# SEMESTER-II
## HONOURS
## CORE COURSE---C 3T
## UNIT-I (MARKS-24)

**UNIT-I**                                           **Dr. Pradip Kumar Gain**

> **Syllabus for Unit-I:** Review of algebraic and order properties of $R$, $\varepsilon-$neighbourhood of a point in $R$. Idea of countable sets, uncountable sets and uncountability of $R$. Bounded above sets, bounded below sets, bounded sets, unbounded sets. Suprema and infima. Completeness property of $R$ and its equivalent properties. The Archimedean property, density property of rational and irrational numbers in $R$. Intervals. Limit point of a set, isolated points, open set, closed set, derived set, illustrations of Bolzano-Weierstrass theorem for sets, compact set in $R$, Heine-Borel Theorem.

**DEFINITION (Natural Numbers) :** The numbers 1,2,3,……….that were discovered in the natural process of counting are known as ***natural numbers***.

The set of all natural numbers is denoted by $N$, i.e., $N = \{1,2,3,……….\}$.

The set of natural numbers can be determined in terms of some axioms, formulated by $G$. Peano in 1889.

**Peano's axioms :**

**Axiom 1.**   $1 \in N$ **.** That is, 1 is a natural number.

**Axiom 2.**   For every natural number $n \in N$**,** there exists a unique natural number $n' \in N$, called the successor of $n$ and $n$ is called the predecessor of $n'$ **.**

**Axiom 3.**   There exist no predecessor of $1 \in N$ **.** That is, 1 is not successor of any natural number.

**Axiom 4.**   If two natural numbers have the same successors then they are themselves equal. That is, $n' = m' \Rightarrow n = m$

**Axiom 5.**   If $G$ be any set of some natural numbers, such that (i) $1 \in G$ and
(ii) $n \in G \Rightarrow n' \in G$, then $G$ contains all natural numbers, i.e., $G = N$
(**Axiom 5 is known as principle of finite or mathematical induction**)

**Well ordering principle of the set of natural numbers :**

**THEOREM 1.1: (Well ordering principle )** *Every non-empty subset of $N$ has a least element*.

The set $N = \{1,2,3,\ldots\ldots\}$ of all natural numbers is closed under addition and multiplication but not closed under the subtraction and division operations, i.e., the difference between any two numbers of $N$ are not always a member of $N$ and quotient of any two natural numbers are not always a member of $N$. That is why it became necessary to extend the system of natural numbers by introducing negative of natural numbers and the number zero 0. Thus to every natural number(positive integer) $n \in N$ there is a unique integer $-n$, called additive inverse of $n$ such that $n + (-n) = 0$, where 0 has the property such that $n + 0 = n$ and $n.0 = 0$ for each integer $n$.

**DEFINITION : (Integers)** The union of set of natural numbers and the set of all negative integers $\{\ldots\ldots -3,-2,-1\}$ and the singleton set $\{0\}$ is called the set of all *integers*. The set of all integers is denoted by $Z$. That is, $Z = \{\ldots\ldots -3,-2,-1,0,1,2,3,\ldots\ldots\}$.

Clearly, $N \subset Z$.

Now the set $Z = \{\ldots\ldots -3,-2,-1,0,1,2,3,\ldots\ldots\}$ of all integers is closed under addition, multiplication and subtraction but not closed under the division operation. That is why it became necessary to extend the system of all integers.

**DEFINITION : (Rational Numbers)** Any number which can be expressed as the ratio $\dfrac{p}{q}$ where $p \in Z$, $q \in N$ and $p$ and $q$ are prime to each other is called a *rational number*. The set of all rational numbers is denoted by $Q$.

Clearly, $N \subset Z \subset Q$.

**EXAMPLE :** i) $5 = \dfrac{5}{1}$, $5 \in Z, 1 \in N$ (ii) $-3 = \dfrac{-3}{1}$, $-3 \in Z, 1 \in N$ (iii) $\dfrac{3}{4} = \dfrac{3}{4}$, $3 \in Z, 4 \in N$ (iv) $-\dfrac{7}{2} = \dfrac{-7}{2}$, $-7 \in Z, 1 \in N$. Clearly $5$, $-3$, $\dfrac{3}{4}$, $-\dfrac{7}{2}$ are examples of rational numbers.

**pecial Characteristics of rational numbers :** Any rational number can be transformed either into a finite decimal form or an infinite recurring decimal form. For example, let us consider the rational numbers $\dfrac{3}{4}$ and $\dfrac{4}{3}$. We see that $\dfrac{3}{4} = .75$ (a finite decimal form) and $\dfrac{4}{3} = 1.33333333\ldots = 1.\dot{3}$ (an infinite recurring decimal).

Now the set $Q$ of all rational numbers is closed under addition, multiplication and subtraction and division.

**ORDER PROPERTIES OF** $Q$ :

    (i) If $a$ and $b$ are two rational numbers, i.e., $a, b \in Q$ then either $a < b$ or $b < a$ ( **law of trichotomy**)

(ii) If $a$, $b$ and $c$ are three rational numbers, i.e., $a,b,c \in Q$ then $a < b$ and $b < c$
$\Rightarrow a < c$ (**transitive law**)

(iii) $a < b \Rightarrow a + c < b + c$ for any $a,b,c \in Q$.

(iv) $a < b$ and $c > 0 \Rightarrow ac < bc$ for any $a,b,c \in Q$.

In view of the above order properties we say that the set $Q$ of all rational numbers is an **ordered set.**

**DEFINITION : (Irrational Numbers)** Any number which is not a rational number is called an *irrational number*. The set of all irrational numbers is denoted by $I$.

**DEFINITION : (Real Numbers)** The union of the set of all rational numbers and the set of all irrational numbers constitute the set of all *real numbers*. The set of all real numbers is denoted by $R$. That is, $R = Z \cup I$.

Clearly, $N \subset Z \subset Q \subset R$.

ORDER PROPERTIES OF $R$ :

(v) If $a$ and $b$ are two real numbers, i.e., $a,b \in R$ then either $a < b$ or $b < a$ ( **law of trichotomy**)

(vi) If $a$, $b$ and $c$ are three real numbers, i.e., $a,b,c \in R$ then $a < b$ and $b < c$
$\Rightarrow a < c$ (**transitive law**)

(vii) $a < b \Rightarrow a + c < b + c$ for any $a,b,c \in R$.

(viii) $a < b$ and $c > 0 \Rightarrow ac < bc$ for any $a,b,c \in R$.

In view of the above order properties we say that the set $R$ of all real numbers is an **ordered set.**

ALGEBRAIC PROPERTIES OF $R$ :

Addition and multiplication are defined on the set $R$.

