 0, \quad c_4 = 0$.
This proves that the set $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is linearly independent.

$V$ is a vector space of dimension 4 and $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a linearly independent set containing 4 vectors of $V$. Therefore the set $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a basis of $V$.

6. Find a basis and the dimension of the subspace $W$ of $\mathbb{R}^3$, where $W = \{(x,y,z) \in \mathbb{R}^3 : x + y + z = 0\}$.

Let $\xi = (a,b,c) \in W$. Then $a, b, c \in \mathbb{R}$ and $a + b + c = 0$.
Therefore $\xi = (a, b, -a-b) = a(1,0,-1) + b(0,1,-1)$. **
Let $\alpha = (1,0,-1), \beta = (0,1,-1)$. Then $\xi = a\alpha + b\beta \in L\{\alpha, \beta\}$.
Therefore $W \subset L\{\alpha, \beta\}$.
Again $\alpha \in W, \beta \in W$ $a\alpha+b\beta\in P$ This implies $L\{\alpha,\beta\} \subset W$, as $W$ is a subspace.
Consequently, $W = L\{\alpha, \beta\}$.

$\alpha, \beta$ are linearly independent in $W$, since none of them is a scalar multiple of the other. [Corollary, Theorem 4.4.5.]
Hence the set $\{\alpha, \beta\}$ is a basis of $W$ and dim $W = 2$.

7. Find a basis and the dimension of the subspace $W$ of $\mathbb{R}^3$, where $W = \{(x,y,z) \in \mathbb{R}^3 : x + 2y + z = 0, 2x + y + 3z = 0\}$.

Let $\xi = (a,b,c)$ be an arbitrary vector of $W$.
Then $a + 2b + c = 0$ and $2a + b + 3c = 0, \; a, b, c \in \mathbb{R}$.
Solving, we have $\frac{a}{5} = \frac{b}{-1} = \frac{c}{-3} = k$, say.

$\xi$ takes the form $k(5,-1,-3)$ where $k$ is an arbitrary real number.
Therefore $W = L\{\alpha\}$ where $\alpha = (5,-1,-3)$. Since $\{\alpha\}$ is a linearly independent set, $\{\alpha\}$ is a basis of $W$ and dim $W = 1$.

** this shows that any vector of W can be expressed in terms of (1,0,-1) and (0,1,-1)
$\therefore W = L\{(1,0,-1), (0,1,-1)\}$

# Exercises 7

1. Show that the sets of vectors are linearly dependent in $\mathbb{R}^3$.
   (i) $\{(2,3,1),(2,1,3),(1,1,1)\}$,
   (ii) $\{(0,-1,3),(3,4,3),(1,1,2)\}$.

2. Show that the sets of vectors are linearly dependent in $\mathbb{R}^4$.
   (i) $\{(1,1,1,0),(1,0,1,1),(1,2,1,2),(1,1,1,1)\}$,
   (ii) $\{(2,3,1,4),(3,2,4,1),(1,1,1,1)\}$.

3. Determine $k$ so that the set $S$ is linearly dependent in $\mathbb{R}^3$.
   (i) $S=\{(1,2,1),(k,3,1),(2,k,0)\}$,
   97 (ii) $S=\{(k,1,1),(1,k,1),(1,1,k)\}$.

4. Find the conditions on $x,y$ so that the set of vectors is linearly depend in $\mathbb{R}^3$.
   (i) $\{(x,y,y),(y,x,y),(y,y,x)\}$;
   (ii) $\{(x,y,1),(y,1,x),(1,x,y)\}$.

5. Show that set of vectors
   $S=\{(1,2,0),(2,1,3),(1,1,1),(2,3,1)\}$ is linearly dependent in $\mathbb{R}^3$. F a linearly independent subset $S_1$ of $S$ such that $L(S_1)=L(S)$.

6. Show that the set of vectors
   $S=\{(1,2,3,0),(2,1,0,3),(1,1,1,1),(2,3,4,1)\}$ is linearly dependent $\mathbb{R}^4$. Find a linearly independent subset $S_1$ of $S$ such that $L(S_1)=L(S)$.

7. Show that the sets of vectors are linearly independent in $\mathbb{R}^3$.
   (i) $\{(1,2,3),(2,3,1),(3,1,2)\}$; (ii) $\{(0,1,1),(1,0,1),(1,1,0)\}$.

8. Show that the sets of vectors are linearly independent in $\mathbb{R}^4$.
   (i) $\{(1,2,3,0),(2,3,0,1),(3,0,1,2)\}$;
   (ii) $\{(1,1,1,0),(1,1,0,1),(1,0,1,1),(0,1,1,1)\}$.

9. For what real values of $k$ does the set $S$ form a basis of $\mathbb{R}^3$ ?
   (i) $S=\{(k,1,k),(0,k,1),(1,1,1)\}$;
   (ii) $S=\{(k,0,1),(1,k+1,1),(1,1,1)\}$.

10. Let $\{\alpha,\beta,\gamma\}$ be a basis of a real vector space $V$ and $c$ be a non-zero number. Prove that
    (i) $\{c\alpha,c\beta,c\gamma\}$ is a basis of $V$,
    (ii) $\{\alpha+c\beta,\beta,\gamma\}$ is a basis of $V$,
    (iii) $\{\alpha+c\beta,\beta+c\gamma,\gamma+c\alpha\}$ may not be a basis of $V$.



