Study Materials

on

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Paper – C9T.

Unit-3 [Marks: 16]

Topic: **Vector Field & Line Integration**

— by

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Three field operators: $\boxed{\text{Gradient, Divergence, \& curl.}}$

① The gradient of a scalar field,

② The divergence of a vector field, and

③ The curl of a vector field.

• **The gradient of a scalar field:—**

If $\phi(\bar{r}) = \phi(x, y, z)$ is a scalar field, i.e., a scalar function of position $\bar{r} = (x, y, z)$ in 3D, then its gradient at any point is defined in Cartesian co-ordinates by

$$\text{grad}\,\phi \equiv \bar{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}.$$

where the vector operator $\bar{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ is called "del" or "nabla".

Note: $\bar{\nabla}\phi$ is a vector field. The gradient of a scalar field tends to point in the direction of greatest change of the field!

• **The significance of grad:—**

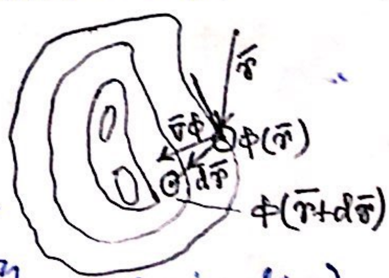
If our current position is $\bar{r}$ in some scalar field ϕ, and we move an infinitesimal distance $d\bar{r}$, we have the change in ϕ is given by—

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz.$$

Also, $d\bar{r} = (\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$ and $\bar{\nabla}\phi = \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right)$.

Then $d\phi = \bar{\nabla}\phi \cdot d\bar{r}$; dividing bothsides by ds,

$$\frac{d\phi}{ds} = \bar{\nabla}\phi \cdot \frac{d\bar{r}}{ds}$$; where we have $|d\bar{r}| = ds$, and $\frac{d\bar{r}}{ds}$ is a unit vector in the direction $d\bar{r}$.



① $\bar{\nabla}\phi$ has the property that the rate of change of ϕ w.r.t. distance in a particular direction $(\hat{a})$ is the projection of $\bar{\nabla}\phi$ onto that direction (or component); which is $\bar{\nabla}\phi \cdot \hat{a}$. (Directional derivative).

The quantity $\frac{d\phi}{ds}$ is called a directional derivative

Note: In general, the directional derivative $\frac{d\phi}{ds}$ has a different value for each direction, and so has no meaning until we specify the direction.

Therefore, we also say that —

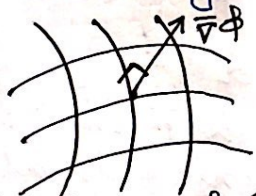
At any point P, $\bar{\nabla}\phi$ (or grad ϕ) points in the direction of greatest change of ϕ at P, and has magnitude equal to the rate of change of ϕ w.r.t. distance in that direction.

② Another property of grad ϕ:

If we consider a surface of constant ϕ, i.e., the locus of (x,y,z) for which $\phi(x,y,z) = $ Constant, then if we move a small amount within that surface, there is no change in ϕ, so $\frac{d\phi}{ds} = 0$.

$\therefore$ for any $\frac{d\bar{r}}{ds}$ in that surface, $\bar{\nabla}\phi \cdot \frac{d\bar{r}}{ds} = 0$.

Since $\frac{d\bar{r}}{ds}$ is a unit tangent vector to the surface, so $\bar{\nabla}\phi$ is everywhere NORMAL to a surface of constant ϕ.



Surface of Constant ϕ : $\phi(x,y,z) = $ Constant.

- The divergence of a vector field

The divergence computes a scalar quantity from a vector field by differentiation.

If $\bar{V}(x,y,z)$ is a vector function of position in 3D, i.e., $\bar{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$, then its divergence at any point is defined in Cartesian co-ordinates by

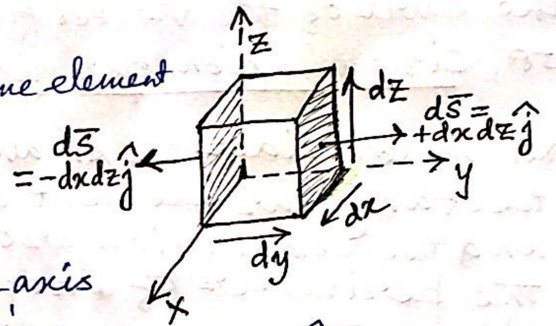
$$\text{div}\,\bar{V} = \bar{\nabla}\cdot\bar{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right)$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}.$$

Note: the divergence of a vector field is a scalar field.

- **The significance of divergence (or, div):–**

Let us consider a typical vector field, water flow, and let it be denoted by $\bar{a}(\bar{r})$. It has magnitude equal to the mass of water crossing a unit area perpendicular to the direction of $\bar{a}$ per unit time.
Let us take an infinitesimal volume element dV and calculate the balance of the flow of $\bar{a}$ in and out of dV.



Let us consider the volume element $dV = dxdydz$.
First, let us consider:
the face of area $dxdz$ perpendicular to the y-axis and facing outwards in the (–ve) y-direction, i.e., $d\bar{s} = -dxdz\,\hat{j}$.
The component of the vector $\bar{a}$ normal to $d\bar{s}\,(=-dxdz\hat{j})$ is $\bar{a}\cdot\hat{j} = a_y$ (say), and is pointing inwards.
So the outward flux from this surface is

$$\bar{a}\cdot d\bar{s} = -a_y(x,y,z)\,dzdx \longrightarrow (1)$$ [flux means mass/time]

From the opposite face, i.e., $d\bar{s} = +dxdz\,\hat{j}$, the outward flux amont will be

$$a_y(x,y+dy,z)\,dzdx = \left(a_y + \frac{\partial a_y}{\partial y}dy\right)dxdz \longrightarrow (2)$$ [by Taylor's theorem]

∴ The total outward amount from there two faces is

$$-a_y\,dzdx + \left(a_y + \frac{\partial a_y}{\partial y}dy\right)dzdx$$
$$= \frac{\partial a_y}{\partial y}dxdydz = \frac{\partial a_y}{\partial y}dV \longrightarrow (3)$$

Summing the other faces gives a total outward flux of

$$\left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}\right)dV = (\bar{\nabla}\cdot\bar{a})\,dV \longrightarrow (4)$$

Therefore,
The divergence of a vector field represents the flux generation per unit volume at each point of the field.

Note: (1) Divergence because it is an efflux not an influx.
(2) the total efflux from the infinitesimal volume is equal to the flux integrated over the surface of the volume

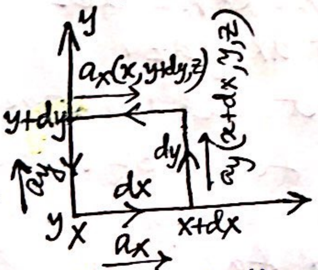
- ## The curl of a vector field

$$\bar{\nabla} \times \bar{a} \equiv \text{curl}(\bar{a}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_x & a_y & a_z \end{vmatrix} = \sum \hat{i}\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right).$$

The circulation of a vector $\bar{a}$ around any closed curve C is defined to be $\oint_C \bar{a} \cdot d\bar{r}$.

The curl of the vector field $\bar{a}$ represents the vorticity, or, circulation per unit area, of the field.

Let us consider the small rectangular element $dxdy$, and the circulation around the perimeter of a rectangular element.



The fields in the x-direction at the bottom & top are:

$$a_x(x,y,z) \quad \text{and} \quad a_x(x,y+dy,z) = a_x(x,y,z) + \frac{\partial a_x}{\partial y}dy. \longrightarrow (1)$$

And the fields in the y-direction at the left and right are:

$$a_y(x,y,z) \quad \text{and} \quad a_y(x+dx,y,z) = a_y(x,y,z) + \frac{\partial a_y}{\partial x}dx. \longrightarrow (2)$$

The total circulation dc starting from the bottom and working round in the anticlockwise sense, we have:

$$\begin{aligned} dc &= +[a_x dx] + [a_y(x+dx,y,z)dy] - [a_x(x,y+dy,z)dx] - [a_y dy] \\ &= [a_x dx] + \left[\left(a_y + \frac{\partial a_y}{\partial x}dx\right)dy\right] - \left[\left(a_x + \frac{\partial a_x}{\partial y}dy\right)dx\right] - [a_y dy] \\ &= \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)dxdy = (\bar{\nabla} \times \bar{a}) \cdot d\bar{s}, \text{ where} \end{aligned}$$

$$d\bar{s} = dxdy\,\hat{k}.$$

Some Important definitions:

(1) A vector field with zero divergence is called Solenoidal.
(2) " " " " curl " " irrotational
(3) " scalar " " " gradient " " Constant.

GRADIENT, DIVERGENCE and CURL.

The vector differential operator DEL (∇),

$$\nabla \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}.$$

The GRADIENT: Let $\phi(x,y,z)$ be defined and differentiable at each point (x,y,z) in a certain region of space.

$$\nabla\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}.$$

$\nabla\phi$ defines a vector field whereas ϕ defines a differentiable scalar field.

Directional derivative of ϕ in the direction of a unit vector $\bar{a}$ is given by $\bar{\nabla}\phi \cdot \bar{a}$.
Physically, this is the rate of change of ϕ at (x,y,z) in the direction $\bar{a}$.

The DIVERGENCE: Let $\bar{V}(x,y,z) = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ be defined and differentiable at each point (x,y,z) in a certain region of space. i.e, $\bar{V}$ defines a differentiable vector field.

$$\bar{\nabla}\cdot\bar{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}.$$

Note that $\bar{\nabla}\cdot\bar{V} \neq \bar{V}\cdot\nabla$

The CURL: If $\bar{V}(x,y,z)$ is a differentiable vector field then

$$\bar{\nabla}\times\bar{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

FORMULAE: If $\bar{A}$ and $\bar{B}$ are differentiable vector functions and ϕ and ψ are differentiable scalar functions of position (x,y,z), then

1. $\bar{\nabla}(\phi+\psi) = \bar{\nabla}\phi + \nabla\psi$
2. $\bar{\nabla}\cdot(\bar{A}+\bar{B}) = \bar{\nabla}\cdot\bar{A} + \bar{\nabla}\cdot\bar{B}$
3. $\bar{\nabla}\times(\bar{A}+\bar{B}) = \bar{\nabla}\times\bar{A} + \bar{\nabla}\times\bar{B}$
4. $\bar{\nabla}\cdot(\phi\bar{A}) = (\bar{\nabla}\phi)\cdot\bar{A} + \phi(\bar{\nabla}\cdot\bar{A})$
5. $\bar{\nabla}\times(\phi\bar{A}) = (\bar{\nabla}\phi)\times\bar{A} + \phi(\bar{\nabla}\times\bar{A})$

6. $\bar\nabla\cdot(\bar A\times\bar B)=\bar B\cdot(\bar\nabla\times\bar A)-\bar A\cdot(\bar\nabla\times\bar B)$

7. $\bar\nabla\times(\bar A\times\bar B)=(\bar B\cdot\bar\nabla)\bar A-\bar B(\bar\nabla\cdot\bar A)-(\bar A\cdot\bar\nabla)\bar B+\bar A(\bar\nabla\cdot\bar B)$

8. $\bar\nabla(\bar A\cdot\bar B)=(\bar B\cdot\bar\nabla)\bar A+(\bar A\cdot\bar\nabla)\bar B+\bar B\times(\bar\nabla\times\bar A)+\bar A\times(\bar\nabla\times\bar B)$

9. $\bar\nabla\cdot(\bar\nabla\phi)=\nabla^2\phi=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$

where $\nabla^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ is called the Laplacian operator.