**Under addition(+)**

(i) $a + b \in R$ for all $a,b \in R$ (Closure property)

(ii) $(a + b) + c = a + (b + c)$ for all $a,b,c \in R$ (Associative property)

(iii) There exists an element 0 in (called the zero element or additive identity) such that
$a + 0 = a$ for all $a \in R$.(existence of additive identity)

(iv) For each $a \in R$ there exists an element $-a \in R$ such that $a + (-a) = 0$ ( existence of additive inverse for each element)

(v) $a + b = b + a$ for all $a,b \in R$ (commutative property)

**Under addition $R$ is an abelian group.**

**Under multiplication(.)**

(vi) $a.b \in R$ for all $a,b \in R$ (Closure property)

(vii) $(a.b).c = a.(b.c)$ for all $a,b,c \in R$ (Associative property)

(viii) There exists an element 1 in (called the unity or multiplicative identity) such that $a.1 = a$ for all $a \in R$.(existence of multiplicative identity)

(ix) For each $a \in R$ there exists an element $\dfrac{1}{a} \in R$ such that $a.\dfrac{1}{a} = 1$ ( existence of multiplicative inverse for each element)

(x) $a.b = b.a$ for all $a, b \in R$ (commutative property)

**Under multiplication $R$ is a commutative group.**

(xi) $a.(b + c) = a.b + a.c$ for all $a, b, c \in R$ (left distributive property)

$(a + b).c = a.c + b.c$ for all $a, b, c \in R$ (right distributive property)

**The set $R$ of real numbers obeys all the field axioms. Also $R$ is an ordered set.**

**Hence $R$ is an ordered field.**

**NOTE :** It should be noted that the set $Q$ of all rational numbers also obeys all the field axioms as well as order properties and hence $Q$ is also an ordered field.

**DEFINITION : (Intervals)** Let $a.b \in R$ such that $a < b$. We write

(i) $[a,b] = \{x : a \leq x \leq b\}$ (called ***closed interval***)

(ii) $(a,b) = \{x : a < x < b\}$ (called ***open interval***)

(iii) $[a,b) = \{x : a \leq x < b\}$ (called ***left closed right open interval***)

(iv) $(a,b] = \{x : a < x \leq b\}$ (called ***right closed left open interval***)

**DEFINITION : (Bounded above set)** A set $S$ of real numbers is called ***bounded above*** if there exists a real number $b$ such that $x \leq b$, $\forall x \in S$. The number $b$ is called ***upper bound*** of $S$.

An upper bound may or may not belong to the set $S$. Every real number greater than an upper bound of a set is also an upper bound of that set. So a bounded above set has infinite number of upper bounds.

**EXAMPLE :** Let us consider the following sets :

$S_1 = [1,2]$ is bounded above set. Here 2 is obviously an upper bound of $S$ since $x \leq 2$, $\forall x \in S$. Every real number greater than 2 is also an upper bound of the set $S_1$. 2 is the **least** of all upper bounds of the set $S_1$.

$S_2 = (1,2)$ is bounded above set. Here 2 is obviously an upper bound of $S$ since $x < 2$, $\forall x \in S$. Every real number greater than 2 is also an upper bound of the set $S_2$. 2 is the **least** of all upper bound of the set $S_2$.

$S_3 = \left\{ 1, \dfrac{1}{2}, \dfrac{1}{3}, \ldots\ldots\ldots \right\}$ is bounded above set. Here 1 is obviously an upper bound of $S_3$ since $x \leq 1$, $\forall x \in S$. Every real number greater than 1 is also an upper bound of the set $S_3$. 1 is the **least** of all upper bounds of the set $S_3$.

$S_4 = \phi$ is bounded above set. Here every real number is an upper bound of $S_4$.

**Least** of all upper bounds is called *least upper bound* or *l.u.b* or *supremum* of the set.

**NOTE :** A set may or may not contain *l.u.b* . In the above example $S_1$ & $S_3$ contain *l.u.b* but $S_2$ does not contain *l.u.b*.

**DEFINITION : (Bounded below set)** A set $S$ of real numbers is called *bounded below* if there exists a real number $a$ such that $x \geq a$, $\forall x \in S$. The number $a$ is called *lower bound* of $S$.

A lower bound may or may not belong to the set $S$. Every real number smaller than a lower bound of a set is also a lower bound of that set. So a bounded below set has infinite number of lower bounds.

**EXAMPLE:** Let us consider the following sets :

$S_1 = [1,2]$. Here 1 is obviously a lower bound of $S$ since $x \geq 1$, $\forall x \in S$. Every real number smaller than 1 is also a lower bound of the set $S_1$. 1 is the **greatest** of all lower bounds of the set $S_1$.

$S_2 = (1,2)$. Here 1 is obviously a lower bound of $S$ since $x \geq 1$, $\forall x \in S$. Every real number smaller than 1 is also a lower bound of the set $S_2$. 1 is the **greatest** of all lower bounds of the set $S_2$.

$S_3 = \left\{ 1, \dfrac{1}{2}, \dfrac{1}{3}, .............. \right\}$. Here 0 is obviously a lower bound of $S_3$ since $x \geq 0$, $\forall x \in S$. Every real number smaller than 0 is also a lower bound of the set $S_3$. 0 is the **greatest** of all lower bounds of the set $S_3$.

$S_4 = \phi$ is bounded below set. Here every real number is a lower bound of $S_4$.

**greatest** of all lower bounds is called **greatest** *lower bound* or *g.l.b* or *infimum* of the set.

**NOTE :** A set may or may not contain *g.l.b*. In the above example $S_1$ contains *g.l.b* but $S_2$ & $S_3$ do not contain *g.l.b* .

**DEFINITION : (Bounded set)** A set $S$ of real numbers is called *bounded* if it is both *bounded above* and *bounded below*.

**DEFINITION : (Suprema) Least** of all upper bounds of a bounded above set is called *least upper bound* or *l.u.b* or *supremum or suprema* of the set.

**EXAMPLE :** Let us consider the following set :

$S = \left\{ 1, \dfrac{1}{2}, \dfrac{1}{3}, ............ \right\}$ is bounded above set. Here 1 is obviously an upper bound of $S$ since $x \leq 1$, $\forall x \in S$. Every real number greater than 1 is also an upper bound of the set $S$. The set of all upper bounds of $S$ is $[1, \infty]$. Clearly 1 is the **least** of all upper bounds of the set $S$. Therefore, **supremum** or **suprema** of $S$ is 1. That is, $\sup S = 1$

**DEFINITION : (Infima) Greatest** of all lower bounds of a bounded below set is called **greatest *lower bound*** or ***g.l.b*** or ***infimum*** or ***infima*** of the set.

**EXAMPLE :** Let us consider the following set :

$S = \left\{ 1, \dfrac{1}{2}, \dfrac{1}{3}, \ldots\ldots\ldots \right\}$ is bounded below set. Here 0 is obviously a lower bound of $S$ since $x \geq 0,\ \forall x \in S$. Every real number smaller than 0 is also a lower bound of the set $S$. The set of all lower bounds of $S$ is $[-\infty, 0]$. Clearly 0 is the **greatest** of all lower bounds of the set $S$. Therefore, **infimum** or **infima** of $S$ is 0. That is, $\inf S = 0$

**DEFINITION : (Greatest member of a set)** A real number $m$ is called the **greastest member** of a set $S$ if (i) $m \in S$ (ii) $m$ is an upper bound of $S$.

**EXAMPLE:** Let us consider the set $S = [1,2]$. Clearly, 2 is the greatest member of the set $S$, since (i) $2 \in S$ and (ii) $2$ is an upper bound of $S$.

Again if we consider the set $S = (1,2)$, we see that this set has no greatest member, for, 2 is obviously an upper bound of $S$ since $x < 2,\ \forall x \in S$ but $2 \notin S$.

**NOTE : An infinite bounded above set may or may not contain greatest member.**

**NOTE : Every finite set has always a greatest member.**

**DEFINITION : (Smallest member of a set)** A real number $n$ is called the **smallest member** of a set $S$ if (i) $n \in S$ (ii) $n$ is a lower bound of $S$.

**EXAMPLE:** Let us consider the set $S = [1,2]$. Clearly, 1 is the smallest member of the set $S$, since (i) $1 \in S$ and (ii) $1$ is a lower bound of $S$.