11. If $\{\alpha_1,\alpha_2,\alpha_3\}$ be a basis of a real vector space $V$ and
    $$\beta_1=\alpha_1+\alpha_3,\ \beta_2=2\alpha_1+3\alpha_2+4\alpha_3,\ \beta_3=\alpha_1+2\alpha_2+3\alpha_3,$$
    prove that $\{\beta_1,\beta_2,\beta_3\}$ is also a basis of $V$.

12. Prove that the set $S=\{(1,1,0),(1,0,1),(0,1,1)\}$ is a basis of the vector space $\mathbb{R}^3$. Show that the vector $(1,1,1)$ may replace any one of the vectors of the set $S$ to form a new basis for $\mathbb{R}^3$, but the same is not true for the vector $(3,1,2)$.

13. Find a basis for the vector space $\mathbb{R}^3$, that contains the vectors
    (i) (1, 2, 1) and (3, 6, 2); (ii) (1, 0, 1) and (1, 1, 1).

14. Find a basis for the vector space $\mathbb{R}^4$, that contains the vectors
    (i) (1, 0, 1, 0) and (0, 1, 0, 1); (ii) (1, 1, 0, 0) and (1, 1, 1, 0).

15. Extend the set $S$ to obtain a basis of the vector space $\mathbb{R}^3$.
    (i) $S=\{(1,2,1),(2,1,1)\}$;
    (ii) $S=\{(1,1,0),(1,1,1)\}$.

16. Extend the set $S$ to obtain a basis of the vector space $\mathbb{R}^4$.
    (i) $S=\{(1,0,1,0),(0,1,0,1)\}$;
    (ii) $S=\{(1,1,0,0),(1,1,1,0)\}$.

17. Let $V$ be a real vector space with a basis $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$. Examine if $\{\alpha_1+\alpha_2,\alpha_2+\alpha_3,\cdots,\alpha_n+\alpha_1\}$ is also a basis of $V$.

18. Find the basis and dimension of the subspace $S$ of $\mathbb{R}^3$ defined by
    (i) $S=\{(x,y,z)\in\mathbb{R}^3 : 2x+y-z=0\}$;
    (ii) $S=\{(x,y,z)\in\mathbb{R}^3 : x+2y=z, 2x+3z=y\}$.

19. Find the dimension of the subspace $S$ of $\mathbb{R}^4$ defined by
    (i) $S=\{(x,y,z,w)\in\mathbb{R}^4 : x+y+z+w=0\}$;
    (ii) $S=\{(x,y,z,w)\in\mathbb{R}^4 : x+2y-z=0, 2x+y+w=0\}$.

20. Find the dimension of the subspace $S\cap T$ of $\mathbb{R}^4$ where $S=\{(x,y,z,w)\in\mathbb{R}^4 : x+y+z+w=0\}$, $T=\{(x,y,z,w)\in\mathbb{R}^4 : 2x+y-z+w=0\}$.

21. Find a basis and determine the dimension of the following subspace $S$ of the vector space $\mathbb{R}_{2\times2}$.
    (i) $S=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\mathbb{R}_{2\times2} : a+b=0\right\}$,
    (ii) $S=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\mathbb{R}_{2\times2} : a=d=0\right\}$

### 4.15. Characteristic equation.

Let $A$ be an $n \times n$ matrix over a field $F$. Then det $(A - xI_n)$ is said to be the *characteristic polynomial* of $A$ and is denoted by $\Psi_A(x)$. The equation $\psi_A(x) = 0$ is said to be the *characteristic equation* of $A$.

Let $A = (a_{ij})_{n,n}$.

$$\text{Then } \Psi_A(x) = \begin{vmatrix} a_{11}-x & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-x & \cdots & a_{2n} \\ \cdots & \cdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-x \end{vmatrix}$$

$= c_0x^n + c_1x^{n-1} + \ldots + c_n$, where $c_0 = (-1)^n$ and

$c_r = (-1)^{n-r}$.[sum of the principal minors of $A$ of order $r$].

In particular, $c_1 = (-1)^{n-1}(a_{11} + a_{22} + \ldots + a_{nn})$
$= (-1)^{n-1}$ trace A,

and $c_n = \det A$.

The degree of the characteristic equation is same as the order of the matrix $A$ and the coefficients are scalars belonging to $F$.

### 4.15.1. Cayley-Hamilton theorem.

**Every square matrix satisfies its own characteristic equation.**

The theorem states that if $A$ be an $n \times n$ matrix and the characteristic polynomial of $A$ be $c_0x^n + c_1x^{n-1} + \cdots + c_n$, then

$$c_0A^n + c_1A^{n-1} + \cdots + c_nI_n = O.$$

*Proof.* det $(A - xI_n) = c_0x^n + c_1x^{n-1} + \cdots + c_n$.

$(A - I_nx)$ is a matric polynomial in $x$ of degree 1 and adj $(A - I_nx)$ is a matric polynomial in $x$ of degree $n-1$, since each element of adj $(A - xI_n)$ (i.e., a cofactor of an element of the matrix $A - I_nx$) is a polynomial in $x$ of degree $n-1$ at most.