10. $\bar\nabla\times(\bar\nabla\phi)=\bar 0$

11. $\bar\nabla\cdot(\bar\nabla\times\bar A)=0$

12. $\bar\nabla\times(\bar\nabla\times\bar A)=\bar\nabla(\bar\nabla\cdot\bar A)-\nabla^2A$

Prove that $\bar\nabla(\bar A\cdot\bar B)=(\bar B\cdot\bar\nabla)\bar A+(\bar A\cdot\bar\nabla)\bar B+\bar B\times(\bar\nabla\times\bar A)+\bar A\times(\bar\nabla\times\bar B)$

Now, $\bar\nabla(\bar A\cdot\bar B)=\sum\hat i\frac{\partial}{\partial x}(\bar A\cdot\bar B)=\sum\hat i\left\{\bar A\cdot\frac{\partial\bar B}{\partial x}+\bar B\cdot\frac{\partial\bar A}{\partial x}\right\}$

$$=\sum\hat i\,\bar A\cdot\frac{\partial\bar B}{\partial x}+\sum\hat i\,\bar B\cdot\frac{\partial\bar A}{\partial x}$$

Now, $\bar A\times\left(\hat i\times\frac{\partial\bar B}{\partial x}\right)=\left(\bar A\cdot\frac{\partial\bar B}{\partial x}\right)\hat i-(\bar A\cdot\hat i)\frac{\partial\bar B}{\partial x}$

or, $\hat i\left(\bar A\cdot\frac{\partial\bar B}{\partial x}\right)=\bar A\times\left(\hat i\times\frac{\partial\bar B}{\partial x}\right)+(\bar A\cdot\hat i)\frac{\partial\bar B}{\partial x}$.

Similarly, $\bar B\times\left(\hat i\times\frac{\partial\bar A}{\partial x}\right)=\left(\bar B\cdot\frac{\partial\bar A}{\partial x}\right)\hat i-(\bar B\cdot\hat i)\frac{\partial\bar A}{\partial x}$

or, $\hat i\left(\bar B\cdot\frac{\partial\bar A}{\partial x}\right)=\bar B\times\left(\hat i\times\frac{\partial\bar A}{\partial x}\right)+(\bar B\cdot\hat i)\frac{\partial\bar A}{\partial x}$.

Hence $\bar\nabla(\bar A\cdot\bar B)=\sum\bar A\times\left(\hat i\times\frac{\partial\bar B}{\partial x}\right)+\sum\bar B\times\left(\hat i\times\frac{\partial\bar A}{\partial x}\right)+\sum(\bar A\cdot\hat i)\frac{\partial\bar B}{\partial x}+\sum(\bar B\cdot\hat i)\frac{\partial\bar A}{\partial x}$.

$$=\bar A\times\sum\hat i\times\frac{\partial\bar B}{\partial x}+\bar B\times\sum\hat i\times\frac{\partial\bar A}{\partial x}+\sum(\bar A\cdot\hat i)\frac{\partial\bar B}{\partial x}+\sum(\bar B\cdot\hat i)\frac{\partial\bar A}{\partial x}$$

$$=\bar A\times(\bar\nabla\times\bar B)+\bar B\times(\bar\nabla\times\bar A)+(\bar A\cdot\bar\nabla)\bar B+(\bar B\cdot\bar\nabla)\bar A$$ proved.

Prove that $\bar\nabla\cdot(\bar A\times\bar B)=\bar B\cdot(\bar\nabla\times\bar A)-\bar A\times(\bar\nabla\times\bar B)$

$\bar\nabla\cdot(\bar A\times\bar B)=\sum\hat i\cdot\frac{\partial}{\partial x}(\bar A\times\bar B)=\sum\hat i\cdot\left\{\frac{\partial\bar A}{\partial x}\times\bar B+\bar A\times\frac{\partial\bar B}{\partial x}\right\}$

$=\sum\hat i\cdot\frac{\partial\bar A}{\partial x}\times\bar B+\sum\hat i\cdot\bar A\times\frac{\partial\bar B}{\partial x}=\sum\left(\hat i\times\frac{\partial\bar A}{\partial x}\right)\cdot\bar B-\sum\left(\hat i\times\frac{\partial\bar B}{\partial x}\right)\cdot\bar A$

$=(\bar\nabla\times\bar A)\cdot\bar B-(\bar\nabla\times\bar B)\cdot\bar A=\bar B\cdot(\bar\nabla\times\bar A)-\bar A\cdot(\bar\nabla\times\bar B)$ proved.

List of some very useful results.

Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\bar{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$; Constant vector. f, g are scalar point functions, both continuously differentiable. $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$.

① $\bar{\nabla} f(u) = f'(u)\,\bar{\nabla} u$. $\left[\bar{\nabla} f(u) = \sum \hat{i}\frac{\partial}{\partial x} f(u) = \sum \hat{i} f'(u)\frac{\partial u}{\partial x} = f'(u)\sum \hat{i}\frac{\partial u}{\partial x} = f'(u)\bar{\nabla} u\right]$

② $\bar{\nabla}(r) = \frac{\bar{r}}{r}$. $\left[\bar{\nabla}(r) = \sum \hat{i}\frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right) = \sum \hat{i}\frac{1}{2r}\frac{\partial}{\partial x}(x^2+y^2+z^2) = \frac{1}{r}\sum \hat{i}x = \frac{\bar{r}}{r}\right]$

③ $\bar{\nabla} r^n = n r^{n-2}\bar{r}$. $\left[\bar{\nabla} r^n = \frac{d}{dr}(r^n)\,\bar{\nabla} r = n r^{n-1}\frac{\bar{r}}{r} = n r^{n-2}\bar{r}\right]$

④ $\bar{\nabla}\cdot\bar{r} = 3$. $\left[\bar{\nabla}\cdot\bar{r} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(x\hat{i} + y\hat{j} + z\hat{k}\right) = 1+1+1 = 3\right]$

⑤ $\bar{\nabla}\cdot\left\{\frac{f(r)}{r}\bar{r}\right\} = \frac{1}{r^2}\frac{d}{dr}(r^2 f)$.

$$\bar{\nabla}\cdot\left\{\frac{f(r)}{r}\bar{r}\right\} = \sum \frac{\partial}{\partial x}\left\{\frac{f(r)}{r}x\right\} = \sum\left\{\frac{f(r)}{r} - \frac{1}{r^2}\frac{\partial r}{\partial x}f(r)x + f'(r)\frac{\partial r}{\partial x}\cdot\frac{x}{r}\right\}$$

$$= \frac{3f(r)}{r} - \sum \frac{x}{r^2}f(r)\cdot\frac{x}{r} + \sum f'(r)\frac{x}{r}\cdot\frac{x}{r}$$

$$= \frac{3f(r)}{r} - \frac{1}{r^3}f(r)(x^2+y^2+z^2) + \frac{f'(r)}{r^2}(x^2+y^2+z^2)$$

$$= \frac{3f(r)}{r} - \frac{1}{r^3}f(r)\cdot r^2 + \frac{f'(r)}{r^2}\cdot r^2 = \frac{2f(r)}{r} + f'(r)$$

$$= \frac{1}{r^2}\frac{d}{dr}\left\{r^2 f(r)\right\}.$$

⑥ $\bar{\nabla}\cdot\bar{c} = 0$. $\left[\bar{\nabla}\cdot\bar{c} = \frac{\partial}{\partial x}(c_1) + \frac{\partial}{\partial y}(c_2) + \frac{\partial}{\partial z}(c_3) = 0+0+0 = 0\right]$

⑦ $\bar{\nabla}\times\bar{r} = \bar{0}$. $\left[\bar{\nabla}\times\bar{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \hat{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \hat{j}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = 0\hat{i} + 0\hat{j} + 0\hat{k} = \bar{0}\right].$

⑧ $\bar{\nabla}\times\bar{c} = \bar{0}$. $\left[\bar{\nabla}\times\bar{c} = \sum \hat{i}\times\frac{\partial}{\partial x}(\bar{c}) = \sum \hat{i}\times\frac{\partial \bar{c}}{\partial x} = \sum \hat{i}\times\bar{0} = \bar{0}\right]$

⑨ $\bar{\nabla}\cdot(\bar{r}\times\bar{c}) = 0$. $\left[\bar{\nabla}\cdot(\bar{r}\times\bar{c}) = \bar{c}\cdot(\bar{\nabla}\times\bar{r}) - \bar{r}\cdot(\bar{\nabla}\times\bar{c}) = \bar{c}\cdot\bar{0} - \bar{r}\cdot\bar{0} = 0\right]$

⑩ $\bar{\nabla}\times(\bar{r}\times\bar{c}) = -2\bar{c}$.

$$\bar{\nabla}\times(\bar{r}\times\bar{c}) = (\bar{c}\cdot\bar{\nabla})\bar{r} - \bar{c}(\bar{\nabla}\cdot\bar{r}) - (\bar{r}\cdot\bar{\nabla})\bar{c} + \bar{r}(\bar{\nabla}\cdot\bar{c})$$

$$= \sum c_1\frac{\partial}{\partial x}(x\hat{i}) - 3\bar{c} - \sum x\frac{\partial}{\partial x}(c_1\hat{i}) + \bar{r}0$$

$$= \sum \hat{i}c_1 - 3\bar{c} - 0 + 0 = \bar{c} - 3\bar{c}$$

$$= -2\bar{c}.$$

(11) $\nabla^2\left(\frac{1}{r}\right) = 0$.

$$\left[\nabla^2\left(\frac{1}{r}\right) = \bar{\nabla}\cdot\bar{\nabla}\left(\frac{1}{r}\right) = \bar{\nabla}\cdot\left\{\frac{d}{dr}\left(\frac{1}{r}\right)\bar{\nabla}r\right\} = \bar{\nabla}\cdot\left\{-\frac{1}{r^2}\frac{\bar{r}}{r}\right\}\right.$$

$$= \bar{\nabla}\cdot\left(-\frac{1}{r^3}\bar{r}\right) = \bar{\nabla}\left(-\frac{1}{r^3}\right)\cdot\bar{r} + \left(-\frac{1}{r^3}\right)(\bar{\nabla}\cdot\bar{r})$$

$$= \frac{d}{dr}\left(-\frac{1}{r^3}\right)(\bar{\nabla}r\cdot\bar{r}) - \frac{3}{r^3}$$

$$= \frac{3}{r^4}\sum\frac{\partial r}{\partial x}x - \frac{3}{r^3} = \frac{3}{r^4}\sum\frac{x}{r}\cdot x - \frac{3}{r^3}$$

$$= \frac{3}{r^4}\sum\frac{x^2}{r} - \frac{3}{r^3} = \frac{3}{r^5}(x^2+y^2+z^2) - \frac{3}{r^3}$$

$$\left.= \frac{3}{r^5}\times r^2 - \frac{3}{r^3} = 0.\right]$$

(12) $\nabla^2(r^n\bar{r}) = n(n+3)r^{n-2}\bar{r}$.

$$\nabla^2(r^n\bar{r}) = \nabla\{\bar{\nabla}\cdot(r^n\bar{r})\}$$

$$= \nabla\{(\bar{\nabla}r^n)\cdot\bar{r} + r^n(\bar{\nabla}\cdot\bar{r})\}$$

$$= \nabla\{nr^{n-2}\bar{r}\cdot\bar{r} + r^n 3\}$$

$$= \nabla\{nr^{n-2}r^2 + 3r^n\}$$

$$= (n+3)\nabla(r^n) = \underline{(n+3)\,n\,r^{n-2}\bar{r}}.$$

Ex (3)(c) Prove that $\text{curl}\left\{\frac{\bar{a}\times\bar{r}}{r^3}\right\} = -\frac{\bar{a}}{r^3} + \frac{3}{r^5}(\bar{a}\cdot\bar{r})\bar{r}$; where $\bar{a}$ is a constant vector.

$$\text{curl}\left\{\frac{\bar{a}\times\bar{r}}{r^3}\right\} = \sum\left\{\hat{i}\times\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right)\right\}$$