Again if we consider the set $S = (1,2)$, we see that this set has no smallest member, for, 1 is obviously a lower bound of $S$ since $x > 1,\ \forall x \in S$ but $1 \notin S$.

**NOTE : An infinite bounded above set may or may not contain smallest member.**

**NOTE : Every finite set has always a smallest member.**

<u>**THEOREM** 1.2:</u> *Let $L$ and $U$ be two proper subsets of $R$ of real numbers such that*
  **(i)** $L \cup U = R$
  **(ii)** $\alpha \in L,\ \beta \in U \Rightarrow \alpha < \beta$
*Then either the subset $L$ has a greastest member or the subset $U$ has a smallest member.*

# <u>Completeness property of $R$</u>

<u>**THEOREM** 1.3:</u> **(Least upper bound axiom)** *The set of all upper bounds of a bounded above set admits of a smallest member.*

**Proof :** Let $S$ be the given bounded above set. Let $R$ be the set of all real numbers. Let us consider two proper subsets $L$ and $U$ of $R$ defined as follows:

$L = \{x : x \text{ is not an upper bound of } S\}$

$U = \{x : x \text{ is an upper bound of } S\}$

Clearly, $L \neq \phi$ and $U \neq \phi$. Also $L \cup U = R$ and $\alpha \in L$, $\beta \in U \Rightarrow \alpha < \beta$. Then by theorem 1.2, we can say that either the subset $L$ has a greastest member or the subset $U$ has a smallest member. Let us suppose that $L$ has a greastest member(say $\xi$). Now $\xi \in L$.Then $\xi \notin U \Rightarrow \xi$ is not an upper bound of $S$. Then there must exist a number $\alpha \in S$ such that $\xi < \alpha$. Let $b$ is a real number such that $\xi < b < \alpha$. Now $\xi < b \Rightarrow b \in U$...........(1)

Again $b < \alpha \Rightarrow b$ is not an upper bound of $S$ and this implies $b \notin U$...........(2)

Here we see that (1) and (2) contradicts each other. So $L$ cannot have greatest member. Therefore, by theorem 1.2, we can say that $U$ has a smallest member.

**THEOREM 1.4:** (greatest lower bound axiom) *The set of all lower bounds of a bounded below set admits of a greatest member.*

**Proof :** Proof is left to the students.

In view of above axioms, i.e., theorem 1.3 & theorem 1.4 , the set $R$ of all real numbers is **complete.**


## ARCHIMEDEAN PROPERTY OF REAL NUMBERS

**THEOREM 1.5: (Archimedean property)** *If $x, y \in R$ and $x > 0$,then there exists a positive integer $n$ such that $nx > y$.*

**Proof :** If possible let $nx \leq y$. Let $A = \{nx : n = 1,2,3,...........\}$. Clearly, $y$ is an upper bound of $x > 0$. Therefore, $A$ is bounded above set. Also $A \neq \phi$. Hence by l.u.b axiom, least upper bound i.e., supremum of $A$ ($\sup A$) exists ($= m$ say), where $m \in R$.

Now $x > 0$

$\Rightarrow -x < 0$

$\Rightarrow m - x < m$

$\Rightarrow m - x$ is not an upper bound of $A$.

Therefore, $m - x < px$, for some integer $p$.

That is, $m < (p+1)x = kx$ (say) (taking $k = p+1$)

$\Rightarrow m$ is not an upper bound of $A$.

$\Rightarrow m$ is not $\sup A$ ............ a contradiction.

Hence $nx > y$.


## DENSITY PROPERTY OF $R$

**THEOREM 1.6: (Density property)** *If $x$ and $y$ are two real numbers such that $x < y$, then there exists a rational number $r$ where $x < r < y$.*

**Proof :** <u>**Case-I**</u> Let us suppose that $x > 0$ and $0 < x < y$. Then $y - x = z > 0$. Then by Archimedean property, there exists a positive integer $n$ such that $nz > 1$. That is, $n(y - x) > 1$. Which implies $\dfrac{1}{n} < (y - x)$.

Let $A = \left\{ m : m \in N, \dfrac{m}{n} > x \right\}$. Clearly, $A \neq \phi$ .(by Archimedean property). By well-principle of natural numbers, we know bthat every non-empty subset of natural numbers has a least element. Let $A$ has the least element $p > 1$. Then $\dfrac{p}{n} > x$ but $\dfrac{p-1}{n} \leq x$. Thus $\dfrac{p}{n} \leq x + \dfrac{1}{n} < x + (y - x)$. This implies $\dfrac{p}{n} < y$. Already we have $\dfrac{p}{n} > x$. So $x < \dfrac{p}{n} < y$.

Therefore, there exists a rational number $\dfrac{p}{n} = r$ (say) contained in $0 < x < y$.

<u>**Case-II**</u> Let us suppose that $x \leq 0 < y$. Then by Archimedean property, there exists a positive integer $n$ such that $ny > 1$. Therefore, $\dfrac{1}{n} < y$. This implies $x < \dfrac{1}{n} < y$. Therefore, there exists a rational number $\dfrac{1}{n} = r$ (say) contained in $x \leq 0 < y$.

<u>**Case-III**</u> Lastly, let $x < y \leq 0$. Then $0 \leq -y < -x$. Then by case-I, there is rational number $r$ such that $-y < r < -x$. This implies $x < -r < y$.

This completes the proof.

**CONCLUSION:** The existence of one rational number between $x$ and $y$ implies the existence of infinitely many rational numbers between $x$ and $y$.

Hence $R$ is dense with rational numbers.


# <u>COUNTABILITY : COUNTABLE SETS</u>


**DEFINITION : (Equivalent set)** For any two sets $A$ and $B$, if there exists a one-to-one mapping from the set $A$ onto the set $B$, we say that $A$ is *equivalent* to $B$. Symbolically, $A \sim B$.


**DEFINITION : (Enumerable/Denumerable set)** An infinite set $A$ is said to be *enumerable* or *denumerable* if $A$ is equivalent to the set $N$ of all natural numbers. In other words $A$ is said to be *enumerable* if there exists a bijective mapping $f : N \to A$.


**DEFINITION : (Countable set/ *Uncountable set*)** A set which is either empty or finite or enumerable is called *countable set*. Otherwise it is *uncountable.*

NOTE : An enumerable set is sometimes called countably infinite set.

**DEFINITION : (Atmost countable set)** A set $A$ is called **atmost countable** if $A$ is either finite or a countable set.