Let adj $(A - I_nx) = B_0x^{n-1} + B_1x^{n-2} + \cdots + B_{n-1}$, where each $B_i$ is an $n \times n$ matrix.

$(A - xI_n)$. adj $(A - xI_n) =$[det $(A - xI_n)$] $I_n$ gives

$(A - I_nx)(B_0x^{n-1} + B_1x^{n-2} + \cdots + B_{n-1}) = (c_0x^n + c_1x^{n-1} + \cdots + c_n)I_n$

or, $A(B_0x^{n-1} + B_1x^{n-2} + \cdots + B_{n-1}) - (B_0x^n + B_1x^{n-1} + \cdots + B_{n-1}x)$
$= (c_0I_n)x^n + (c_1I_n)x^{n-1} + \cdots + (c_nI_n)$.

Equating coefficients of like powers of $x$, we have



$$\begin{aligned} -B_0 &= c_0I_n, \\ AB_0 - B_1 &= c_1I_n, \\ AB_1 - B_2 &= c_2I_n, \\ \cdots \quad \cdots & \\ AB_{n-2} - B_{n-1} &= c_{n-1}I_n, \\ AB_{n-1} &= c_nI_n. \end{aligned}$$

Pre-multiplying the relations by $A^n, A^{n-1}, A^{n-2}, \ldots, A, I_n$ respectively and adding, we have $c_0A^n + c_1A^{n-1} + \cdots + c_{n-1}A + c_nI_n = O$.

This completes the proof.

Cayley-Hamilton theorem gives a computation procedure for obtaining $A^{-1}$ when $A$ is a non-singular matrix.

Let the characteristic equation of $A$ be $c_0x^n + c_1x^{n-1} + \cdots + c_n = 0$. By Cayley-Hamilton theorem, $c_0A^n + c_1A^{n-1} + \cdots + c_{n-1}A + c_nI_n = O$.

Since $c_n = \det A \neq 0$, $c_n^{-1}$ exists in $F$. Multiplying by $-c_n^{-1}$, we have
$-c_n^{-1}(c_0A^n + c_1A^{n-1} + \cdots + c_{n-1}A) - I_n = O$
or, $-c_n^{-1}(c_0A^{n-1} + c_1A^{n-2} + \cdots + c_{n-1}I_n)A = I_n$.

From the definition and uniqueness of an inverse it follows that $A^{-1} = -c_n^{-1}(c_0A^{n-1} + c_1A^{n-2} + \cdots + c_{n-1}I_n)$.

Thus $A^{-1}$ is expressed as a polynomial in $A$ with scalar coefficients.

**Worked Examples.**

1. Use Cayley-Hamilton theorem to find $A^{-1}$, where $A = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$.

The characteristic equation of $A$ is $\begin{vmatrix} 2-x & 1 \\ 3 & 5-x \end{vmatrix} = 0$

or, $x^2 - 7x + 7 = 0$.

By Cayley-Hamilton theorem, $A^2 - 7A + 7I_2 = O$

or, $A(A - 7I_2) = -7I_2$ or, $-\frac{1}{7}A(A - 7I_2) = I_2$.

This gives $A^{-1} = -\frac{1}{7}(A - 7I_2) = \frac{1}{7}\begin{pmatrix} 5 & -1 \\ -3 & 2 \end{pmatrix}$.

2. Use Cayley-Hamilton theorem to find $A^{50}$, where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

The characteristic equation of $A$ is $x^2 - 2x + 1 = 0$.

By Cayley-Hamilton theorem, $A^2 - 2A + I_2 = O$ or, $A^2 - A = A - I_2$.

Therefore $A^3 - A^2 = A^2 - A = A - I_2, \cdots, A^{50} - A^{49} = A - I_2$.

Adding, we have $A^{50} = 50A - 49I_2 = \begin{pmatrix} 1 & 50 \\ 0 & 1 \end{pmatrix}$.

## 4.16. Eigen value of a matrix.

A root of the characteristic equation of a square matrix $A$ is said to be an *eigen value* (or a *characteristic value*) of $A$.

Although the coefficients of $\Psi_A(x)$ are elements of $F$, the eigen values of $A$ may not be all elements of $F$. But they all belong to a suitable algebraic extension of the field $F$.

For example, if the ground field of $A$ be $\mathbb{R}$ then $\Psi_A(x)$ is a real polynomial but the roots of $\Psi_A(x) = 0$ may not be all real. They are all elements of the field $\mathbb{C}$ which is an algebraic extension of $\mathbb{R}$.

A root of $\Psi_A(x) = 0$ of multiplicity $r$ is said to be an *$r$-fold* eigen value of $A$.

**Theorem 4.16.1.** The product of the eigen values of a square matrix $A$ is det $A$.

*Proof.* Let $A$ be an $n \times n$ matix and let the characteristic equation of $A$ be $c_0x^n + c_1x^{n-1} + c_2x^{n-2} + \cdots + c_n = 0$.

Then $c_0 = (-1)^n$, $c_n =$det $A$.

The product of the roots of the equation is $(-1)^n \frac{c_n}{c_0} = c_n =$ det $A$.