Now, $$\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = -\frac{3}{r^4}\frac{\partial r}{\partial x}(\bar{a}\times\bar{r}) + \frac{1}{r^3}\left(\bar{a}\times\frac{\partial\bar{r}}{\partial x}\right) + \frac{1}{r^3}\left(\frac{\partial\bar{a}}{\partial x}\times\bar{r}\right)$$

$$= -\frac{3}{r^4}\left(\frac{x}{r}\right)(\bar{a}\times\bar{r}) + \frac{1}{r^3}(\bar{a}\times\hat{i}) + \frac{1}{r^3}(\bar{0}\times\bar{r})$$

$$= -\frac{3}{r^4}\frac{x}{r}(\bar{a}\times\bar{r}) + \frac{1}{r^3}(\bar{a}\times\hat{i}) + \bar{0}$$

$$\therefore\ \hat{i}\times\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = \frac{-3x}{r^5}[\hat{i}\times(\bar{a}\times\bar{r})] + \frac{1}{r^3}[\hat{i}\times(\bar{a}\times\hat{i})].$$

$$= -\frac{3x}{r^5}[(\hat{i}\cdot\bar{r})\bar{a} - (\hat{i}\cdot\bar{a})\bar{r}] + \frac{1}{r^3}[(\hat{i}\cdot\hat{i})\bar{a} - (\hat{i}\cdot\bar{a})\hat{i}]$$

$$= -\frac{3x}{r^5}x\bar{a} + \frac{3x}{r^5}a_1\bar{r} + \frac{1}{r^3}\bar{a} - \frac{1}{r^3}a_1\hat{i}$$

$$\sum\left\{\hat{i}\times\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right)\right\} = \left(-\frac{3}{r^5}\sum x^2\right)\bar{a} + \left(\frac{3}{r^5}\sum a_1x\right)\bar{r} + \frac{3}{r^3}\bar{a} - \frac{1}{r^3}\bar{a}.$$

$$= -\frac{3}{r^5}r^2\bar{a} + \frac{3}{r^5}(\bar{r}\cdot\bar{a})\bar{r} + \frac{2}{r^3}\bar{a}$$

$$\therefore\ \text{curl}\left\{\frac{\bar{a}\times\bar{r}}{r^3}\right\} = -\frac{\bar{a}}{r^3} + \frac{3}{r^5}(\bar{a}\cdot\bar{r})\bar{r}.$$

Prove: $\bar{\nabla}\times(\bar{\nabla}\times\bar{A}) = \bar{\nabla}(\bar{\nabla}\cdot\bar{A}) - \nabla^2\bar{A}$

Let $\bar{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, where A_1, A_2, A_3 are real-valued differentiable functions of real variables x, y, z.

$$\bar{\nabla}\times\bar{A} = \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k}$$

$$\bar{\nabla}\times(\bar{\nabla}\times\bar{A}) = \left[\frac{\partial}{\partial y}\left\{\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right\} - \frac{\partial}{\partial z}\left\{\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right\}\right]\hat{i} + \left[\frac{\partial}{\partial z}\left\{\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right\} - \frac{\partial}{\partial x}\left\{\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right\}\right]\hat{j}$$

$$+ \left[\frac{\partial}{\partial x}\left\{\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right\} - \frac{\partial}{\partial y}\left\{\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right\}\right]\hat{k}$$

$$= \sum\left[\hat{i}\left\{\frac{\partial}{\partial y}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\right\}\right]$$

$$= \sum\left[\hat{i}\left(\frac{\partial^2 A_2}{\partial y\partial x} + \frac{\partial^2 A_3}{\partial z\partial x}\right) - \left(\frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2}\right)\right\}\Bigg]$$

$$= \sum\hat{i}\left\{\frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) - \left(\frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2}\right)\right\}$$

$$= \sum\hat{i}\left\{\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) - \left(\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2}\right)\right\}$$

$$= \sum\hat{i}\left\{\frac{\partial}{\partial x}(\bar{\nabla}\cdot\bar{A}) - \nabla^2 A_1\right\} = \sum\hat{i}\left\{\frac{\partial}{\partial x}(\bar{\nabla}\cdot\bar{A})\right\} - \nabla^2\sum A_1\hat{i}$$

$$= \bar{\nabla}(\bar{\nabla}\cdot\bar{A}) - \nabla^2\bar{A} = \text{R.H.S. proved.}$$

Prob: Show that $\bar{\nabla}\phi$ is a vector normal to the surface $\phi(x,y,z) = \text{constant}$.

Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the p.v. to any point $P(x,y,z)$ on the surface.

Then $d\bar{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$ lies on the tangent plane to the surface at P.

But $d\phi = 0 = \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$

$\Rightarrow \bar{\nabla}\phi\cdot d\bar{r} = 0$.

$\therefore \bar{\nabla}\phi$ is $\perp^r$ to $d\bar{r}$ and therefore to the surface $\phi(x,y,z) = c$

① Find the unit vector $\perp^r$ to the surface of the paraboloid of revolution $z = x^2 + y^2$ at the point $(1,2,5)$.

Ans: $\dfrac{2\hat{i} + 4\hat{j} - \hat{k}}{\pm\sqrt{21}}$, find: $\dfrac{\nabla\phi}{|\nabla\phi|}$ at $(1,2,5)$

② Find equations for the tangent plane and normal line to the surface $xz^2 + x^2y = z - 1$ at the point $(1,-3,2)$

$\phi = xz^2 + x^2y - z = -1$ (constant)

Tangent plane: $(\bar{r} - \bar{r}_0)\cdot\bar{N} = 0$; $\quad \bar{N} = \bar{\nabla}\phi$ at $(1,-3,2) = (z^2 + 2xy,\ x^2,\ 2xz - 1)\big|_{(1,-3,2)}$

$= (-2, 1, 3)$.

Normal line: $(\bar{r} - \bar{r}_0)\times\bar{N} = \bar{0}$; $\quad r_0 = (1,-3,2)$

Ans: Tangent plane: $2x - y - 3z + 1 = 0$. $\Leftarrow \{(x,y,z) - (1,-3,2)\}\cdot(-2,1,3)$

Normal line: $\dfrac{x-1}{-2} = \dfrac{y+3}{1} = \dfrac{z-2}{3}$

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(3) Let $\phi(x,y,z)$ and $\phi(x+\Delta x, y+\Delta y, z+\Delta z)$ be the temperatures at two nbd. points $P(x,y,z)$ and $Q(x+\Delta x, y+\Delta y, z+\Delta z)$ of a certain region.

(a) Interpret physically the quantity:

$$\frac{\Delta\phi}{\Delta s} = \frac{\phi(x+\Delta x, y+\Delta y, z+\Delta z) - \phi(x,y,z)}{\Delta s}, \text{ where } \Delta s = PQ.$$

(b) Evaluate: $\lim_{\Delta s\to 0} \frac{\Delta\phi}{\Delta s} = \frac{d\phi}{ds}$ and interpret physically

(c) Show that $\frac{d\phi}{ds} = \nabla\phi \cdot \frac{d\bar{r}}{ds}$.

(a) Since $\Delta\phi$ is the change in temperature between P & Q and Δs is the distance between these points,

$\frac{\Delta\phi}{\Delta s}$ represents the average rate of change in temp. per unit distance in the direction from P to Q.

(b) From the calculus,

$\Delta\phi = \frac{\partial\phi}{\partial x}\Delta x + \frac{\partial\phi}{\partial y}\Delta y + \frac{\partial\phi}{\partial z}\Delta z +$ infinitesimal of order higher than $\Delta x, \Delta y, \Delta z$.

Then $\lim_{\Delta s\to 0}\frac{\Delta\phi}{\Delta s} = \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial z}\frac{dz}{ds} = \frac{d\phi}{ds}$.

(c) $\boxed{\frac{d\phi}{ds} = \nabla\phi\cdot\frac{d\bar{r}}{ds}}$ → Directional derivative

fml → The rate of change of ϕ at P.

$\frac{d\bar{r}}{ds}$ is a unit vector. $\frac{d\phi}{ds}$ is called the directional derivative of ϕ.

Directional derivative: $\bar\nabla\phi\cdot\frac{d\bar{r}}{ds}$ is the component of $\bar\nabla\phi$ in the direction of the unit vector $\frac{d\bar{r}}{ds}$.

Show that the maximum directional derivative, takes place in the direction of and has the magnitude of the vector $\bar\nabla\phi$.

Now, $\frac{d\phi}{ds} = \bar\nabla\phi\cdot\frac{d\bar{r}}{ds}$ is the projection of $\bar\nabla\phi$ in the direction $\frac{d\bar{r}}{ds}$. This will be maximum when $\bar\nabla\phi$ & $\frac{d\bar{r}}{ds}$ have the same direction. Then the max. value of $\frac{d\phi}{ds}$ takes place in the direction of $\bar\nabla\phi$ and its magnitude is $|\bar\nabla\phi|$.

Eg. (62) Find the directional derivative of $\phi = 4xz^3 - 3x^2y^2z$ at $(2,-1,2)$ in the direction $2\hat{i} - 3\hat{j} + 6\hat{k}$.

$\nabla\phi\cdot\hat{a}$, where $\hat{a} = \frac{2\hat{i}-3\hat{j}+6\hat{k}}{|2\hat{i}-3\hat{j}+6\hat{k}|} = \frac{2\hat{i}-3\hat{j}+6\hat{k}}{\sqrt{4+9+36}} = \frac{1}{7}(2\hat{i}-3\hat{j}+6\hat{k})$

$\nabla\phi|_{(2,-1,2)} = (4z^3 - 6xy^2z, -6x^2yz, 12xz^2 - 3x^2y^2)|_{(2,-1,2)} = 4(2,12,24)$

Directional derivative $= \nabla\phi\cdot\hat{a} = \frac{4}{7}\times(4-36+126) = \frac{4\times 94}{7} = \frac{376}{7}$.

(67) Find the constants a, b so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1,-1,2)$. Ans. $a = 5/2, b = 1$.

$$\nabla\phi_1 \cdot \nabla\phi_2 = |\nabla\phi_1|\,|\nabla\phi_2| \cos 90^\circ = 0.$$

(74) Prove $\nabla^2 r^n = n(n+1) r^{n-2}$, n is a constant.

Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$.

$$\nabla^2 r^n = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(x^2+y^2+z^2)^{n/2}$$

$$= \sum \frac{\partial^2}{\partial x^2}(x^2+y^2+z^2)^{n/2} = \sum \frac{\partial}{\partial x}\left\{\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1} \times 2x\right\}$$

$$= \sum \frac{\partial}{\partial x}\left\{n x \times (x^2+y^2+z^2)^{\frac{n}{2}-1}\right\} = \sum\left\{n(x^2+y^2+z^2)^{\frac{n}{2}-1} + n\left(\frac{n}{2}-1\right)x \cdot 2x (x^2+y^2+z^2)^{\frac{n}{2}-2}\right\}$$

$$= \sum\left[n(x^2+y^2+z^2)^{n/2-1} + n(n-2)x^2(x^2+y^2+z^2)^{n/2-2}\right]$$

$$= \sum n(x^2+y^2+z^2)^{n/2-1} + \sum n(n-2)x^2(x^2+y^2+z^2)^{n/2-2}$$

$$= 3n(x^2+y^2+z^2)^{\frac{n-2}{2}} + n(n-2)(x^2+y^2+z^2)^{\frac{n-4}{2}}(x^2+y^2+z^2)$$

$$= 3n(x^2+y^2+z^2)^{\frac{n-2}{2}} + n(n-2)(x^2+y^2+z^2)^{\frac{n-2}{2}}$$

$$= n(x^2+y^2+z^2)^{\frac{n-2}{2}}\left[3 + (n-2)\right] = n(n+1) r^{n-2}.$$ proved

(83) (a) Prove $\nabla^2 f(r) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$. (b) find $f(r)$ s.t. $\nabla^2 f(r) = 0$.

(a) $$\nabla^2 f(r) = \sum \frac{\partial^2}{\partial x^2} f(r) = \sum \frac{\partial}{\partial x}\left(\frac{df}{dr}\cdot\frac{\partial r}{\partial x}\right) = \sum \frac{\partial}{\partial x}\left\{\frac{df}{dr}\cdot x(x^2+y^2+z^2)^{-1/2}\right\}$$

$r = (x^2+y^2+z^2)^{1/2}$

$\frac{\partial r}{\partial x} = x(x^2+y^2+z^2)^{-1/2} = \frac{x}{r}$.

$$= \sum \frac{\partial}{\partial x}\left\{\frac{df}{dr}\cdot\left(\frac{x}{r}\right)\right\} = \sum\left\{\frac{d^2 f}{dr^2}\cdot\frac{\partial r}{\partial x}\cdot\left(\frac{x}{r}\right) + \frac{df}{dr}\cdot\frac{1}{r} + x\frac{df}{dr}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right)\right\}$$

$$= \sum\left\{\frac{d^2 f}{dr^2}\cdot\left(\frac{x}{r}\right)^2 + \frac{df}{dr}\cdot\frac{1}{r} - \frac{x^2}{r^3}\frac{df}{dr}\right\}$$

$$= \frac{d^2 f}{dr^2}\cdot\frac{1}{r^2}(x^2+y^2+z^2) + \frac{3}{r}\frac{df}{dr} - \frac{df}{dr}\cdot\frac{1}{r^3}(x^2+y^2+z^2)$$

$$= \frac{d^2 f}{dr^2} + \frac{3}{r}\frac{df}{dr} - \frac{1}{r}\frac{df}{dr} = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}.$$ proved.