## EXAMPLES OF COUNTABLE SETS

(A) The set $S = \{2,4,6,\ldots\ldots\}$ of all even natural numbers is enumerable(i.e., countable) because there exists a a one-to-one onto mapping( bijective) $f : N \to S$ defined by $f(n) = 2n,\ n \in N$ ( In other words $S \sim N$ )

(B) The set $S = \{0,2,4,6,\ldots\ldots\}$ is enumerable(i.e., countable) because there exists a a one-to-one onto mapping( bijective) $f : N \to S$ defined by $f(n) = 2n,-2\ n \in N$ ( In other words $S \sim N$ )

(C) The set $S = \{1^2,2^2,3^2\ldots\ldots\}$ is enumerable(i.e., countable) because there exists a a one-to-one onto mapping( bijective) $f : N \to S$ defined by $f(n) = n^2,\ n \in N$ ( In other words $S \sim N$ )

(D) The set $N = \{1,2,3,\ldots\ldots\}$ is enumerable(i.e., countable) because there exists a a one-to-one onto mapping( bijective) $f : N \to N$ defined by $f(n) = n,\ n \in N$ ( In other words $N \sim N$ )

(E) The set $Z = \{0,\pm1,\pm2,\pm3,\ldots\ldots\}$ is enumerable(i.e., countable) because there exists a a one-to-one onto mapping( bijective) $f : N \to Z$ defined by

$$f(n) = \frac{n}{2} \text{ if } n \text{ is even}$$

$$= \frac{1-n}{2} \text{ if } n \text{ is odd ( In other words } Z \sim N )$$

## THEOREMS ON COUNTABILITY

**THEOREM : 1.7** An infinite subset of an enumerable set is enumerable.

**COROLLARY :** A subset of an enumerable set is either finite or enumerable.

**THEOREM : 1.8** The union of a finite set and an enumerable set is enumerable.

**THEOREM : 1.9** The union of two enumerable sets is enumerable.

**THEOREM : 1.10** The union an enumerable numbers of enumerable sets is enumerable.

**THEOREM : 1.11** Every superset of an uncountable set is uncountable.

**THEOREM : 1.12** If $f : A \to B$ is a bijective mapping( i.e.,one-to one onto) and $B$ is enumerable then $A$ is also enumerable.

**P-I :** *Show that the set $Q$ of rational numbers is countable*.

**Solution:** Let $Q^+$ be the set of all positive rational numbers, $Q^-$ be the set of all negative rational numbers. Then obviously, $Q = Q^+ \cup Q^- \cup \{0\}$.

Let us consider the following collection of sets:

$$A_1 = \left\{ \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \ldots\ldots\ldots, \frac{n}{1}, \ldots\ldots\ldots\ldots \right\},$$

$$A_2 = \left\{ \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \ldots\ldots\ldots, \frac{n}{2}, \ldots\ldots\ldots\ldots \right\},$$

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$$A_n = \left\{ \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots\ldots\ldots, \frac{n}{n}, \ldots\ldots\ldots\ldots \right\}$$

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Clearly, $Q^+ = A_1 \cup A_2 \cup A_3 \cup \ldots\ldots\ldots \cup A_n \cup \ldots\ldots\ldots\ldots = \bigcup_{n=1}^{\infty} A_n$. Now $A_1$ is enumerable since there exists a bijective mapping ( i.e.,one-to one onto) $f : N \to A_1$ defined by $f(n) = \dfrac{n}{1}$, $n \in N$. Similarly, $A_2$ is enumerable since there exists a bijective mapping ( i.e.,one-to one onto) $f : N \to A_2$ defined by $f(n) = \dfrac{n}{2}$, $n \in N$.

Thus we can see that each of the sets $A_1, A_2, A_3, \ldots\ldots\ldots, A_n, \ldots\ldots\ldots$ is an enumerable set. Also the collection $A_1, A_2, A_3, \ldots\ldots\ldots, A_n, \ldots\ldots\ldots$ is an enumerable collection sets.

Therefore, $Q^+ = A_1 \cup A_2 \cup A_3 \cup \ldots\ldots\ldots \cup A_n \cup \ldots\ldots\ldots\ldots = \bigcup_{n=1}^{\infty} A_n$ is the union of an enumerable collection of enumerable sets. Hence $Q^+$ is enumerable. Similarly, it can be shown that $Q^-$ is enumerable. [ Alternative proof : Since $Q^+$ is enumerable and there exists a bijective mapping ( i.e.,one-to one onto) $f : Q^- \to Q^+$ defined by $f(x) = -x$, $x \in Q^-$, $Q^-$ is also enumerable.]

Therefore, $Q^+ \cup Q^-$ is enumerable because the union of two enumerable sets is enumerable.

Since $Q^+ \cup Q^-$ is enumerable and $\{0\}$ is finite set, $Q^+ \cup Q^- \cup \{0\}$ is enumerable. That is, $Q \left(= Q^+ \cup Q^- \cup \{0\}\right)$ is enumerable.

## P-II : *Show that the set* $(0,1)$ *is not enumerable. Hence show that the set* $R$ *of real numbers is not enumerable.*

**Solution:** If possible let the set $(0,1)$ is enumerable. Then the numbers in the set $(0,1)$ can be described as $x_1, x_2, x_3, \ldots\ldots\ldots$ . That is, they can be arranged in a succession. Also each real number in $(0,1)$ can be written as infinite decimal forms in the following way.

$x_1 = 0.\, a_1 a_2 a_3 a_4 a_5 \ldots\ldots\ldots$

$x_2 = 0.\, b_1 b_2 b_3 b_4 b_5 \ldots\ldots\ldots$

$x_3 = 0.\, c_1 c_2 c_3 c_4 c_5 \ldots\ldots\ldots$

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Let us consider a real number $x_\lambda = 0.\, pqrst\ldots\ldots\ldots$

where $p = 6(say)$ if $a_1 \neq 6$ $\qquad$ $q = 5(say)$ if $b_2 \neq 5$

$\qquad\quad = 7(say)$ if $a_1 = 6$ $\qquad\quad = 6(say)$ if $b_2 = 5$ $\qquad$ and so on.

According to construction of $x_\lambda$ in the way described above, $x_\lambda \notin (0,1)$. But According to the numerical value of $x_\lambda$, $x_\lambda \in (0,1)$. We thus arrived at a contradiction.

Therefore, the set $(0,1)$ is not enumerable.

**2$^{nd}$ part :** $(0,1) \subset R$. $R$ i.e., the superset of a non-enumerable set $(0,1)$ is not enumerable.

**COROLLARY : *The set* $I$ *of all irrational numbers is non-enumerable.***
**Proof :** Let $I$ be the set of all irrational numbers. Then $Q \cup I = R$. We know tha the union of two enumerable sets is enumerable. Since $R$ is not enumerable, $I$ cannot be enumerable although $Q$ is enumerable.

**DEFINITION : (Nested intervals)** If $\{I_1, I_2, I_3, \ldots\ldots\ldots\}$ be a family of intervals such that $I_{n+1} \subset I_n$, $n \in N$ then the family $\{I_1, I_2, I_3, \ldots\ldots\ldots\}$ is said to be a family of *nested intervals*.

**EXAMPLE :** Let $I_1 = (0,1)$, $I_2 = \left(0, \dfrac{1}{2}\right)$, $I_3 = \left(0, \dfrac{1}{3}\right), \ldots\ldots\ldots$ be a family of intervals. Clearly, $I_1 \supset I_2 \supset I_3 \supset \ldots\ldots\ldots$ Therefore, the family $\{I_1, I_2, I_3, \ldots\ldots\ldots\}$ is a family of nested intervals.

**THEOREM : 1.13** $\quad$ If $\{[a_1, b_1], [a_2, b_2], [a_3, b_3], \ldots\ldots\ldots\}$ be a family of nested closed and bounded intervals then $[a_1, b_1] \cap [a_2, b_2] \cap [a_3, b_3] \cap \ldots\ldots\ldots \neq \phi$

**P-III** : *Show that the closed interval $[a,b]$ is not enumeable. Hence show that the set $R$ of real numbers is not enumerable.*

**Solution:** If possible let the set $I = [a,b]$ is enumerable. Then the elements $I$ of can be described as $x_1, x_2, x_3, \ldots\ldots\ldots\ldots$ . That is, they can be arranged in a succession. That is, $I = [a,b] = \{x_1, x_2, x_3, \ldots\ldots\ldots\}$. Let us divide the interval $I = [a,b]$ into three subintervals $[a,c]$, $[c,d]$, $[d,b]$(say). Atleast one these three subintervals does not contain $x_1$. Let us call that subinterval by $I_1$. Again Let us divide the interval $I_1$ into three subintervals such that atleast one these three subintervals of $I_1$ does not contain $x_2$. Let us call that subinterval by $I_2$. So $I_2$ does not contain both $x_1$ and $x_2$. Continuing this process we can obtain a family of closed and bounded subintervals $I_3$, $I_4$ , $I_5$,……………………………………………such that

$I_1$ does not contain $x_1$

$I_2$ does not contain $x_1$ and $x_2$

$I_3$ does not contain $x_1$, $x_2$, $x_3$

.