Hence, the product of the eigen values of $A$ is det $A$.

Note. det $A = \lambda_1\lambda_2 \cdots \lambda_n$

**Theorem 4.16.2.** If $A$ be a singular matrix, 0 is an eigen value of $A$.

*Proof.* Since $A$ is singular, det $A = 0$.

Let $\Psi_A(x) = c_0x^n + c_1x^{n-1} + c_2x^{n-2} + \cdots + c_n$.

Then $c_n =$ det $A = 0$. Consequently, 0 is a root of the characteristic equation of $A$ and therefore 0 is an eigen value of $A$.

Note: det $A \neq 0 \Rightarrow \lambda \neq 0$.

**Theorem 4.16.3.** The eigen values of a diagonal matrix are its diagonal elements.

*Proof.* Let $A =$ diag $(d_1, d_2, \ldots, d_n)$.

Then det $(A - xI_n) = (d_1 - x)(d_2 - x)\ldots(d_n - x)$.

So the roots of the characteristic equation of $A$ are $d_1, d_2, \ldots, d_n$ and hence the eigen values of $A$ are $d_1, d_2, \ldots, d_n$.

**Theorem 4.16.4.** If $\lambda$ be an $r$-fold eigen value of $A$, 0 is an $r$-fold eigen value of the matrix $A - \lambda I_n$.

*Proof.* Let $\Psi_A(x) =$ det $(A - xI_n) = (x - \lambda)^r\phi(x)$ where $\phi$ is a polynomial of degree $n - r$ and $\phi(\lambda) \neq 0$.

The characteristic polynomial of $A - \lambda I_n$ is det $(A - \lambda I_n - xI_n)$.

$$\begin{aligned} \det(A - \lambda I_n - xI_n) &= \det[A - (\lambda + x)I_n] \\ &= (\lambda + x - \lambda)^r\phi(\lambda + x) \\ &= x^r\mu(x), \text{ where } \mu(0) = \phi(\lambda) \neq 0. \end{aligned}$$





This proves that 0 is a root of multiplicity $r$ of the characteristic equation of the matrix $A - \lambda I_n$.

**Theorem 4.16.5.** If $\lambda$ be an eigen value of a non-singular matrix $A$, then $\lambda^{-1}$ is an eigen value of $A^{-1}$.

*Proof.* Let the order of $A$ be $n$. Since $A$ is non-singular, $A^{-1}$ exists. Also $\lambda \neq 0$. Therefore $\lambda^{-1}$ exists.

Since $\lambda$ is an eigen value of $A$, det $(A - \lambda I_n) = 0$.

$$\begin{aligned} \det(A^{-1} - \lambda^{-1}I_n) &= (\det A)^{-1} \det(AA^{-1} - \lambda^{-1}A) \\ = \det\{A^{-1}(I_n - \lambda^{-1}A)\} &= (\det A)^{-1}(\lambda^{-1})^n \det(\lambda I_n - A) \\ = \det\{-A^{-1}\lambda^{-1}(A - \lambda I_n)\} &= (\det A)^{-1}(\lambda^{-1})^n(-1)^n \det(A - \lambda I_n) \\ = \det(-A^{-1}\lambda^{-1}) \det(A - \lambda I_n) = 0 &= 0, \text{ since } \det(A - \lambda I_n) = 0. \end{aligned}$$

This proves that $\lambda^{-1}$ is an eigen value of $A^{-1}$.

**Theorem 4.16.6.** If $A$ and $P$ be both $n \times n$ matrices and $P$ be non-singular, then $A$ and $P^{-1}AP$ have the same eigen values.

*Proof.* The characteristic polynomial of $P^{-1}AP$ is det $(P^{-1}AP - xI_n)$.

$$\begin{aligned} \det(P^{-1}AP - xI_n) &= \det[P^{-1}AP - P^{-1}(xI_n)P], \\ &\quad \text{since } P^{-1}(xI_n)P = xI_n \\ &= \det[P^{-1}(A - xI_n)P] \\ &= \det P^{-1} \det(A - xI_n) \det P \\ &= \det(A - xI_n) \det(P^{-1}P) \\ &= \det(A - xI_n) \det(I_n) \\ &= \det(A - xI_n). \end{aligned}$$

Therefore the matrices $P^{-1}AP$ and $A$ have the same characteristic polynomial and so they have the same eigen values.

Thus let $A$ be a square matrix of order $n$ over a field $F$. prove that $A$ and $A^t$ have same eigen values.

**Examples.**

1. Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

The characteristic equation of $A$ is $\begin{vmatrix} -x & -1 \\ 1 & -x \end{vmatrix} = 0$

or, $x^2 + 1 = 0$.

The eigen values of $A$ are $i, -i$.

2. Let $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{pmatrix}$.

The characteristic equation of $A$ is $\begin{vmatrix} 1-x & -1 & 0 \\ 1 & 2-x & -1 \\ 3 & 2 & -2-x \end{vmatrix} = 0$

or, $(1-x)(x^2-2)+(1-x)=0$
or, $(1-x)(x^2-1)=0$.
The eigen values of $A$ are $1, 1, -1$.