(b) $\nabla^2 f(r) = 0$

$$\Rightarrow \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} = 0 \Rightarrow r^2\frac{d^2 f}{dr^2} + 2r\frac{df}{dr} = 0.$$

$$\Rightarrow \frac{d^2 f}{dt^2} - \frac{df}{dt} + 2\frac{df}{dt} = 0 \Rightarrow \frac{d^2 f}{dt^2} + \frac{df}{dt} = 0.$$

Let $r = e^t$
$t = \log r$
$dt = \frac{1}{r}dr$.
$\frac{df}{dr} = \frac{df}{dt}\cdot\frac{dt}{dr} = \frac{1}{r}\frac{df}{dt}$.

Let $f = Ce^{mt}$ be the trial soln.

$m^2 + m = 0 \Rightarrow m(m+1) = 0$
$\Rightarrow m = 0, m = -1.$

$f(r) = A + Be^{-t} = A + \frac{B}{r}$, A, B are arbitrary constants.

102. Show that $\bar{A} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$ is irrotational. Find ϕ s.t. $\bar{A} = \nabla\phi$. Ans: $\phi = 3x^2y + xz^3 - yz +$ Constant

109. If $\bar{A}$ and $\bar{B}$ are irrotational, prove that $\bar{A}\times\bar{B}$ is solenoidal.

Problem ① Show that if $\phi(x,y,z)$ is any solution of Laplace's eqn then $\bar{\nabla}\phi$ is a vector which is both solenoidal and irrotational.

Problem ② Determine the constant a so that $\bar{V}=(x+3y)\hat{i}+(y-2z)\hat{j}+(x+az)\hat{k}$ is solenoidal. (Ans: $a=-2$).

Solution of ①: If $\phi(x,y,z)$ be any solution of Laplace's equation, then $\nabla^2\phi=0$

$$\Rightarrow \bar{\nabla}\cdot\bar{\nabla}\phi=0 \Rightarrow \bar{\nabla}\phi \text{ is solenoidal.}$$

Now
$$\bar{\nabla}\times\bar{\nabla}\phi=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z}\end{vmatrix}=\left(\frac{\partial^2\phi}{\partial y\partial z}-\frac{\partial^2\phi}{\partial z\partial y}\right)\hat{i}+\left(\frac{\partial^2\phi}{\partial z\partial x}-\frac{\partial^2\phi}{\partial x\partial z}\right)\hat{j}+\left(\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\phi}{\partial y\partial x}\right)\hat{k}.$$

For a vector $\bar{A}$, Condition of solenoidal $\bar{\nabla}\cdot\bar{A}=0$. Condition of irrotational $\bar{\nabla}\times\bar{A}=0$

$$=0\hat{i}+0\hat{j}+0\hat{k}$$
$$=\bar{0}$$

[Since $\frac{\partial^2\phi}{\partial y\partial z}=\frac{\partial^2\phi}{\partial z\partial y}$, ... etc. ϕ has continuous 2nd order partial derivatives.]

$\therefore \bar{\nabla}\times\bar{\nabla}\phi=\bar{0}\Rightarrow\bar{\nabla}\phi$ is irrotational.

$\therefore \bar{\nabla}\phi$ is both solenoidal & irrotational.

Solution of ②: If $\bar{V}=(x+3y)\hat{i}+(y-2z)\hat{j}+(x+az)\hat{k}$ be solenoidal, then $\bar{\nabla}\cdot\bar{V}=0$

$$\Rightarrow \frac{\partial}{\partial x}(x+3y)+\frac{\partial}{\partial y}(y-2z)+\frac{\partial}{\partial z}(x+az)=0$$
$$\Rightarrow 1+1+a=0\Rightarrow \underline{a=-2}.$$

Solution of (10②) (from previous page);

Given $\bar{A}=(6xy+z^3)\hat{i}+(3x^2-z)\hat{j}+(3xz^2-y)\hat{k}$

To show $\bar{A}$ is irrotational, we calculate

$$\bar{\nabla}\times\bar{A}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ (6xy+z^3) & (3x^2-z) & (3xz^2-y)\end{vmatrix}=(-1+1)\hat{i}+(3z^2-3z^2)\hat{j}+(6x-6x)\hat{k}$$
$$=0\hat{i}+0\hat{j}+0\hat{k}$$
$$=\bar{0}$$

$\Rightarrow \bar{A}$ is irrotational.

Now to find ϕ s.t. $\bar{A} = \bar{\nabla}\phi$.

i.e., $\underbrace{\frac{\partial\phi}{\partial x} = 6xy + z^3}_{①}$, $\underbrace{\frac{\partial\phi}{\partial y} = 3x^2 - z}_{②}$, $\underbrace{\frac{\partial\phi}{\partial z} = 3xz^2 - y}_{③}$

Integrating partially

① w.r.t. x, keeping y & z fixed; $\phi = 3x^2y + xz^3 + f_1(y,z)$.

② " y, " x & z " ; $\phi = 3x^2y - zy + f_2(x,z)$.

③ " z, " x & y " ; $\phi = xz^3 - zy + f_3(x,y)$.

where f_1, f_2, f_3 are arbitrary functions.

If we choose $f_1(y,z) = -zy + c_1$

$f_2(x,z) = xz^3 + c_2$

$f_3(x,y) = 3x^2y + c_3$.

Then $\phi = 3x^2y + xz^3 - yz + c$, [where $c = c_1 + c_2 + c_3$], c being an arbitrary constant.

Solution of (104): (from previous page).

If $\bar{A}$ & $\bar{B}$ are irrotational then $\bar{A}\times\bar{B}$ is solenoidal — prove it.

$\bar{\nabla}\times\bar{A} = \bar{0} = \bar{\nabla}\times\bar{B}$. [Condition for irrotational]

Now $\bar{\nabla}\cdot(\bar{A}\times\bar{B}) = \bar{B}\cdot(\bar{\nabla}\times\bar{A}) - \bar{A}\cdot(\bar{\nabla}\times\bar{B})$

$= \bar{B}\cdot\bar{0} - \bar{A}\cdot\bar{0} = 0 - 0 = 0$.

$\Rightarrow$ $\bar{A}\times\bar{B}$ is solenoidal.

Solution of (67): [from previous page].

Let the surface $\phi_1(x,y,z) = ax^2 - byz - (a+2)x = 0$.

& " " $\phi_2(x,y,z) = 4x^2y + z^3 = 4$

Normal to the surface ϕ_1 at $(1,-1,2) = \bar{\nabla}\phi_1\big|_{(1,-1,2)}$

$= \left[2ax - (a+2), -bz, -by\right]_{(1,-1,2)}$

$= (a-2, -2b, b)$.

Normal to ϕ_2 at $(1,-1,2) = \bar{\nabla}\phi_2\big|_{(1,-1,2)} = (8xy, 4x^2, 3z^2)_{(1,-1,2)}$

$= (-8, 4, 12)$.

Now, $\bar{\nabla}\phi_1\cdot\bar{\nabla}\phi_2 = 0$ [orthogonality condition]

$\Rightarrow -8(a-2) + 4(-2b) + 12b = 0$

$\Rightarrow 8a - 4b = 16 \Rightarrow 2a - b = 4 \longrightarrow ①$.

The surface $\phi_1(x,y,z) = ax^2 - byz - (a+2)x = 0$ passes at the point $(1,-1,2)$; i.e.,

$$a + 2b - (a+2) = 0$$

$$\Rightarrow \quad 2b = 2 \Rightarrow b = 1 \longrightarrow (2).$$

From (1) & (2), solving for a & b, we get

$2a - 1 = 4;\ b = 1$

$\Rightarrow \quad \underline{a = 5/2,\ b = 1}$ (Ans.)

(76) If ω is a constant vector and $\bar{v} = \bar{\omega} \times \bar{r}$, prove that $\operatorname{div} \vec{v} = 0$.

$$\operatorname{div} \vec{v} = \bar{\nabla} \cdot (\bar{\omega} \times \bar{r}) = \bar{r} \cdot (\bar{\nabla} \times \bar{\omega}) - \bar{\omega} \cdot (\bar{\nabla} \times \bar{r})$$

$$= \bar{r} \cdot \bar{0} - \bar{\omega} \cdot \bar{0}$$

$$= 0 - 0$$

$$= 0 \text{ (proved)}$$

$\left[\because \bar{\omega} \text{ is a constant vector, } \bar{\nabla} \times \bar{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k} = \bar{0} \right.$

(61) Find equations for the tangent plane and normal line to the surface $z = x^2 + y^2$ at the point $(2,-1,5)$.

Let the surface be $\phi(x,y,z) \equiv x^2 + y^2 - z = 0$

Normal vector $\bar{N}$ to the surface $\phi(x,y,z)$ at $(2,-1,5)$ is given by $\bar{N} = \bar{\nabla}\phi \Big|_{(2,-1,5)}$

or, $\bar{N} = (2x, 2y, -1)\Big|_{(2,-1,5)} = (4,-2,-1)$.

Now the eqn of the tangent plane:

$$(\bar{r} - \bar{r}_0) \cdot \bar{N} = 0, \text{ where } \bar{r}_0 = (2,-1,5)$$

or, $\{(x,y,z) - (2,-1,5)\} \cdot (4,-2,-1) = 0$

or, $(x-2) \cdot 4 - 2(y+1) - 1(z-5) = 0$

or, $\underline{4x - 2y - z = 5}$

And the eqn of the normal line:

$(\bar{r} - \bar{r}_0) \times \bar{N} = \bar{0}$ or, $\underline{\dfrac{x-2}{4} = \dfrac{y+1}{-2} = \dfrac{z-5}{-1}}$,

or, $x = 4t + 2,\ y = -2t - 1,\ z = -t + 5$, t being a parameter

Ex: Show that $\bar{\nabla} r^n = n r^{n-2} \bar{r}$; where $r = |\bar{r}|$.

$$\bar{\nabla} r^n = \sum \hat{i} \frac{\partial}{\partial x}(r^n) = \sum \hat{i} (n r^{n-1}) \frac{\partial r}{\partial x} = n r^{n-1} \sum \hat{i} \frac{\partial r}{\partial x}$$
$$= n r^{n-1} \bar{\nabla} r$$
$$= n r^{n-1} \sum \hat{i} \frac{\partial}{\partial x}(\sqrt{x^2+y^2+z^2})$$
$$= n r^{n-1} \sum \hat{i} \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot \frac{\partial}{\partial x}(x^2+y^2+z^2)$$
$$= n r^{n-1} \sum \hat{i} \frac{1}{2r} \times 2x = n r^{n-1} \sum \hat{i} \frac{x}{r}$$
$$= n r^{n-1} \frac{\bar{r}}{r} = \underline{n r^{n-2} \bar{r}}$$

Show that $\bar{\nabla} f(u) = f'(u) \bar{\nabla} u$.

$$\bar{\nabla} f(u) = \sum \hat{i} \frac{\partial}{\partial x} f(u) = \sum \hat{i} f'(u) \cdot \frac{\partial u}{\partial x} = f'(u) \sum \hat{i} \frac{\partial u}{\partial x}$$
$$= f'(u) \bar{\nabla} u \longrightarrow (1)$$

(74) Prove $\nabla^2 r^n = n(n+1) r^{n-2}$, where n is a constant

$$\nabla^2 r^n = \bar{\nabla} \cdot (\bar{\nabla} r^n) = \bar{\nabla} \cdot \left[\frac{d}{dr}(r^n) \bar{\nabla} r\right], \text{ using } (1).$$
$$= \bar{\nabla} \cdot (n r^{n-1} \bar{\nabla} r).$$
$$= \bar{\nabla} \cdot \left[n r^{n-1} \sum \hat{i} \frac{\partial}{\partial x}(r)\right]$$
$$= \bar{\nabla} \cdot \left[n r^{n-1} \sum \hat{i} \left(\frac{x}{r}\right)\right] \; ; \left[\because \frac{\partial r}{\partial x} = \frac{x}{r}\right]$$
$$= \bar{\nabla} \cdot \left[n r^{n-1} \frac{\bar{r}}{r}\right]$$
$$= n \bar{\nabla} \cdot (r^{n-2} \bar{r}) = n\left[(\bar{\nabla} r^{n-2}) \cdot \bar{r} + r^{n-2} (\bar{\nabla} \cdot \bar{r})\right]$$
$$= n\left[\frac{d}{dr}(r^{n-2})(\bar{\nabla} r \cdot \bar{r}) + r^{n-2}(1+1+1)\right]$$
$$= n\left[(n-2) r^{n-3} \left(\frac{\bar{r}}{r} \cdot \bar{r}\right) + 3 r^{n-2}\right]$$
$$= n\left[(n-2) \frac{r^{n-3}}{r} (x^2+y^2+z^2) + 3 r^{n-2}\right]$$
$$= n\left[(n-2) r^{n-4} \cdot r^2 + 3 r^{n-2}\right]$$
$$= n[n-2+3] r^{n-2}$$
$$= \underline{n(n+1) r^{n-2}}$$

proved.