.

$I_n$ does not contain $x_1$, $x_2$, $x_3$,……………., $x_n$

.

.

.

Now by the construction of $I_1$, $I_2$ , $I_3$ ,……, $I_1 \cap I_2 \cap I_3 \cap \ldots\ldots \cap I_n \cap \ldots\ldots\ldots = \phi$ ………(1)

Again as $I_1 \supset I_2 \supset I_3 \supset \ldots\ldots\ldots\ldots\ldots$, by the theorem of nested interval , we must have $I_1 \cap I_2 \cap I_3 \cap \ldots\ldots \cap I_n \cap \ldots\ldots\ldots \neq \phi$ ………….(2). Clearly, (1) and (2) contradicts each other. Hence $I = [a,b]$ is not enumerable.

**2$^{nd}$ part :** $[a,b] \subset R$. $R$ i.e., the superset of a non-enumerable set $[a,b]$ is not enumerable.


**P-IV** : *Show that the set $R$ of all real numbers is not enumerable*(**i.e., not countable).**

**Solution:** If possible let the set $R$ is enumerable (i.e., countable). Then the elements $R$ of can be described as $x_1, x_2, x_3, \ldots\ldots\ldots$ . That is, they can be arranged in a succession. That is, $R = \{x_1, x_2, x_3, \ldots\ldots\ldots\}$. Let us consider the following open intervals :

$$I_1 = \left( x_1 - \frac{1}{2^2}, x_1 + \frac{1}{2^2} \right)$$

$$I_2 = \left( x_2 - \frac{1}{2^3}, x_2 + \frac{1}{2^3} \right)$$

.

$$I_n = \left( x_n - \frac{1}{2^{n+1}}, x_n + \frac{1}{2^{n+1}} \right)$$

.

.

.

Now $x_1 \in I_1 \Rightarrow \{x_1\} \subset I_1$

$\quad\quad x_2 \in I_2 \Rightarrow \{x_2\} \subset I_2$

.

.

.

$\quad\quad x_n \in I_n \Rightarrow \{x_n\} \subset I_n$

.

.

.

Therefore, $\{x_1\} \cup \{x_2\} \cup \{x_3\} \cup \text{.......} \cup \{x_n\} \cup \text{.............} \subset I_1 \cup I_2 \cup I_3 \cup \text{........} \cup I_n \cup \text{.............}$

That is, $R \subset I_1 \cup I_2 \cup I_3 \cup \text{........} \cup I_n \cup \text{.............}$ $\quad$ ...............(A).

Length of $I_1 = \dfrac{1}{2}$, length of $I_2 = \dfrac{1}{2^2}$ ,........., length of $I_n = \dfrac{1}{2^n}$ ,................... . $\quad$ The total length

of $I_1$, $I_2$ , $I_3$,...... $= \dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \text{.........} = \dfrac{\dfrac{1}{2}\left(1 - \dfrac{1}{2^n}\right)}{1 - \dfrac{1}{2}} = 1 - \dfrac{1}{2^n} = 1$ $\quad$ when $n \to \infty$

Relation (A) shows that the whole real line ( whose length is infinite ) is contained in the union of intervals whose length add upto 1 which is clearly, not possible.

Hence $R$ is not enumerable (i.e., not countable).

**DEFINITION : ($\varepsilon - $ neighbourhood)** Let $\xi$ be a real number. Then any open interval of which $\xi$ is a member is called a ***neighbourhood*** of $\xi$ .

In particular, for any $\varepsilon > 0$, the open interval $(\xi - \varepsilon, \xi + \varepsilon)$ is called $\varepsilon - $***neighbourhood*** of $\xi$ .

$\varepsilon - $**neighbourhood** of $\xi$ is denoted by $N_\zeta(\varepsilon)$

**EXAMPLE :** By $.01 - $neighbourhood of $5$ we mean the open interval (5-.01 , 5+.01), i.e., we mean the open interval (4.99 , 5.01). Symbolically, $N_5(.01) = (4.99 , 5.01)$.

**DEFINITION : (Interior point)** Let $A$ be any non empty linear point set. Any element $x \in A$ is said to be an ***interior point*** of $A$ if there exists some neighbourhoods of $x$ which lie wholly in $A$ .

**EXAMPLE :** Every point of $(2,3)$ is an interior point. Where as every point of $[2,3]$ except 2 and 3 is an interior point.

**DEFINITION : (Interior of a set)** The set of all interior points of a set $A$ is called interior of $A$.

**DEFINITION : (Open set)** A linear point set $A$ is called an **open set** if and only if every point of $A$ is an interior point of $A$.

**EXAMPLES : (i)** Any open interval is an open set. **(ii)** $\phi$ is an open set. **(iii)** The set $R = (-\infty, \infty)$ an open set. **(iv)** A finite set is not an open set.
**(v)** The set $Q$ of all rational numbers is not an open set.

**P-V** *Prove that the set $Q$ of all rational numbers is not an open set.*

**Solution:** Let $x \in Q$. Let $N_x(\varepsilon)$ be $\varepsilon-$neighbourhood of $x$, where $\varepsilon > 0$, $\varepsilon$ is arbitrary. Since $N_x(\varepsilon)$ contains rational and as well as irrational numbers, $N_x(\varepsilon) \not\subset Q$. Since $\varepsilon$ is arbitrary, there is no such neighbourhood of $x$ which lie wholly in $Q$. So, $x$ is not an interior point of $Q$. Hence $Q$ is not an open set.

**DEFINITION : (Boundary point)** A real number $x$ is said to be a **boundary point** or a **frontier point** of a set $A$ if every neighbourhood of $x$ contains atleast one point of $A$ and atleast one point that does not belong to $A$.
**EXAMPLE :** 2 and 3 are two boundary points of the set $A = [2,3]$.

**DEFINITION : (Exterior point)** A real number $x$ is said to be a **exterior point** of a set $A$ if there exists some neighbourhood of $x$ which entirely lie outside of the set $A$.

## PROPERTIES OF OPEN SETS

**THEOREM 1.14:** *The union of an arbitrary collection of open sets is an open set.*
**Proof :** Let $S_1, S_2, S_3, \ldots\ldots\ldots$ be an arbitrary collection of open sets. Let $S = S_1 \cup S_2 \cup S_3 \cup \ldots\ldots \cup S_n \cup \ldots\ldots = \bigcup_n S_n$. We shall show that $S$ is open set. Let $x \in S$. Then $x$ be a member of $S_n$ for some $n$. Let $x \in S_k$ $(n = k)$. Since every $S_n$ is open set, $S_k$ is also open. Since $x \in S_k$, $x$ is an interior point of $S_k$. So there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset S_k$. As each $S_k \subset S$, $(x - \varepsilon, x + \varepsilon) \subset S$. Thus $x$ is an interior point of $S$. Since $x$ is an arbitrary point of $S$, every member of $S$ is an interior point of $S$. Consequently, $S$ is open.