## 4.17. Eigen vectors of a matrix.

**Definition.** Let $A$ be $n \times n$ matrix over a field $F$. A non-null vector $X$ belonging to $V_n(F)$ is said to be an *eigen vector* or a *characteristic vector* of $A$ if there exists a scalar $\lambda$ belonging to $F$ such that $AX = \lambda X$ holds.

Let there exist an eigen vector $X$ of the matrix. Then for some suitable scalar $\lambda$, $AX = \lambda X$ holds. That is, $(A - \lambda I_n)X = O$.

This is a homogeneous system of $n$ equations in $n$ unknowns. Since there exists a non-null solution of the system, $\det(A - \lambda I_n) = 0$.

This implies that $\lambda$ is an eigen value of $A$. Thus for an eigen vector, if it exists, there corresponds an eigen value of the matrix.

**Theorem 4.17.1.** Let $A$ be an $n \times n$ matrix over a field $F$. To an eigen vector of $A$ there corresponds a unique eigen value of $A$.

*Proof.* Let there be two distinct eigen values $\lambda_1, \lambda_2$ of $A$ corresponding to an eigen vector $X$. Then $AX = \lambda_1 X$ and $AX = \lambda_2 X$

Therefore $(\lambda_1 - \lambda_2)X = O$. But this is a contradiction, since $X$ is a non-null vector and $\lambda_1 - \lambda_2 \neq 0$. Hence the theorem.

**Theorem 4.17.2.** Let $A$ be an $n \times n$ matrix over a field $F$ and $\lambda$ be an eigen value belonging to $F$. To each such eigen value of $A$ there corresponds at least one eigen vector.

*Proof.* Since $\lambda$ is an eigen value, $\det(A - \lambda I_n) = 0$. Therefore the homogeneous system of equations $(A - \lambda I_n)X = O$ has a non-null solution, say $X = X_1$ where $X_1 \in V_n(F)$.

Then $(A - \lambda I_n)X_1 = O$ or, $AX_1 = \lambda X_1$.

This shows that $X_1$ is an eigen vector of $A$ corresponding to $\lambda$.

This proves the theorem.

**Note.** In fact, there are many eigen vectors of $A$ corresponding to an eigen value $\lambda$ belonging to $F$, because $\det(A - \lambda I_n) = 0$ implies that there are infinite number of non-null solutions of the homogeneous system $(A - \lambda I_n)X = 0$ and each such non-null solution gives an eigen vector of $A$ corresponding to $\lambda$.



**Examples.**

1. Let $A = \begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}$. The eigen values of $A$ are $-1, 7$.

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be an eigen vector corresponding to $-1$.

Then $AX = -X$ and this gives $\begin{aligned} 2x_1 + 3x_2 &= 0 \\ 4x_1 + 6x_2 &= 0. \end{aligned}$

The equivalent system is $x_1 + \frac{3}{2}x_2 = 0$.
The solution of the system is $k(-\frac{3}{2}, 1)$, where $k \in \mathbb{R}$.

The eigen vectors are $k\begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix}$ or equivalently, $c\begin{pmatrix} 3 \\ -2 \end{pmatrix}$, where $c$ is a *non-zero* real number.

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be an eigen vector corresponding to 7.

Then $AX = 7X$ and this gives $\begin{aligned} -6x_1 + 3x_2 &= 0 \\ 4x_1 - 2x_2 &= 0. \end{aligned}$

The system is equivalent to $x_1 - \frac{1}{2}x_2 = 0$.
The solution of the system is $c(1, 2)$, where $c \in \mathbb{R}$.

Therefore the eigen vectors are $c\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, where $c$ is a *non-zero* real number.

2. Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The eigen values of $A$ are $i, -i$.

$A$ is a real matrix and the eigen values of $A$ are not real numbers. Therefore the real matrix $A$ has no eigen vector.

But if $A$ be considered as a complex matrix, then the eigen vectors of $A$ corresponding to the eigen values $i$ and $-i$ can be obtained.

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be an eigen vector corresponding to $i$.

Then $AX = iX$ and this gives $\begin{aligned} -ix_1 - x_2 &= 0 \\ x_1 - ix_2 &= 0. \end{aligned}$

The equivalent system is $x_1 - ix_2 = 0$.
The solution is $k(i, 1)$, where $k \in \mathbb{C}$.

The eigen vectors are $k\begin{pmatrix} i \\ 1 \end{pmatrix}$, where $k$ is a *non-zero* complex number.

The eigen vectors corresponding to $-i$ are $c\begin{pmatrix} 1 \\ i \end{pmatrix}$, where $c$ is a *non-zero* complex number.

So the eigen vectors are $c \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, where $c$ is a non-zero real number.

The rank of the characteristic subspace is 1 and therefore the geometric multiplicity of the eigen value 1 is 1. In this case,
the geometric multiplicity = the algebraic multiplicity.

**Theorem 4.17.8.** The eigen values of a real symmetric matrix are all real.

*Proof.* Let $A$ be an $n \times n$ real symmetric matrix. The characteristic equation of $A$ is an equation with real coefficients. So the eigen values of $A$ are complex numbers, some or all of which may be purely real.

Let $\lambda$ be an eigen value of $A$. Then $\det(A-\lambda I_n) = 0$. Therefore there exist non-null solutions of the homogeneous system $(A-\lambda I_n)X = O$. Let $X_1$ be one such solution.
Then $(A-\lambda I_n)X_1 = O$. That is, $AX_1 = \lambda X_1$.