Ex: 58 (a) Prove $\nabla^2 f(r) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$. (b) Find $f(r)$ s.t. $\nabla^2 f(r) = 0$.

$$\nabla^2 f(r) = \bar{\nabla}\cdot\bar{\nabla} f(r) = \bar{\nabla}\cdot f'(r)\,\bar{\nabla} r \quad [\because \nabla f(u) = f'(u)\,\bar{\nabla} u].$$

$$= \bar{\nabla}\cdot \frac{f'(r)}{r}\bar{r} \quad [\because \bar{\nabla} r = \frac{\bar{r}}{r}].$$

$$= \frac{f'(r)}{r}\bar{\nabla}\cdot\bar{r} + \bar{r}\cdot\bar{\nabla}\frac{f'(r)}{r} \quad [\because \bar{\nabla}(f\bar{F}) = f(\bar{\nabla}\cdot\bar{F}) + \bar{F}\cdot(\bar{\nabla} f)]$$

$$= \frac{3f'(r)}{r} + \bar{r}\cdot\left\{-\frac{f'(r)}{r^2} + \frac{f''(r)}{r}\right\}\bar{\nabla} r$$

$$= \frac{3f'(r)}{r} + \bar{r}\cdot\left\{\frac{f''(r)}{r} - \frac{f'(r)}{r^2}\right\}\frac{\bar{r}}{r}$$

$$= \frac{3f'(r)}{r} + \left\{\frac{f''(r)}{r^2} - \frac{f'(r)}{r^3}\right\}(\bar{r}\cdot\bar{r})$$

$$= \frac{3f'(r)}{r} + \left\{\frac{f''(r)}{r^2} - \frac{f'(r)}{r^3}\right\} r^2 = \frac{3f'(r)}{r} + f''(r) - \frac{f'(r)}{r}$$

$$\therefore \nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}.$$

For $\nabla^2 f(r) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} = 0, \longrightarrow (1)$

we substitute $z = \log r$. Then $\frac{dz}{dr} = \frac{1}{r}$.

Now $\frac{df}{dr} = \frac{df}{dz}\cdot\frac{dz}{dr} = \frac{1}{r}\frac{df}{dz} \Rightarrow$

or, $\frac{d^2 f}{dr^2} = \frac{d}{dr}\left(\frac{1}{r}\frac{df}{dz}\right) = -\frac{1}{r^2}\frac{df}{dz} + \frac{1}{r}\frac{d^2 f}{dz^2}\cdot\frac{dz}{dr}$

$$= -\frac{1}{r^2}\frac{df}{dz} + \frac{1}{r^2}\frac{d^2 f}{dz^2} = \frac{1}{r^2}\left(\frac{d^2 f}{dz^2} - \frac{df}{dz}\right).$$

From (1), we get,

$$\frac{1}{r^2}\left(\frac{d^2 f}{dz^2} - \frac{df}{dz}\right) + \frac{2}{r}\cdot\frac{1}{r}\frac{df}{dz} = 0$$

or, $\frac{d^2 f}{dz^2} + \frac{df}{dz} = 0 \Rightarrow \frac{\frac{d^2 f}{dz^2}}{\frac{df}{dz}} = -1$

Integrating both sides w.r.t. z, we get

$\log\left(\frac{df}{dz}\right) = -z + \log C$, or, $\frac{df}{dz} = C\cdot e^{-z}$

Again integrating w.r.t. z, we get

$f = -Ce^{-z} + D$; or, $f(r) = D - \frac{C}{r}$; where C & D are arbitrary constants.

Ex (105) If $f(r)$ is differentiable, prove that $f(r)\bar{r}$ is irrotational

$$\bar{\nabla}\times\{f(r)\bar{r}\} = \bar{\nabla}f(r)\times\bar{r} + f(r)\bar{\nabla}\times\bar{r} \quad [\because \bar{\nabla}\times(\phi\bar{A}) = \bar{\nabla}\phi\times\bar{A}+\phi(\bar{\nabla}\times\bar{A})]$$

$$=\{f'(r)\bar{\nabla}r\}\times\bar{r} + f(r)\bar{0} \quad [\because \bar{\nabla}f(u) = f'(u)\bar{\nabla}u]$$

$$=\left\{f'(r)\frac{\bar{r}}{r}\right\}\times\bar{r} + \bar{0} \quad \left[\because \bar{\nabla}r = \sum\hat{i}\frac{\partial r}{\partial x} = \sum\hat{i}\frac{x}{r} = \frac{\bar{r}}{r}\right]$$

$$=\frac{f'(r)}{r}(\bar{r}\times\bar{r}) + \bar{0}$$

$$=\frac{f'(r)}{r}\bar{0} + \bar{0}$$

$$=\bar{0}+\bar{0}$$

$$=\bar{0}$$

$\therefore$ $f(r)\bar{r}$ is irrotational. proved.

Maity & Ghosh

(16) Find the angle of intersection at the point $(2,-1,2)$ of the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$.

The angle of intersection of the given surfaces is equal to the angle of intersection of their normals.

Let $\phi_1(x,y,z) = x^2+y^2+z^2 = 9 \longrightarrow (1)$

& $\phi_2(x,y,z) = x^2+y^2-z = 3 \longrightarrow (2)$,

$$\bar{\nabla}\phi_1\Big]_{(2,-1,2)} = (2x, 2y, 2z)\Big]_{(2,-1,2)} = (4,-2,4) \longrightarrow \text{Normal to } \phi_1$$

$$\bar{\nabla}\phi_2\Big]_{(2,-1,2)} = (2x, 2y, -1)\Big]_{(2,-1,2)} = (4,-2,-1) \longrightarrow \text{Normal to } \phi_2$$

$\therefore$ Angle between their normals at $(2,-1,2)$ is given by

$$\cos\theta = \frac{\bar{\nabla}\phi_1\cdot\bar{\nabla}\phi_2}{|\bar{\nabla}\phi_1||\bar{\nabla}\phi_2|} = \frac{16+4-4}{\sqrt{36}\cdot\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}.$$

a, $\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$, which is the required angle.

Ex. (106) Is there a differentiable vector function $\bar{v}$ such that curl $\bar{v} = \bar{r}$?

Let curl $\bar{v} = \bar{r}$, then $\bar{\nabla}\cdot(\bar{\nabla}\times\bar{v}) = \bar{\nabla}\cdot\bar{r}$

Now, L.H.S. $= \bar{\nabla}\cdot(\bar{\nabla}\times\bar{v}) = \bar{\nabla}\cdot\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$ [Let $\bar{v}=(v_1,v_2,v_3)$]

$$= \bar{\nabla}\cdot\left\{\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)\hat{i}+\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)\hat{j}+\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)\hat{k}\right\}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)$$

$$= \frac{\partial^2 v_3}{\partial x\partial y}-\frac{\partial^2 v_2}{\partial x\partial z}+\frac{\partial^2 v_1}{\partial y\partial z}-\frac{\partial^2 v_3}{\partial y\partial x}+\frac{\partial^2 v_2}{\partial z\partial x}-\frac{\partial^2 v_1}{\partial z\partial y}$$

$$= 0$$

R.H.S. $= \bar{\nabla}\cdot\bar{r} = \frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z) = 1+1+1 = 3$.

$\therefore$ L.H.S $= 0 \neq 3 =$ R.H.S.

$\therefore$ There is no differentiable vector function $\bar{v}$ s.t. curl $\bar{v} = \bar{r}$.

Ex. (92) For what value of the constant a will the vector $\bar{A} = (axy-z^3)\hat{i}+(a-2)x^2\hat{j}+(1-a)xz^2\hat{k}$ have its curl identically equal to zero?

$$\bar{\nabla}\times\bar{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (axy-z^3) & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} = 0\hat{i}+[-3z^2-(1-a)z^2]\hat{j}+[2x(a-2)-ax]\hat{k}$$

$$= (-4z^2+az^2)\hat{j}+(xa-4x)\hat{k}$$

$$= z^2(a-4)\hat{j}+x(a-4)\hat{k}$$

Now $\bar{\nabla}\times\bar{A} = \bar{0}$

$\Rightarrow z^2(a-4)\hat{j}+x(a-4)\hat{k} = 0\hat{i}+0\hat{j}+0\hat{k}$.

$\Rightarrow a-4 = 0 \Rightarrow \underline{a=4}$. (Ans.)

VECTOR INTEGRATION [Line Integration]

Ordinary Integrals

Let $\bar{R}(u) = R_1(u)\hat{i} + R_2(u)\hat{j} + R_3(u)\hat{k}$ be a vector function of u, where $R_1(u), R_2(u), R_3(u)$ are continuous in a specified interval $a \le u \le b$. Then

$$\int_{u=a}^{b} \bar{R}(u)\,du = \hat{i}\int_a^b R_1(u)\,du + \hat{j}\int_a^b R_2(u)\,du + \hat{k}\int_a^b R_3(u)\,du$$

$$= \int_{u=a}^{b} \frac{d}{du}\left(\bar{S}(u)\right)du = \left[\bar{S}(u) + \bar{c}\right]_{u=a}^{b} = \bar{S}(b) - \bar{S}(a)$$

① **LINE INTEGRALS** :- Let $\bar{r}(u) = x(u)\hat{i} + y(u)\hat{j} + z(u)\hat{k}$ be the position vector of (x,y,z), define a curve C joining points $P_1\,(u=u_1)$ and $P_2\,(u=u_2)$.

Let $\bar{r}(u)$ have continuous derivative along C.

Let $\bar{A}(x,y,z) = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ be a vector function of position and it be continuous along C.

Then the integral of the tangential component of $\bar{A}$ along C from P_1 to P_2 as a Line Integral:

$$\int_{P_1}^{P_2} \bar{A}\cdot d\bar{r} = \int_C \bar{A}\cdot d\bar{r} = \int_C A_1\,dx + A_2\,dy + A_3\,dz$$

If C be a simple closed curve, then the line integral:

$$\oint \bar{A}\cdot d\bar{r} = \oint A_1\,dx + A_2\,dy + A_3\,dz.$$

Work done $= \int_C \bar{F}\cdot d\bar{r}$, where $\bar{A} = \bar{F}(x,y,z)$, force acting on the particle moving along C.

Theorem: If $\bar{A} = \bar{\nabla}\phi$ everywhere in a region $R: \{a_1 \le x \le a_2,\ b_1 \le y \le b_2,\ c_1 \le z \le c_2\}$ where $\phi(x,y,z)$ is single-valued and has continuous derivatives in R, then

1. $\int_{P_1}^{P_2} \bar{A}\cdot d\bar{r}$ is independent of the path C in R joining P_1 & P_2

2. $\oint_C \bar{A}\cdot d\bar{r} = 0$ around any closed curve C in R.

$\bar{A} \longrightarrow$ conservative vector field iff $\bar{\nabla}\times\bar{A} = \bar{0}$ or, $\bar{A} = \bar{\nabla}\phi$

$\phi \rightarrow$ its scalar potential

For such case, $\bar{A}\cdot d\bar{r} = \bar{\nabla}\phi\cdot d\bar{r}$

$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi$

an exact differential.