**<u>THEOREM</u> 1.15:** *The intersection of finite number of open sets is an open set.*

**Proof :** Let $S_1, S_2, S_3, \ldots\ldots\ldots\ldots\ldots\ldots S_n$ be a finite collection of open sets. Let $S = S_1 \cap S_2 \cap S_3 \cap \ldots\ldots \cap S_n \cap \ldots\ldots = \bigcap\limits_{k=1}^{n} S_k$. We shall show that $S$ is open set. If $S$ is empty, i.e., $S = \phi$ then there is nothing to prove because empty set is an open set. Let $S \neq \phi$. Let $x \in S$. Then $x \in S_k$ for every $k = 1,2,3,\ldots\ldots.,n$. So is $x$ an interior point of $S_k$ for every $k = 1,2,3,\ldots\ldots.,n$. Therefore, there must exist $\varepsilon_k > 0, k = 1,2,3,\ldots\ldots.,n$, such that $(x - \varepsilon_k, x + \varepsilon_k) \subset S_k$ for every $k = 1,2,3,\ldots\ldots.,n$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots\ldots, \varepsilon_n\}$. Then clearly, $(x - \varepsilon, x + \varepsilon) \subset S_k$ for every $k = 1,2,3,\ldots\ldots.,n$. Hence $(x - \varepsilon, x + \varepsilon) \subset \bigcap\limits_{k=1}^{n} S_k = S$. This shows that $x$ is an interior point of $S$. Since $x$ is an arbitrary point of $S$, every member of $S$ is an interior point of $S$. Consequently, $S$ is open.

**<u>P-VI</u> :** *Show by an example that intersection of an arbitrary collection of open sets may not be open.*

**Solution:** Let $S_1 = \left(\dfrac{-1}{1}, \dfrac{1}{1}\right)$, $S_2 = \left(\dfrac{-1}{2}, \dfrac{1}{2}\right)$, $S_3 = \left(\dfrac{-1}{3}, \dfrac{1}{3}\right),\ldots\ldots\ldots\ldots$ be an arbitrary collection of open sets. Then $S = S_1 \cap S_2 \cap S_3 \cap \ldots\ldots\ldots\ldots\ldots = \{0\}$ which is a finite set and hence $S$ is not open.

Again let $S_1 = \left(\dfrac{-1}{2}, \dfrac{1}{2}\right)$, $S_2 = \left(\dfrac{-2}{3}, \dfrac{2}{3}\right)$, $S_3 = \left(\dfrac{-3}{4}, \dfrac{3}{4}\right),\ldots\ldots\ldots\ldots$ be an arbitrary collection of open sets. Then $S = S_1 \cap S_2 \cap S_3 \cap \ldots\ldots\ldots\ldots\ldots = \left(\dfrac{-1}{2}, \dfrac{1}{2}\right)$ which is an open interval and hence open.

**DEFINITION : (Limit point)** A real number $x$, may or may not belong to a set $A$ is said to be a *limit point* or a *accumulation point or a cluster point* of a set $A$, if every neighbourhood of $x$ contains atleast one point of $A$ other than $x$.

Infact, every neighbourhood of $x$ contains infinitely many point of the set $A$.

**EXAMPLES :**

   A) Every member of the set $[2,3]$ is a limit point of this set. There is no other limit point of this lying outside of this set.

   B) Every member of the set $(2,3)$ is a limit point of this set. There are another two limit points of this lying outside of this set, namely, 2 and 3.

**C)** The set $\left\{1, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \ldots\ldots\right\}$ has only one limit point, namely, $0$ but $0$ does not

belong this set.

**D)** The set $\left\{\dfrac{3}{2}, \dfrac{-4}{3}, \dfrac{5}{4}, \dfrac{-6}{5}, \dfrac{7}{6}, \dfrac{-8}{7}, \ldots\ldots\ldots\right\}$ has two limit points, namely, $-1$ and $1$

but they do not belong this set.

**E)** A finite set has no limit point but it does not necessarily mean that an infinite set has always a limit point. The set $N = \{1,2,3\ldots\ldots\}$ of all natural numbers or the set $Z = \{\ldots\ldots -3,-2,-1,0,1,2,3,\ldots\ldots\ldots\}$ of all integers has no limit point.

**DEFINITION : (Isolated point)** A real number $x \in S$ is said to be an ***isolated point*** of $S$ if there exists a neighbourhood of $x$ which contains $x$ but no other point of $S$.

**EXAMPLE :**

- A finite set contains isolated points only.
- Each point of the set of all integers is an isolated point.
- The set $[2,3]$ or the set $(2,3)$ has no isolated point.

**DEFINITION : (Derived set)** The set of all limit points of a set $S$ is called ***derived set*** of that set. The derived set of a set $S$ is denoted by $S'$.

**DEFINITION : (Closed set)** A set $S$ is called ***closed*** if and only if every limit point of $S$ is member of $S$. That is, $S' \subset S$.

**EXAMPLES :.**

- The derived set of the set $S = \left\{0, \dfrac{1}{1}, \dfrac{1}{2}, \dfrac{1}{3}, \ldots\ldots\right\}$ is $S' = \{0\}$ ( the only limit point is $0$ ).

  Clearly, $S' \subset S$. Hence the set $S$ is closed. Whereas the set $S = \left\{\dfrac{1}{1}, \dfrac{1}{2}, \dfrac{1}{3}, \ldots\ldots\right\}$ is not

  closed, since the only limit point of $S$ is $0$ and $S' = \{0\} \not\subset S$.
- The derived set of the set $S = (2,3)$ is $S' = [2,3]$. As $S' \not\subset S$, the set $S$ is not closed.
- The derived set of the set $S = [1,2]$ is $S' = [1,2]$. As $S' \subseteq S$, the set $S$ is closed.
- The empty set $\phi$ is closed because $\phi' \subset \phi$.
- The set $R$ of all real numbers is closed because every limit point of $R$ is a member of $R$. That is, $R' \subset R$.
- The set $Q$ of all rational numbers is not closed because every rational number as well as every irrational number is a limit point of $Q$ and irrational limit points do not belong to $Q$. That is, $Q' \not\subset Q$. Infact, $Q' = R \not\subset Q$.

**DEFINITION : (Dense-In-Itself set)** A set $S$ is called ***Dense-In-Itself*** if and only if every member of $S$ is a limit point of $S$. That is, $S \subset S'$.

**EXAMPLES** :.

- The set $S = (2,3)$ is Dense-In-Itself because $S' = [2,3]$ and $S \subset S'$.
- The set $S = [1,2]$ is Dense-In-Itself because $S' = [1,2]$ and $S \subseteq S'$.
- The set $Q$ of all rational numbers is Dense-In-Itself because $Q' = R$ and $Q \subset R = Q'$.
- The set $R$ of all real numbers is Dense-In-Itself because every member of $R$ is a limit point of $R$. That is, $R \subset R'$.

**DEFINITION : (Perfect set)** A set $S$ is called **Perfect set** if $S = S'$. That is, $S$ is both closed and Dense-In-Itself ( $S' \subset S$ & $S \subset S'$ ).

**EXAMPLES** :.

- The set $R$ of all real numbers is perfect because $R' \subset R$ & $R \subset R'$.
- The set $S = [1,2]$ is perfect because $S' \subset S$ & $S \subset S'$.

**DEFINITION : (Closure of a set)** The union of a set $S$ and its derived set $S'$ is called closure of $S$. The closure of a set is denoted by $\overline{S}$. That is, $\overline{S} = S \cup S'$.