[Note that this $X_1$ is not an eigen vector of $A$ unless $\lambda$ is purely real.]

Taking transpose of the conjugate, we have

$(\overline{AX_1})^t = (\overline{\lambda X_1})^t$

or, $(\bar{X}_1)^t(\bar{A})^t = \bar{\lambda}(\bar{X}_1)^t$, since $\lambda$ is a scalar

or, $(\bar{X}_1)^t A = \bar{\lambda}(\bar{X}_1)^t$, since $\bar{A}^t = A^t = A$.

Multiplying by $X_1$ from the right, we have

$(\bar{X}_1)^t AX_1 = \bar{\lambda}(\bar{X}_1)^t X_1$

or, $(\bar{X}_1)^t \lambda X_1 = \bar{\lambda}(\bar{X}_1)^t X_1$

or, $\lambda(\bar{X}_1)^t X_1 = \bar{\lambda}(\bar{X}_1)^t X_1$

or, $(\lambda - \bar{\lambda})(\bar{X}_1)^t X_1 = 0$.

But $(\bar{X}_1)^t X_1 \neq 0$, since $X_1$ is non-null.
It follows that $\lambda = \bar{\lambda}$ and therefore $\lambda$ is purely real.

This proves the theorem.

**Theorem 4.17.9.** The eigen values of a real skew symmetric matrix are purely imaginary or zero.

*Proof.* Let $A$ be an $n \times n$ real skew symmetric matrix. Following the same argument as in the previous theorem, we have
$(\lambda + \bar{\lambda})(\bar{X}_1)^t X_1 = 0$, since $\bar{A}^t = A^t = -A$.
Since $X_1$ is non-null, $\lambda + \bar{\lambda} = 0$. That is, $\lambda = -\bar{\lambda}$.
Therefore $\lambda$ is purely imaginary or zero and the theorem is proved.

Note. The eigen values of a Hermitian matrix are all real.

Let $\lambda$ be an eigen value of a Hermitian matrix $A$ and $X$ be an eigen vector corresponding to $\lambda$. Then $AX = \lambda X$. Using $\bar{A}^t = A$ and proceeding in a similar manner as in the theorem 4.17.8, the assertion can be established.

**Theorem 4.17.10.** The eigen vectors corresponding to two distinct eigen values of a real symmetric matrix are orthogonal.

*Proof.* Let $A$ be a real symmetric matrix. Let $X_1, X_2$ be two eigen vectors of $A$ corresponding to two distinct eigen values $\lambda_1$ and $\lambda_2$.

Then $AX_1 = \lambda_1 X_1$ and $AX_2 = \lambda_2 X_2$.

Now $AX_1 = \lambda_1 X_1 \quad \Rightarrow (AX_1)^t = \lambda_1 X_1^t$, since $\lambda_1$ is real

or, $X_1^t A = \lambda_1 X_1^t$, since $A = A^t$.

Multiplying by $X_2$ from the right, we have

$X_1^t A X_2 = \lambda_1 X_1^t X_2$

or, $X_1^t \lambda_2 X_2 = \lambda_1 X_1^t X_2$

or, $(\lambda_2 - \lambda_1) X_1^t X_2 = 0$

or, $X_1^t X_2 = 0$, since $\lambda_1 \neq \lambda_2$.

Since $X_1 \neq 0$ and $X_2 \neq 0$, it follows that $X_1$ is orthogonal to $X_2$.

This proves the theorem.

**Theorem 4.17.11.** Each eigen value of a real orthogonal matrix has unit modulus.

*Proof.* Let $A$ be an $n \times n$ real orthogonal matrix. Then $AA^t = I_n$. The eigen values of $A$ are in general, complex numbers, some of which may be purely real. Let $\lambda$ be an eigen value of $A$. Then det $(A - \lambda I_n) = 0$.

Therefore there exists a non-null solution of the homogeneous system $(A - \lambda I_n)X = O$. Let $X_1$ be one such solution.

Then $(A - \lambda I_n)X_1 = O$. That is, $AX_1 = \lambda X_1$.

[Note that this $X_1$ is not an eigen vector of $A$ unless $\lambda$ is purely real.]

Taking transpose of the conjugate, we have

$\overline{(AX_1)}^t = \overline{(\lambda X_1)}^t$

or, $(\bar{X}_1)^t (\bar{A})^t = \bar{\lambda}(\bar{X}_1)^t$, since $\lambda$ is a scalar

or, $(\bar{X}_1)^t A^t = \bar{\lambda}(\bar{X}_1)^t$, since $(\bar{A})^t = A^t$.

Multiplying by $AX_1$ from the right, we have

$(\bar{X}_1)^t A^t (AX_1) = \bar{\lambda}(\bar{X}_1)^t (AX_1)$

or, $(\bar{X}_1)^t (A^t A) X_1 = \bar{\lambda}(\bar{X}_1)^t \lambda X_1$



or, $(\bar{X}_1)^t X_1 = \bar{\lambda}\lambda(\bar{X}_1)^t X_1$, since $AA^t = I_n \Rightarrow A^t A = I_n$.