Problem: (a) If $\bar{F} = \bar{\nabla}\phi$ (ϕ is single-valued and having continuous partial derivatives), show that the work-done in moving a particle from one point $P_1(x_1, y_1, z_1)$ in this field to another point $P_2(x_2, y_2, z_2)$ is independent of the path joining two points

(b) Conversely, if $\int \bar{F}\cdot d\bar{r}$ is independent of the path C joining two points, show that $\exists$ a function ϕ s.t. $\bar{F} = \bar{\nabla}\phi$.

(a) Work done $= \int_{P_1}^{P_2} \bar{F}\cdot d\bar{r} = \int_{P_1}^{P_2} \bar{\nabla}\phi\cdot d\bar{r} = \int_{P_1}^{P_2} \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$

$$= \int_{P_1}^{P_2} \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = \int_{P_1}^{P_2} d\phi = \phi(P_2) - \phi(P_1)$$

$$= \phi(x_2, y_2, z_2) - \phi(x_1, y_1, z_1)$$

$\therefore$ Work done depends on two points P_1 and P_2 only, and not on the path joining them.

This is true only if $\phi(x, y, z)$ is single-valued at all points P_1 and P_2.

(b) Let $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ and given that $\int_C \bar{F}\cdot d\bar{r}$ is independent of the path C joining any two points, say (x_1, y_1, z_1) and (x, y, z) respectively.

Then $\phi(x, y, z) = \int_{(x_1,y_1,z_1)}^{(x,y,z)} \bar{F}\cdot d\bar{r} = \int_{(x_1,y_1,z_1)}^{(x,y,z)} F_1\,dx + F_2\,dy + F_3\,dz$

is independent of the path joining (x_1, y_1, z_1) and (x, y, z)

Now, $\phi(x, y, z) = \int_{(x_1,y_1,z_1)}^{(x,y,z)} \bar{F}\cdot\frac{d\bar{r}}{ds}\,ds$

Differentiating w.r.t. s, we get

$\frac{d\phi}{ds} = \bar{F}\cdot\frac{d\bar{r}}{ds}$; But $\frac{d\phi}{ds} = \nabla\phi\cdot\frac{d\bar{r}}{ds}$.

So that $(\bar{\nabla}\phi - \bar{F})\cdot\frac{d\bar{r}}{ds} = 0$. Since this must hold irrespective of $\frac{d\bar{r}}{ds}$, we have $\bar{F} = \bar{\nabla}\phi$.

Prob: (a) If $\bar{F}$ is a conservative field, prove that curl $\bar{F} = \bar{\nabla} \times \bar{F} = \bar{0}$, i.e., $\bar{F}$ is irrotational.

(b) Conversely, if $\bar{\nabla} \times \bar{F} = \bar{0}$, Prove that $\bar{F}$ is conservative

(a) $\bar{F}$ is conservative means $\int_c \bar{F} \cdot d\bar{r}$ is independent of the path c joining two points, and hence by previous problem, $\bar{F} = \bar{\nabla}\phi$.

Thus curl $\bar{F} = \bar{\nabla} \times \bar{F} = \bar{\nabla} \times \bar{\nabla}\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$

$\therefore \bar{\nabla} \times \bar{F} = \hat{i}.0 + \hat{j}.0 + \hat{k}.0$

$\therefore \bar{\nabla} \times \bar{F} = \bar{0} \Rightarrow \bar{F}$ is irrotational.

(b) If $\bar{\nabla} \times \bar{F} = \bar{0}$, then $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \bar{0}$

$$\Rightarrow \left.\begin{aligned} \frac{\partial F_3}{\partial y} &= \frac{\partial F_2}{\partial z} \\ \frac{\partial F_1}{\partial z} &= \frac{\partial F_3}{\partial x} \\ \frac{\partial F_2}{\partial x} &= \frac{\partial F_1}{\partial y} \end{aligned}\right\}$$

We must show that $\bar{F} = \bar{\nabla}\phi$.

The workdone in moving a particle from (x_1, y_1, z_1) to (x, y, z) in the force field F is

$$\int_c F_1(x,y,z)\,dx + F_2(x,y,z)\,dy + F_3(x,y,z)\,dz$$

where c is the path joining (x_1, y_1, z_1) and (x, y, z). Let us choose as a particular path the straight line segments from (x_1, y_1, z_1) to (x, y_1, z_1) to (x, y, z_1) to (x, y, z) and call $\phi(x,y,z)$ is called the workdone along this particular path.

Then $\phi(x,y,z) = \int_{x_1}^{x} F_1(x, y_1, z_1)\,dx + \int_{y_1}^{y} F_2(x, y, z_1)\,dy + \int_{z_1}^{z} F_3(x,y,z)\,dz$

$$\therefore \frac{\partial \phi}{\partial z} = F_3(x,y,z) \longrightarrow (1)$$

$$\frac{\partial \phi}{\partial y} = F_2(x,y,z_1) + \int_{z_1}^{z} \frac{\partial F_3}{\partial y}(x,y,z)\,dz = F_2(x,y,z_1) + \int_{z_1}^{z} \frac{\partial F_2}{\partial z}(x,y,z)\,dz$$

$$= F_2(x,y,z_1) + F_2(x,y,z)\Big|_{z=z_1}^{z}$$

$$= F_2(x,y,z_1) + F_2(x,y,z) - F_2(x,y,z_1)$$

$$\therefore \frac{\partial \phi}{\partial y} = F_2(x,y,z) \longrightarrow (2)$$

$$\frac{\partial \phi}{\partial x} = F_1(x,y_1,z_1) + \int_{y_1}^{y} \frac{\partial F_2}{\partial x}(x,y,z_1)\,dy + \int_{z_1}^{z} \frac{\partial F_3}{\partial x}(x,y,z)\,dz$$

$$= F_1(x,y_1,z_1) + \int_{y_1}^{y} \frac{\partial F_1}{\partial y}(x,y,z_1)\,dy + \int_{z_1}^{z} \frac{\partial F_1}{\partial z}(x,y,z)\,dz$$

i.e., $\frac{\partial\phi}{\partial x} = F_1(x,y_1,z_1) + F_1(x,y,z_1)\Big|_{y=y_1}^{y} + F_1(x,y,z)\Big|_{z=z_1}^{z}$

$= F_1(\cancel{x,y_1,z_1}) + F_1(\cancel{x,y},z_1) - F_1(\cancel{x,y_1,z_1}) + F_1(x,y,z) - F_1(\cancel{x,y,z_1})$

$\therefore \frac{\partial\phi}{\partial x} = F_1(x,y,z) \longrightarrow (3)$

$\therefore$ From (1), (2) & (3), we get

$\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k} = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} = \bar{\nabla}\phi.$

Thus, a necessary and sufficient condition that a field $\bar{F}$ be conservative is that curl $\bar{F} = \bar{\nabla}\times\bar{F} = \bar{0}$

Prob: Show that a necessary and sufficient condition that $F_1dx + F_2dy + F_3dz$ be an exact differential is that $\bar{\nabla}\times\bar{F} = \bar{0}$ where $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

Suppose, $F_1dx + F_2dy + F_3dz = d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$, an exact differential.

Then x, y, z are independent variables,

$F_1 = \frac{\partial\phi}{\partial x},\ F_2 = \frac{\partial\phi}{\partial y},\ F_3 = \frac{\partial\phi}{\partial z}$

$\therefore \bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k} = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} = \bar{\nabla}\phi$

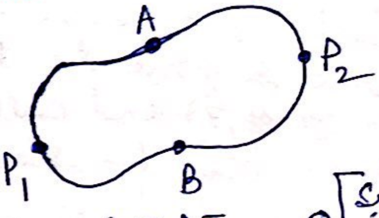
Thus $\bar{\nabla}\times\bar{F} = \bar{\nabla}\times\bar{\nabla}\phi = \bar{0}$

Conversely, if $\bar{\nabla}\times\bar{F} = \bar{0}$, then $\bar{F} = \bar{\nabla}\phi$, so that

$\bar{F}\cdot d\bar{r} = \bar{\nabla}\phi\cdot d\bar{r} = d\phi$, i.e., $F_1dx + F_2dy + F_3dz = d\phi$, an exact differential.

Prob: Prove that if $\int_{P_1}^{P_2}\bar{F}\cdot d\bar{r}$ is independent of the path joining any two points P_1 and P_2 in a given region, then $\oint\bar{F}\cdot d\bar{r} = 0$ for all closed paths in the region and conversely.

Let $P_1AP_2BP_1$ be a closed curve.



Then $\oint\bar{F}\cdot d\bar{r} = \int_{P_1AP_2BP_1}\bar{F}\cdot d\bar{r}$

$= \int_{P_1AP_2}\bar{F}\cdot d\bar{r} + \int_{P_2BP_1}\bar{F}\cdot d\bar{r} = \int_{P_1AP_2}\bar{F}\cdot d\bar{r} - \int_{P_1BP_2}\bar{F}\cdot d\bar{r} = 0$ [Since it is independent of the path]

$\therefore \oint\bar{F}\cdot d\bar{r} = 0$

Conversely, if $\oint\bar{F}\cdot d\bar{r} = 0$, then $\int_{P_1AP_2BP_1}\bar{F}\cdot d\bar{r} = \int_{P_1AP_2}\bar{F}\cdot d\bar{r} + \int_{P_2BP_1}\bar{F}\cdot d\bar{r}$

$= \int_{P_1AP_2}\bar{F}\cdot d\bar{r} - \int_{P_1BP_2}\bar{F}\cdot d\bar{r} = 0$

$\therefore \int_{P_1AP_2}\bar{F}\cdot d\bar{r} = \int_{P_1BP_2}\bar{F}\cdot d\bar{r}$

$\Rightarrow \int_{P_1}^{P_2}\bar{F}\cdot d\bar{r}$ is independent the path joining P_1 & P_2.

LINE INTEGRALS

Prob. 37 If $\bar{A} = (2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}$, evaluate $\int_C \bar{A}\cdot d\bar{r}$ along the following paths C:

(a) $x = 2t^2,\ y = t,\ z = t^3$ from $t=0$ to $t=1$,

(b) The straight lines from $(0,0,0)$ to $(0,0,1)$, then to $(0,1,1)$, and then to $(2,1,1)$

(c) the straight line joining $(0,0,0)$ and $(2,1,1)$.

(a) $\bar{A}\cdot d\bar{r} = (2y+3)\,dx + xz\,dy + (yz-x)\,dz$

$x = 2t^2,\quad dx = (4t)\,dt$

$y = t,\quad dy = dt$

$z = t^3,\quad dz = 3t^2\,dt$

$$\int_C \bar{A}\cdot d\bar{r} = \int_{t=0}^{1} (2t+3)\cdot 4t\,dt + 2t^2\cdot t^3\cdot dt + (t\cdot t^3 - 2t^2)\cdot 3t^2\,dt$$

$$= \int_{t=0}^{1} (8t^2 + 12t + 2t^5 + 3t^6 - 6t^4)\,dt = \left[\frac{8}{3}t^3 + \frac{12}{2}t^2 + \frac{2}{6}t^6 + \frac{3}{7}t^7 - \frac{6t^5}{5}\right]_{t=0}^{1}$$

$$= \frac{8}{3} + 6 + \frac{1}{3} + \frac{3}{7} - \frac{6}{5} = 9 + \frac{3}{7} - \frac{6}{5} = \frac{315+15-42}{35} = \frac{288}{35}.$$

(b) Straight line C_1: from $(0,0,0)$ to $(0,0,1)$,

here $x = 0,\ y = 0,\ z = 0$ to 1.

$\therefore dx = 0,\ dy = 0$.

$$\therefore \int_{C_1} \bar{A}\cdot d\bar{r} = \int_{z=0}^{1} (yz - x)\,dz = \int_{z=0}^{1} (0.z - 0)\,dz = 0.$$

Straight line C_2: from $(0,0,1)$ to $(0,1,1)$.

$x = 0,\ y = 0$ to $1,\ z = 1$; $\therefore dx = 0,\ dz = 0$.

$$\int_{C_2} \bar{A}\cdot d\bar{r} = \int_{y=0}^{1} xz\,dy = \int_0^1 0.1\,dy = 0.$$

Straight line C_3: from $(0,1,1)$ to $(2,1,1)$.