In case of closed set $S$, $\overline{S} = S$.

**THEOREM 1.16:** *If a set $S$ is open then its complement $S^c$ is closed. Conversely, if a set $S$ is closed then its complement $S^c$ is open.*

**Proof :** Let *a set* $S$ is open. We are to prove $S^c$ is closed**.** That is, we shall prove $S^{c'} \subset S^c$**.** Let $x \in S^{c'}$. We have show $x \in S^c$. If possible let, $x \notin S^c$. Then $x \in S$. Since $S$ is open, $x$ is an interior point of $S$. That is, there exists a neighbourhood of $x$ lie wholly in $S$. So all the elements of that particular neighbourhood of $x$ will be in the set $S$ not in $S^c$ ……………….(i). Again, $x \in S^{c'}$ (assumption) implies $x$ is a limit point of $S^c$. That is, every neighbourhood of $x$ will contain at least one element of $S^c$ other than $x$. Which implies that some points of every neighbourhood of $x$ will be in the set $S^c$ ……………….(ii). Clearly, the statements (i) & (ii) contradicts each other. So, our assumption was wrong. Hence $x \in S^c$. Therefore, $S^{c'} \subset S^c$. This implies $S^c$ is closed**.**

Conversely, let the *set* $S$ is closed**.** We are to prove $S^c$ is open**.** Let $x \in S^c$. Then $x \notin S$. Since the *set* $S$ is closed and $x \notin S$, $x$ is not a limit point of $S$. So there exist some neighbourhood of $x$ which do not contain any point of $S$ other than $x$. That is, there exist some neighbourhood of $x$ which lie entirely in $S^c$. This implies $x$ is an interior point $S^c$. Since $x$ is arbitrary it follows that every point in $S^c$ is an interior point of $S^c$. Hence $S^c$ is open**.**

## PROPERTIES OF CLOSED SETS

**THEOREM 1.17:** *The union of a finite collection of closed sets is a closed set.*

**Proof :** Let $S_1, S_2, S_3, \ldots, S_n$ be a finite collection of closed sets. Then $S_1^c, S_2^c, S_3^c, \ldots, S_n^c$ is a finite collection of open sets.

Therefore, $\left(S_1 \cup S_2 \cup S_3 \cup \text{.......} \cup S_n\right)^c = S_1^c \cap S_2^c \cap S_3^c \cap \text{........} \cap S_n^c$ (By De Morgan's Law). Right hand side of the above relation is the intersection of finite number of open sets and hence the intersection is open ( by theorem 1.15). Thus $\left(S_1 \cup S_2 \cup S_3 \cup \text{.......} \cup S_n\right)^c$ is open. Clearly this implies $S_1 \cup S_2 \cup S_3 \cup,\text{......},\cup S_n$ is closed.

**THEOREM 1.18:** *The intersection of an arbitrary collection of closed sets is a closed set.*

**Proof :** Let $S_1, S_2, S_3, \text{............}$ be an arbitrary collection of closed sets. Then $S_1^c, S_2^c, S_3^c, \text{..........}$ is an arbitrary collection of open sets.

Therefore, $\left(S_1 \cap S_2 \cap S_3 \cap \text{.........}\right)^c = S_1^c \cup S_2^c \cup S_3^c \cup \text{..........}$ (By De Morgan's Law). Right hand side of the above relation is the union of an arbitrary collection of open sets and hence the union is open ( by theorem 1.14). Thus $\left(S_1 \cap S_2 \cap S_3 \cap \text{.........}\right)^c$ is open. Clearly this implies $S_1 \cap S_2 \cap S_3 \cap,\text{...........}$ is closed.

**P-VII :** *Show by an example that union of an arbitrary collection of closed sets may not be closed.*

**Solution :** Let $S_1 = \left[-1,1\right]$, $S_2 = \left[\dfrac{-1}{2}, \dfrac{1}{2}\right]$, $S_3 = \left[\dfrac{-1}{3}, \dfrac{1}{3}\right], \text{..........}$ be an arbitrary collection of closed sets. Then $S_1 \cup S_2 \cup S_3 \cup,\text{...........} = \left[-1,1\right]$ which is a closed interval and hence closed.

Again let $S_1 = \left[1,2\right]$, $S_2 = \left[\dfrac{1}{2},2\right]$, $S_3 = \left[\dfrac{1}{3},2\right]$, $\text{.........}$ be an arbitrary collection of closed sets. Then $S_1 \cup S_2 \cup S_3 \cup,\text{...........} = \left(0,2\right]$ which is not a closed set.

## BOLZANO WEIERSTRASS THEOREM

**Theorem 1.19:** *Every bounded infinite subset of the set of all real numbers has atleast one limit point.*

**DEFINITION : (Covering of a set/ open covering/ subcovering)** Let $S$ be a set of some real numbers. A collection $F$ of sets is said to be a *Covering* of $S$ if every $x \in S$ is contained in some sets in $F$.

If all the sets in $F$ are open sets, then the collection $F$ is called *open covering* of $S$.

If a subcollection of $F$ is also a covering of $S$ then this subcollection is called *subcovering* of $S$.

**EXAMPLE :** Let $S = \left(0,1\right)$. Let us consider the following collection of sets :

$F = \left\{ \left( \frac{1}{2}, 1 \right), \left( \frac{1}{3}, \frac{2}{3} \right), \left( \frac{1}{4}, \frac{2}{4} \right), \ldots \ldots \right\}$. This collection of sets clearly covers the set $S$. It is also an open covering.

# HEINE-BOREL THEOREM

**Theorem 1.20: (Heine-Borel Theorem)** Every open covering of a closed and bounded set $S$ of real numbers has a finite subcovering.

      If a set $S$ is such that every open covering of $S$ has a finite subcovering then the set $S$ is said to possess the **Heine-Borel property.**

**DEFINITION : (Compact Set)** A set $S$ is said to be *Compact* if and only if the set possesses the Heine-Borel property. That is, the set $S$ is *Compact* if and only if every open covering of $S$ contains a finite subcovering.

**NOTE :** Heine-Borel Theorem is not true if the set $S$ is not closed and bounded.

**SOME EXAMPLES ON DERIVED SET**

**EX :** Find the derived set of the set $S = \left\{ \frac{1}{n} : n \in N \right\}$.

**SOLUTION :** Let $\varepsilon > 0$. By Archimedean property there exist a positive integer $n$ such that

$0 < \dfrac{1}{n} < \varepsilon$

$\Rightarrow -\varepsilon < \dfrac{1}{n} < \varepsilon$

$\Rightarrow 0 - \varepsilon < \dfrac{1}{n} < 0 + \varepsilon$

$\Rightarrow \dfrac{1}{n} \in (0 - \varepsilon, 0 + \varepsilon) \Rightarrow \varepsilon - $ neighbourhood of $0$ contains points of the set $S$. Since $\varepsilon$ is arbitrary, we can say that any neighbourhood of $0$ contains points of the set $S$. So, $0$ is the only limit point of the set $S$. Therefore, the derived set of the set $S$ is $S' = \{0\}$.