This implies $(\bar{X}_1)^t X_1 (1 - \bar{\lambda}\lambda) = 0$.

Since $X_1$ is non-null, $(\bar{X}_1)^t X_1 \neq 0$. It follows that $\bar{\lambda}\lambda = 1$, i.e., $|\lambda| = 1$.

This proves the theorem.

## Worked Examples.

1. If $\lambda$ be an eigen value of a real orthogonal matrix $A$, prove that $\frac{1}{\lambda}$ is also an eigen value of $A$. VH-10, 12

Let $A$ be an orthogonal matrix of order $n$. Then $AA^t = I_n$ and $A$ is non-singular. Since $A$ is non-singular, $\lambda \neq 0$.

Since $\lambda$ is an eigen value of $A$, det $(A - \lambda I_n) = 0$

or, det $(A - \lambda A A^t) = 0$

or, det $A$. det $(I_n - \lambda A^t) = 0$

or, det $(I_n - \lambda A^t) = 0$ , since det $A \neq 0$

or, $(-1)^n \lambda^n$ det $(A^t - \frac{1}{\lambda} I_n) = 0$

or, $(-1)^n \lambda^n$ det $(A - \frac{1}{\lambda} I_n) = 0$, since det $(A^t - \frac{1}{\lambda} I_n)$ =det $(A - \frac{1}{\lambda} I_n)^t$ =det $(A - \frac{1}{\lambda} I_n)$.

This proves that $\frac{1}{\lambda}$ is an eigen value of $A$.

2. A is a $3 \times 3$ real matrix having the eigen values $2, 3, 1$. $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ are the eigen vectors of A corresponding to the eigen values $2, 3, 1$ respectively. Find the matrix A.

Let $X_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, X_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Then $AX_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}, AX_2 = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}, AX_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Let $P = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Then $AP = \begin{pmatrix} 2 & 0 & 1 \\ 4 & 3 & 1 \\ 2 & 3 & 1 \end{pmatrix}$.

The column vectors of $P$ are the eigen vectors of $A$ corresponding to three distinct eigen values. Therefore P is non-singular.

$$P^{-1} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}.$$

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 4 & 3 & 1 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ -2 & 3 & 0 \\ -2 & 1 & 2 \end{pmatrix}.$$

3. If $S$ be a real skew symmetric matrix of order $n$ prove that

(i) $I_n + S$ is non-singular,

(ii) $(I_n+S)^{-1}(I_n-S)$ is orthogonal,

(iii) if $X$ be an eigen vector of $S$ with eigen value $\lambda$ then $X$ is also an eigen vector of the matrix $(I_n+S)^{-1}(I_n-S)$ with eigen value $\frac{1-\lambda}{1+\lambda}$,

(iv) if $\bar{S} = (I_n+S)^{-1}(I_n-S)$ then $I_n+\bar{S}$ is also non-singular and $\bar{\bar{S}} = S$.

(i) Since $S$ is a real skew symmetric matrix, its eigen values are imaginary or zero. Therefore -1 is not an eigen value of $S$. So -1 is not a root of the characteristic equation $\det(S - xI_n) = 0$.

It follows that $\det(S+I_n) \neq 0$. That is, $I_n+S$ is non-singular.

(ii) Let $P = (I_n+S)^{-1}(I_n-S)$.

$$\begin{aligned}
\text{Then } PP^t &= (I_n+S)^{-1}(I_n-S)[(I_n+S)^{-1}(I_n-S)]^t \\
&= (I_n+S)^{-1}(I_n-S)(I_n-S)^t\{(I_n+S)^{-1}\}^t \\
&= (I_n+S)^{-1}(I_n-S)(I_n+S)\{(I_n+S)^t\}^{-1} \\
&= (I_n+S)^{-1}\{(I_n+S)(I_n-S)\}(I_n-S)^{-1} \\
&\quad [\text{ since } (I_n-S)(I_n+S) = (I_n+S)(I_n-S)] \\
&= \{(I_n+S)^{-1}(I_n+S)\}\{(I_n-S)(I_n-S)^{-1}\} \\
&= I_n \cdot I_n = I_n.
\end{aligned}$$

This proves that $P$ is orthogonal.

(iii) $SX = \lambda X$.

$$\begin{aligned}
\text{Therefore } (I_n+S)^{-1}(I_n-S)X &= (I_n+S)^{-1}(1-\lambda)X \\
&= (1-\lambda)(I_n+S)^{-1}X.
\end{aligned}$$

Again $(I_n+S)X = (1+\lambda)X$

or, $X = (I_n+S)^{-1}(1+\lambda)X = (1+\lambda)(I_n+S)^{-1}X$.

So we have $\frac{1}{1+\lambda}X = (I_n+S)^{-1}X$ since $\lambda+1 \neq 0$. ie $[\lambda \neq -1]$

Therefore $(I_n+S)^{-1}(I_n-S)X = (1-\lambda)\frac{1}{1+\lambda}X = \frac{1-\lambda}{1+\lambda}X$.

This proves that $X$ is an eigen vector of $(I_n+S)^{-1}(I_n-S)$ with eigen value $\frac{1-\lambda}{1+\lambda}$.