$x = 0$ to $2,\ y = 1,\ z = 1$, $\therefore dy = 0,\ dz = 0$.

$$\int_{C_3} \bar{A}\cdot d\bar{r} = \int_{x=0}^{2} (2y+3)\,dx = \int_{x=0}^{2} (2.1+3)\,dx = \left[5x\right]_0^2 = 10.$$

$$\therefore \int_C \bar{A}\cdot d\bar{r} = \int_{C_1} \bar{A}\cdot d\bar{r} + \int_{C_2} \bar{A}\cdot d\bar{r} + \int_{C_3} \bar{A}\cdot d\bar{r} = 0 + 0 + 10 = 10$$

(c) Straight line joining $(0,0,0)$ to $(2,1,1)$ is given by

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = t\ (\text{say}) \Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{1} = t$$

$\therefore x = 2t,\ y = t,\ z = t$. $\quad dx = 2\,dt,\ dy = dt,\ dz = dt$

$$\int_C \bar{A}\cdot d\bar{r} = \int_{t=0} (2t+3)\cdot 2\,dt + 2t\cdot t\,dt + (t\cdot t - 2t)\,dt = \int_{t=0}^{1} (3t^2 + 2t + 6)\,dt$$

$= 1 + 1 + 6 = 8$

Prob. (39). If $\bar{F} = (2x+y)\hat{i} + (3y-x)\hat{j}$, evaluate $\int \bar{F}.d\bar{r}$ where c is the curve in the xy plane consisting of the straight lines from $(0,0)$ to $(2,0)$ and then to $(3,2)$.

Solution: $F.d\bar{r} = (2x+y)dx + (3y-x)dy$

The straightline C_1: from $(0,0)$ to $(2,0)$, $x = 0$ to 2, $y = 0$.

$\therefore dy = 0$.

$$\therefore \int_{C_1} \bar{F}.d\bar{r} = \int_{x=0}^{2} (2x+y)dx = \int_0^2 2x\,dx = x^2\Big]_0^2 = 4.$$

The straightline C_2: from $(2,0)$ to $(3,2)$.

$$\therefore \frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = t \Rightarrow \frac{x-2}{3-2} = \frac{y-0}{2-0} = t$$

t varies from 0 to 1. $\Rightarrow x = t+2,\ y = 2t.$

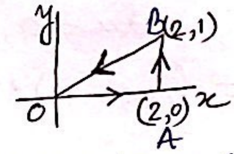
$dx = dt,\ dy = 2dt.$

$$\therefore \int_{C_2} \bar{F}.d\bar{r} = \int_{t=0}^{1} \{2(t+2)+2t\}dt + (3.2t-t-2).2dt$$

$$= \int_{t=0}^{1} 14t\,dt = 7t^2\Big]_{t=0}^{1} = 7$$

$$\therefore \int_C \bar{F}.d\bar{r} = \int_{C_1} \bar{F}.d\bar{r} + \int_{C_2} \bar{F}.d\bar{r} = 4+7 = 11.$$

P. 43. If $F = (2x+y^2)\hat{i} + (3y-4x)\hat{j}$, evaluate $\oint_c F.d\bar{r}$ around the triangle C:



$F.d\bar{r} = (2x+y^2)dx + (3y-4x)dy.$

Along the straight line $\overrightarrow{OA}$: $x = 0$ to 2, $y = 0$, $\therefore dy = 0$.

$$\int_{\overrightarrow{OA}} F.d\bar{r} = \int_{x=0}^{2} (2x+y^2)dx = \int_0^2 2x\,dx = [x^2]_0^2 = 4.$$

Along the Straight line $\overrightarrow{AB}$: $x = 2$, $y = 0$ to 1, $dx = 0$.

$$\int_{\overrightarrow{AB}} F.d\bar{r} = \int_{y=0}^{1} (3y-4x)dy = \int_0^1 (3y-4.2)dy = \left[\frac{3y^2}{2} - 8y\right]_{y=0}^{1} = \left(3.\frac{1}{2} - 8\right)$$

$$= \frac{3}{2} - 8 = \frac{-13}{2}$$

Along the Straight line $\overrightarrow{BO}$: $\frac{x-2}{0-2} = \frac{y-1}{0-1} = t$

$x = 2-2t,\ y = 1-t,\ dx = -2dt,\ dy = -dt,\ t = 0$ to 1

$$\int_{\overrightarrow{BO}} \bar{F}.d\bar{r} = \int_{t=0}^{1} \{2(2-2t)+(1-t)^2\}(-2)\,dt + \{3(1-t) - 4(2-2t)\}.(-dt)$$

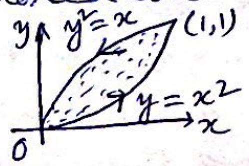
$$= \int_{t=0}^{1} \{(t^2-6t+5)\times(-2) + (5t-5)\times(-1)\}dt = \int_{t=0}^{1} (-2t^2+7t-5)\,dt = -\frac{2}{3} + \frac{7}{2} - 5$$

$$= \frac{-4+21-30}{6}$$

$$= \frac{-13}{6}$$

$$\therefore \oint_c F.d\bar{r} = \int_{OA} + \int_{AB} + \int_{BO} = 4 - \frac{13}{2} - \frac{13}{6} = \frac{24-39-13}{6} = -\frac{28}{6} = \frac{-14}{3}$$

44. Evaluate $\oint_C \bar{A}\cdot d\bar{r}$ around the closed curve C if
$\bar{A} = (x-y)\hat{i} + (x+y)\hat{j}$



Soln: Along $y = x^2$, $dy = 2x\,dx$.

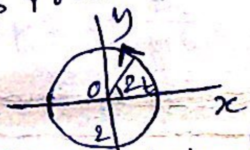
Now, $\bar{A}\cdot d\bar{r} = (x-y)\,dx + (x+y)\,dy$.

$\therefore \int_{(y=x^2)} \bar{A}\cdot d\bar{r} = \int_{x=0}^{1} (x-x^2)\,dx + (x+x^2)\,2x\,dx = \int_0^1 (2x^3+x^2+x)\,dx$
$= \frac{2}{4}+\frac{1}{3}+\frac{1}{2} = 1+\frac{1}{3} = 4/3.$

$\int_{y^2=x} \bar{A}\cdot d\bar{r} = \int_{y=1}^{0} (y^2-y)\,2y\,dy + (y^2+y)\,dy = \int_{y=1}^{0} (2y^3-y^2+y)\,dy$
$= -\frac{2}{4}+\frac{1}{3}-\frac{1}{2} = -1+\frac{1}{3} = -2/3.$

$\therefore \oint_C \bar{A}\cdot d\bar{r} = \frac{4}{3}-\frac{2}{3} = \frac{2}{3}.$ (Ans.).

45. If $\bar{A} = (y-2x)\hat{i} + (3x+2y)\hat{j}$, compute the circulation of $\bar{A}$ about a circle C in the xy plane with centre at the origin and radius 2, if C is traversed in the +ve direction.



$\bar{A}\cdot d\bar{r} = (y-2x)\,dx + (3x+2y)\,dy$

Circle C: $x^2+y^2 = 2^2$, $x = 2\cos t$, $y = 2\sin t$, t is a parameter

t runs for 0 to 2π.

$\therefore \int_C \bar{A}\cdot d\bar{r} = \int_{t=0}^{2\pi} (2\sin t - 4\cos t)(-2\sin t)\,dt + (3\cdot 2\cos t + 2\cdot 2\sin t)(2\cos t)\,dt = \int_0^{2\pi} (-4\sin^2 t + 16\sin t\cos t + 12\cos^2 t)\,dt$

$= \int_{t=0}^{2\pi} (-4\sin^2 t + 8\sin t\cos t + 12\cos^2 t + 8\sin t\cos t)\,dt = \int_0^{2\pi} (4 + 8\cos 2t + 8\sin 2t)\,dt$

$= \int_0^{2\pi} [-2(1-\cos 2t) + 8\sin 2t + 6(1+\cos 2t)]\,dt$

$= \int_0^{2\pi} 4\,dt + 8\int_0^{2\pi} \cos 2t\,dt + 8\int_0^{2\pi} \sin 2t\,dt = 4\times 2\pi + 8\left[\frac{\sin 2t}{2} - \frac{\cos 2t}{2}\right]_{t=0}^{2\pi}$

$= 8\pi + 4[1-1] = 8\pi$ **Ans.**

47. Prove that $\bar{F} = (y^2\cos x + z^3)\hat{i} + (2y\sin x - 4)\hat{j} + (3xz^2+2)\hat{k}$ is a conservative force field.

(i) Find the scalar potential for $\bar{F}$

(ii) Find the work done in moving an object in this field from $(0,1,-1)$ to $(\pi/2,-1,2)$.

Ans: A necessary and sufficient condition that a force $\bar{F}$ will be conservative is that curl $\bar{F} = \bar{\nabla}\times\bar{F} = \bar{0}$

Now $\bar{\nabla}\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y^2\cos x+z^3) & (2y\sin x-4) & (3xz^2+2) \end{vmatrix} = 0\hat{i}+0\hat{j}+0\hat{k} = \bar{0}$

$\therefore$ F is a conservative force field.

(i) $\therefore$ $F = \bar{\nabla}\phi$, ϕ is a scalar potential for F

$\therefore$
$$\frac{\partial\phi}{\partial x} = y^2\cos x + z^3 \Rightarrow \phi(x,y,z) = y^2\sin x + xz^3 + f(y,z)$$
$$\frac{\partial\phi}{\partial y} = 2y\sin x - 4 \Rightarrow \phi(x,y,z) = y^2\sin x - 4y + g(x,z)$$
$$\frac{\partial\phi}{\partial z} = 3xz^2 + 2 \Rightarrow \phi(x,y,z) = xz^3 + 2z + h(x,y)$$

Let us choose $f(y,z) = -4y+2z$, $g(x,z) = xz^3+2z$, & $h(x,y) = y^2\sin x - 4y$.

Then $\phi(x,y,z) = y^2\sin x + xz^3 - 4y + 2z + \text{Constant}$.

(ii) The Work done $= \int_{P_1}^{P_2}\bar{F}\cdot d\bar{r} = \int_{P_1}^{P_2}\bar{\nabla}\phi\cdot d\bar{r} = \int_{P_1}^{P_2} d\phi = \phi(P_2)-\phi(P_1)$

$= \phi(\pi/2,-1,2) - \phi(0,1,-1)$

$= 1 + 4\pi + 4 + 4 + \cancel{c} + 4 + 2 - \cancel{c} = \underline{15+4\pi}$ Ans.

53. Show that $(2x\cos y + z\sin y)dx + (xz\cos y - x^2\sin y)dy + x\sin y\,dz$ is an exact differential. Hence solve the corresponding D.E.

Ans: Let $F = (2x\cos y + z\sin y)\hat{i} + (xz\cos y - x^2\sin y)\hat{j} + x\sin y\hat{k}$

A necessary and sufficient condition that $\bar{F}\cdot d\bar{r}$ is an exact differential $= d\phi$, if $F = \bar{\nabla}\phi$ or curl $\bar{F} = \bar{0}$,

Now, we see that curl $\bar{F} = \bar{0}$, Hence it is exact differential

Now, Solve an exact differential eqn

Ans: $x^2\cos y + xz\sin y = \text{Constant}$,

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40. Find the workdone in moving a particle in the force field $\bar{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ along

(a) the straight line from $(0,0,0)$ to $(2,1,3)$.

(b) the space curve $x = 2t^2$, $y = t$, $z = 4t^2 - t$ from $t=0$ to $t=1$.

(c) the curve defined by $x^2 = 4y$, $3x^3 = 8z$ from $x=0$ to $x=2$.

Solution:

(a) The parametric equations of the straight line from $(0,0,0)$ to $(2,1,3)$ is given by:

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} = t \text{ (say); or, } x = 2t, y = t, z = 3t$$

where $t : 0 \to 1$.

Now workdone $= \int \bar{F}\cdot d\bar{r} = \int 3x^2dx + (2xz - y)dy + zdz$

$$= \int_{t=0}^{1}\left[3\cdot(2t)^2(2dt) + (2\cdot 2t\cdot 3t - t)\,dt + 3t\,(3dt)\right]$$

$$= \int_{t=0}^{1}\left[24t^2 + 12t^2 - t + 9t\right]dt = \int_{t=0}^{1}(36t^2 + 8t)dt = \left[12t^3 + 4t^2\right]_{t=0}^{1}$$

$= 16.$

$\therefore$ workdone $= \int \bar{F}\cdot d\bar{r} = 16$. (Ans.)