**EX :** Find the derived set of the set $S = \left\{ \dfrac{(-1)^m}{m} + \dfrac{1}{n} : m \in N, n \in N \right\}$.

**SOLUTION :** Let $\varepsilon > 0$. By Archimedean property there exist a positive integer $m$ such that

$0 < \dfrac{1}{m} < \varepsilon$

$\Rightarrow -\varepsilon < \dfrac{1}{m} < \varepsilon$

$$\Rightarrow -\varepsilon < \frac{(-1)^m}{m} < \varepsilon$$

$$\Rightarrow \frac{1}{n} - \varepsilon < \frac{(-1)^m}{m} + \frac{1}{n} < \frac{1}{n} + \varepsilon$$

$$\Rightarrow \frac{(-1)^m}{m} + \frac{1}{n} \in \left(\frac{1}{n} - \varepsilon, \frac{1}{n} + \varepsilon\right) \Rightarrow \varepsilon - \text{neighbourhood of } \frac{1}{n} \text{ contains points of the set } S.$$

Since $\varepsilon$ is arbitrary, we can say that any neighbourhood of $\frac{1}{n}$ contains points of the set $S$.

So, $\frac{1}{n}$ , $n \in N$ is limit point of the set $S$.

Again let $\varepsilon > 0$. By Archimedean property there exist a positive integer $n$ such that

$$0 < \frac{1}{n} < \varepsilon$$

$$\Rightarrow -\varepsilon < \frac{1}{n} < \varepsilon \quad \Rightarrow \frac{(-1)^m}{m} - \varepsilon < \frac{(-1)^m}{m} + \frac{1}{n} < \frac{(-1)^m}{m} + \varepsilon$$

$$\Rightarrow \frac{(-1)^m}{m} + \frac{1}{n} \in \left(\frac{(-1)^m}{m} - \varepsilon, \frac{(-1)^m}{m} + \varepsilon\right) \Rightarrow \varepsilon - \text{neighbourhood of } \frac{(-1)^m}{m} \text{ contains points of}$$

the set $S$. Since $\varepsilon$ is arbitrary, we can say that any neighbourhood of $\frac{(-1)^m}{m}$ contains points

of the set $S$. So, $\frac{(-1)^m}{m}$ is a limit point of the set $S$.

Again let $\varepsilon > 0$. By Archimedean property there exist a positive integer $n$ such that

$$0 < \frac{(-1)^m}{m} + \frac{1}{n} < \varepsilon$$

$$\Rightarrow -\varepsilon < \frac{(-1)^m}{m} + \frac{1}{n} < \varepsilon \quad \Rightarrow 0 - \varepsilon < \frac{(-1)^m}{m} + \frac{1}{n} < 0 + \varepsilon$$

$$\Rightarrow \frac{(-1)^m}{m} + \frac{1}{n} \in (0 - \varepsilon, 0 + \varepsilon) \Rightarrow \varepsilon - \text{neighbourhood of } 0 \text{ contains points of the set } S. \text{ Since}$$

$\varepsilon$ is arbitrary, we can say that any neighbourhood of $0$ contains points of the set $S$. So, $0$ is a limit point of the set $S$.

Therefore, the derived set of the set $S$ is $S' = \left\{\frac{1}{n}, n \in N\right\} \cup \left\{\frac{(-1)^m}{m}, m \in N\right\} \cup \{0\}$.

That is, $S' = \left\{0, \pm 1, \frac{1}{2}, \pm \frac{1}{3}, \ldots\ldots\ldots\right\}$.

**EX :** Find the derived set of the set $S = \left\{\frac{2}{m} + \frac{3}{n} : m \in N, n \in N\right\}$.

**SOLUTION :** Left as an exercise.

**EX :** Find the derived set of the set $S = \left\{(-1)^m + \dfrac{1}{n} : m \in N, n \in N\right\}$.

**SOLUTION :** Clearly, $S = \left\{(-1)^{2k} + \dfrac{1}{n} : m \in N, n \in N\right\} \cup \left\{(-1)^{2k-1} + \dfrac{1}{n} : m \in N, n \in N\right\}$

$$= A \cup B.$$

Since, $\left\{\dfrac{1}{n}, n \in N\right\}$ has only limit point $0$, the set $A$ has only limit point $1$ and the set $B$ has only limit point $-1$. Therefore, the derived set of the set $S$ is $S' = (A \cup B)' = A' \cup B' = \{-1,1\}$.

**EX :** Find the derived set of the set $S = \left\{(-1)^{n-1} \dfrac{n}{n-1} : n \in N\right\}$.

**SOLUTION :** Clearly, $S = \left\{(-1)^{n-1} + \dfrac{(-1)^{n-1}}{n-1} : n \in N\right\}$. Let $\varepsilon > 0$. By Archimedean property there exist a positive integer $n$ such that $0 < \dfrac{1}{n-1} < \varepsilon$

$$\Rightarrow -\varepsilon < \dfrac{1}{n-1} < \varepsilon$$

$$\Rightarrow -\varepsilon < \dfrac{(-1)^{n-1}}{n-1} < \varepsilon$$

$$\Rightarrow (-1)^{n-1} - \varepsilon < (-1)^{n-1} + \dfrac{(-1)^{n-1}}{n-1} < (-1)^{n-1} + \varepsilon.$$ Therefore, $\varepsilon$ – neighbourhood of $(-1)^{n-1}$ contains points of the set $S$. So, $(-1)^{n-1}$, i.e., $1$ and $-1$ are the limit points of the set $S$ (according as $n$ is odd or even). Therefore, the derived set of the set $S$ is $S' = \{-1,1\}$.

# QUESTIONS

- Define "limit point" of a set. Show that a finite set cannot have any limit point. Give an example of an infinite set which does not have any limit point.
- State Bolzano Weierstrass theorem. Verify this theorem for the set $S = \left\{1 + \dfrac{(-1)^n}{n}, n \in N\right\}$.
- Define interior of a set. Find the interior of the set $S = \{x \in R : 1 < x < 3\}$.
- State Peano's axioms for natural numbers. Using this axioms prove that the set of all natural numbers has no upper bound.
- Prove that $\sqrt{17}$ is not rational number.
- Prove that the set $Q$ of all rational numbers is not complete.

- If $x, y \in R$ and $x > 0$, then show that there exists a positive integer $n$ such that $nx > y$. (Archimedean property)

- If $x$ and $y$ are two real numbers such that $x < y$, then show that there exists a rational number $r$ where $x < r < y$. (Density property)

- Define enumerable set. Prove that the set $Q$ of all rational numbers is enumerable(countable).

- Is the set $Z = \{0, \pm 1, \pm 2, \pm 3, \ldots \ldots \ldots\}$ of all integers enumerable(countable) ? -----justify

- Show that the set $(0,1)$ is not enumerable. Hence show that the set $R$ of real numbers is not enumerable.

- Show that the closed interval $[a, b]$ is not enumeable. Hence show that the set $R$ of real numbers is not enumerable.

- Show that the set $R$ of all real numbers is not enumerable(i.e., not countable).

- Define "open set" , "closed set".

- Prove that the set $Q$ of all rational numbers is neither open nor closed.

- Prove that the intersection of finite collection of open sets is open. Show by an example that the intersection of an infinite collection open sets may not be open.

- Prove that the union of finite collection of closed sets is closed. Show by an example that the union of an infinite collection closed sets may not be closed.

- Find the derived set of the set $S = \left\{ \dfrac{(-1)^m}{m} + \dfrac{1}{n} : m \in N, n \in N \right\}$.

- Find the derived set of the set $S = \left\{ \dfrac{1}{m} + \dfrac{1}{n} : m \in N, n \in N \right\}$.

- Find the derived set of the set $S = \left\{ \dfrac{2}{m} + \dfrac{3}{n} : m \in N, n \in N \right\}$.

- Find the derived set of the set $S = \left\{ (-1)^{n-1} \dfrac{n}{n-1} : n \in N \right\}$.