(iv) $\bar{S} = (I_n+S)^{-1}(I_n-S)$.

$$\begin{aligned}
I_n+\bar{S} &= (I_n+S)^{-1}(I_n+S) + (I_n+S)^{-1}(I_n-S) \\
&= (I_n+S)^{-1}\{(I_n+S)+(I_n-S)\} \\
&= 2(I_n+S)^{-1}.
\end{aligned}$$

Therefore $(I_n+\bar{S})^{-1} = \frac{1}{2}(I_n+S)$, proving that $I_n+\bar{S}$ is non-singular.



$$\begin{aligned}
\text{Also } I_n-\bar{S} &= (I_n+S)^{-1}(I_n+S) - (I_n+S)^{-1}(I_n-S) \\
&= (I_n+S)^{-1}\{(I_n+S)-(I_n-S)\} \\
&= 2(I_n+S)^{-1}S.
\end{aligned}$$

$$\begin{aligned}
\text{Therefore } \bar{\bar{S}} &= (I_n+\bar{S})^{-1}(I_n-\bar{S}) \\
&= \tfrac{1}{2}(I_n+S).2(I_n+S)^{-1}.S = S.
\end{aligned}$$

## Exercises 13

1. Verify Cayley-Hamilton theorem for the matrix $A$. Express $A^{-1}$ as a polynomial in $A$ and then compute $A^{-1}$.

(i) $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix}$; (ii) $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix}$.

2. Use Cayley-Hamilton theorem to find $A^{100}$, where $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

3. (i) Find the eigen values and the corresponding eigen vectors of the matrix $I_3$. Generalise the result for the matrix $I_n$.

(ii) Find the eigen values and the corresponding eigen vectors of the scalar matrix $cI_3, c \in \mathbb{R}$. Generalise the result for the matrix $cI_n$.

(iii) Find the eigen values and the corresponding eigen vectors of the diagonal matrix diag $(d_1, d_2, d_3)$. Generalise the result for the matrix diag $(d_1, d_2, \ldots, d_n)$.

4. If $\lambda$ be an eigen value of an $n \times n$ matrix $A$, prove that

(i) $\lambda$ is also an eigen value of the matrix $A^t$;

(ii) $k\lambda$ is an eigen value of the matrix $kA$, $k$ being a scalar;

(iii) $\lambda^2$ is an eigen value of the matrix $A^2$.

5. $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of a non-singular matrix $A$ of order $n$. Find the eigen value of the matrix (i) $A^{-1}$, (ii) adj $A$. P-739

6. Find the eigen values and the corresponding eigen vectors of the following real matrices.

(i) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, (ii) $\begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{pmatrix}$,

(iii) $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$, (iv) $\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$.

7. Find the eigen values and the corresponding eigen vectors of the following complex matrices.

(i) $\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$, (ii) $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, (iii) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

8. Find the algebraic and the geometric multiplicities of each eigen value of the matrix $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{pmatrix}$.

9. If $\lambda$ be an eigen value of an $n \times n$ matrix $A$, prove that $\lambda^m$ is an eigen value of the matrix $A^m$, where $m$ is a positive integer.

[Hint. $AX = \lambda X$ implies $A^2X = \lambda(AX) = \lambda^2 X$ etc.]

10. If $\lambda$ be an eigen value of an $n \times n$ non-singular matrix $A$, prove that $\lambda^{-m}$ is an eigen value of the matrix $A^{-m}$, where $m$ is a positive integer.

11. If $A$ and $B$ are square matrices of the same order over a field $F$ and $A$ is non-singular, prove that

(i) the matrices $AB$ and $BA$ have the same eigen values;

(ii) the matrices $B$ and $ABA^{-1}$ have the same eigen values.

12. If $\lambda$ be an eigen value of an $n \times n$ idempotent matrix $A$, prove that $\lambda$ is either 1 or 0.

[Hint. $A^2 = A$. Let $AX = \lambda X$ for some $X \neq 0$. Then $\lambda X = AX = A^2X = \lambda(AX) = \lambda^2 X$.]

13. If $\lambda$ be an eigen value of a real skew symmetric matrix, prove that $\left|\frac{1-\lambda}{1+\lambda}\right| = 1$.

B of Th: 4.17.9 $\Rightarrow$ $\lambda = 0$ or $ib$

14. If $S$ be a real skew symmetric matrix of order $n$, prove that the matrices $I_n + S$ and $I_n - S$ are both non-singular.

15. If $A$ be a real non-singular symmetric matrix, prove that the matrices $A$ and $A^{-1}$ have the same set of eigen vectors.

16. Let $X$ be an eigen vector of an $n \times n$ matrix $A$ associated with an eigen value $\lambda$. Prove that $P^{-1}X$ is an eigen vector of the matrix $P^{-1}AP$ associated with $\lambda$.

17. Show that the characteristic equation of an orthogonal matrix is a reciprocal equation.

[Hint. Let $A$ be orthogonal and $\psi(x) = \det(A - xI_n)$. Then $\psi(x) = \pm x^n \psi(\frac{1}{x})$.]

18. $P$ is a real orthogonal matrix with $\det P = -1$. Prove that $-1$ is an eigen value of $P$.

[Hint. Consider the product $(P + I)P^t$.]