(b) $$\int \bar{F}\cdot d\bar{r} = \int_{t=0}^{1}\left[3\cdot(2t^2)^2 d(2t^2) + \{2\cdot 2t^2\cdot(4t^2 - t) - t\}dt + (4t^2 - t)\,d(4t^2 - t)\right]$$

$$= \int_{t=0}^{1}\left[48t^5 + 16t^4 - 4t^3 - t + (4t^2 - t)(8t - 1)\right]dt$$

$$= \int_{t=0}^{1}\left[48t^5 + 16t^4 + 28t^3 - 12t^2\right]dt$$

$$= \left[8t^6 + \frac{16}{5}t^5 + 7t^4 - 4t^3\right]_{t=0}^{1}$$

$$= 8 + \frac{16}{5} + 7 - 4 = 11 + \frac{16}{5} = \frac{71}{5}.$$

$\therefore$ Workdone $= \frac{71}{5}$ (Ans)

(c) The given curve: $x^2 = 4y$, $3x^3 = 8z$; or, $y = x^2/4$, $z = \frac{3}{8}x^3$.

Workdone $= \int \bar{F}\cdot d\bar{r} = \int 3x^2dx + (2xz - y)\,dy + zdz$

$$= \int_{x=0}^{2} 3x^2dx + \left(2x\cdot\frac{3}{8}x^3 - \frac{x^2}{4}\right)d\left(\frac{x^2}{4}\right) + \frac{3}{8}x^3 d\left(\frac{3}{8}x^3\right)$$

$$= \int_{x=0}^{2}\left[3x^2 + \left(\frac{3}{4}x^4 - \frac{x^2}{4}\right)\cdot\frac{x}{2} + \frac{27}{64}x^5\right]dx$$

$$= \int_{x=0}^{2}\left[3x^2 + \left(\frac{3}{8} + \frac{27}{64}\right)x^5 - \frac{x^3}{8}\right]dx = \left[x^3 + \frac{1}{6}\left(\frac{3}{8} + \frac{27}{64}\right)x^6 - \frac{x^4}{32}\right]_{x=0}^{2}$$

$$= 8 + \frac{51}{64}\cdot\frac{2^6}{6} - \frac{2^4}{32} = 8 + \frac{17}{2} - \frac{1}{2} = 16. \text{ (Ans.)}$$

Ghosh & Maity Book [C]

(2) Show that $\bar{F} = (2xy+z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is a conservative force field. Find the scalar potential. Find also the workdone in moving an object in this field from $(1,-2,1)$ to $(3,1,4)$.

1st part: The condition for $\bar{F}$ to be a conservative force field is that $\bar{\nabla}\times\bar{F} = \bar{0}$.

Now
$$\bar{\nabla}\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix} = \hat{i}\left\{\frac{\partial}{\partial y}(3xz^2) - \frac{\partial}{\partial z}(x^2)\right\} + \hat{j}\left\{\frac{\partial}{\partial z}(2xy+z^3) - \frac{\partial}{\partial x}(3xz^2)\right\} + \hat{k}\left\{\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(2xy+z^3)\right\}$$
$$= 0\hat{i} + \hat{j}(3z^2-3z^2) + \hat{k}(2x-2x)$$
$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = \bar{0}$$

$\bar{F}$ is conservative. (proved)

2nd part: Since $\bar{F}$ is conservative, there exists a scalar potential ϕ s.t. $\bar{F} = \bar{\nabla}\phi$

i.e., $(2xy+z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k} = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$.

This give:
$$\frac{\partial\phi}{\partial x} = 2xy+z^3 \longrightarrow (1)$$
$$\frac{\partial\phi}{\partial y} = x^2 \longrightarrow (2)$$
$$\frac{\partial\phi}{\partial z} = 3xz^2 \longrightarrow (3).$$

Integrating (1), (2), (3) partially w.r.t. x, y, z respectively, we obtain;
$$\left.\begin{aligned} \phi &= x^2y + xz^3 + f(y,z) \\ \phi &= x^2y \qquad\quad\;\; + g(x,z) \\ \phi &= xz^3 \qquad\quad\;\; + h(x,y) \end{aligned}\right\}$$
where $f(y,z)$, $g(x,z)$ and $h(x,y)$ are arbitrary functions.

Let us choose $f(y,z) = c_1$, $g(x,z) = xz^3 + c_2$, $h(x,y) = x^2y + c_3$

Then $\phi = x^2y + xz^3 + c$, where $c = c_1+c_2+c_3$, an arbitrary constant.

3rd part: Workdone $= \int \bar{F}\cdot d\bar{r} = \int \bar{\nabla}\phi\cdot d\bar{r}$
$$= \int\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$
$$= \int \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = \int_{(1,-2,1)}^{(3,1,4)} d\phi$$
$$= [\phi]_{(1,-2,1)}^{(3,1,4)} = [x^2y + xz^3 + c]_{(1,-2,1)}^{(3,1,4)}$$
$$= \underline{202}.$$

④ Show that $(z^2-2xy)dx - x^2dy + 2xz\,dz$ is an exact differential. Find a function $\phi(x,y,z)$ such that $d\phi$ = the given differential.

The necessary and sufficient condition that $F_1dx + F_2dy + F_3dz$ be an exact differential is that $\bar{\nabla}\times\bar{F} = \bar{0}$ where $\bar{F} = (F_1, F_2, F_3)$.

$= (z^2-2xy, -x^2, 2xz)$, (given).

$$\therefore \bar{\nabla}\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (z^2-2xy) & (-x^2) & 2xz \end{vmatrix} = \hat{i}\left\{\frac{\partial}{\partial y}(2xz) - \frac{\partial}{\partial z}(-x^2)\right\} + \hat{j}\left\{\frac{\partial}{\partial z}(z^2-2xy) - \frac{\partial}{\partial x}(2xz)\right\} + \hat{k}\left\{\frac{\partial}{\partial x}(-x^2) - \frac{\partial}{\partial y}(z^2-2xy)\right\}$$

$$= \hat{i}\,0 + \hat{j}(2z-2z) + \hat{k}(-2x+2x)$$

$$= \bar{0}$$

$\therefore$ The given differential is an exact differential.

Let $d\phi = (z^2-2xy)dx - x^2dy + 2xz\,dz$

or, $\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z} = (z^2-2xy)dx - x^2dy + 2xz\,dz$.

Since x, y and z are independent variables,

$$\frac{\partial\phi}{\partial x} = z^2-2xy, \quad \frac{\partial\phi}{\partial y} = -x^2, \quad \frac{\partial\phi}{\partial z} = 2xz$$

Integrating partially w.r.t. x, y, z respectively,

$\phi = xz^2 - x^2y + f(y,z)$, $\phi = -x^2y + g(x,z)$, $\phi = xz^2 + h(x,y)$

where $f(y,z)$, $g(x,z)$ and $h(x,y)$ are arbitrary functions. Let us choose $f(y,z) = c_1$, $g(x,z) = xz^2 + c_2$, $h(x,y) = -x^2y + c_3$.

Then $\underline{\phi(x,y,z) = xz^2 - x^2y + c}$, where $c = c_1+c_2+c_3$ an arbitrary constant

⑥ Evaluate $\int yz\,dx + (xz+1)dy + xy\,dz$ along any path Γ joining the points $(1,0,0)$ and $(2,1,4)$.

First of all we have to check whether $yz\,dx + (xz+1)dy + xy\,dz$ is an exact differential.

Let $\bar{F} = (yz, xz+1, xy)$. Then $\bar{\nabla}\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & xz+1 & xy \end{vmatrix} = (x-x)\hat{i} + (y-y)\hat{j} + (z-z)\hat{k} = \bar{0}$

$\therefore$ It is an exact differential, so there exists a scalar function ϕ such that

$yz\,dx + (xz+1)dy + xy\,dz = d\phi$ (Exact differential)

or, $\int yz\,dx + (xz+1)dy + xy\,dz = \int d\phi = \phi + \text{Constant}$.

Now we have:

$$\frac{\partial \phi}{\partial x} = yz, \quad \frac{\partial \phi}{\partial y} = xz+1, \quad \frac{\partial \phi}{\partial z} = xy.$$

Integrating w.r.t. x, y, z partially respectively,

$$\phi = xyz + f(y,z), \quad \phi = xyz + y + g(x,z), \quad \phi = xyz + h(x,y).$$

Let us choose $f(y,z) = y + c_1$, $g(x,z) = c_2$, $h(x,y) = y + c_3$

Then $\phi(x,y,z) = xyz + y + c$, where $c_1 + c_2 + c_3 = c$.

$$\therefore \int_\Gamma yz\,dx + (xz+1)\,dy + xy\,dz = \Big[\phi(x,y,z)\Big]_{(1,0,0)}^{(2,1,4)}$$

$$= \Big[xyz + y + c\Big]_{(1,0,0)}^{(2,1,4)}$$

$$= 8 + 1 - 0 = \underline{9} \quad \text{(Ans).}$$

Spiegel

(38) If $\bar{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}$, evaluate $\int_C \bar{F}\cdot d\bar{r}$ along the curve C in the xy plane, $y = x^3$ from the point $(1,1)$ to $(2,8)$.

$y = x^3$, then $dy = 3x^2\,dx$.

$$\int_C \bar{F}\cdot d\bar{r} = \int_C (5xy - 6x^2)\,dx + (2y - 4x)\,dy.$$

$$= \int_{x=1}^{2} \left[(5x.x^3 - 6x^2) + (2x^3 - 4x)3x^2\right]dx$$

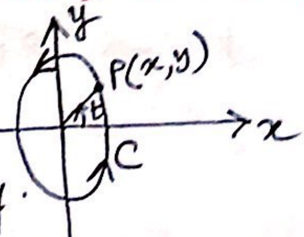
$$= \int_{x=1}^{2} (5x^4 - 6x^2 + 6x^5 - 12x^3)\,dx$$

$$= \Big[x^5 - 2x^3 + x^6 - 3x^4\Big]_{x=1}^{2}$$

$$= (32 - 16 + 64 - 48) - (1 - 2 + 1 - 3)$$

$$= 32 + 3 = \underline{35}. \quad \text{(Ans.)}$$

(41) Evaluate $\oint_C \bar{F}\cdot d\bar{r}$, where $\bar{F} = (x - 3y)\hat{i} + (y - 2x)\hat{j}$ and C is the closed curve in the xy plane, $x = 2\cos t$, $y = 3\sin t$ from $t = 0$ to $t = 2\pi$.



Let C be a closed curve which is traversed in the (+ve) counterclockwise direction.

$$\oint_C \bar{F}\cdot d\bar{r} = \oint_C (x - 3y)\,dx + (y - 2x)\,dy.$$

$$= \int_{t=0}^{2\pi} \left[(2\cos t - 9\sin t)\,d(2\cos t) + (3\sin t - 4\cos t)\,d(3\sin t)\right]$$

$$= \int_{t=0}^{2\pi} \left[-2\sin 2t + 18\sin^2 t + \frac{9}{2}\sin 2t - 12\cos^2 t\right]dt = \int_{t=0}^{2\pi} \left[\frac{5}{2}\sin 2t - 12\cos 2t + 3(1 - \cos 2t)\right]dt$$

$$= \left[-\frac{5}{4}\cos 2t - 6\sin 2t + 3t - \frac{3}{2}\sin 2t\right]_{t=0}^{2\pi} = \underline{6\pi}. \quad \text{(Ans.)}$$

Spiegel (53), Maity & Ghosh (10) — Show that $(2x\cos y + z\sin y)dx + (xz\cos y - x^2\sin y)dy + x\sin y\, dz$ is an exact differential. Hence solve the corresponding differential equation.

The necessary and sufficient condition that $F_1 dx + F_2 dy + F_3 dz$ an exact differential is $\bar{\nabla}\times\bar{F} = \bar{0}$. Here $\bar{F} = (2x\cos y + z\sin y)\hat{i} + (xz\cos y - x^2\sin y)\hat{j} + x\sin y\,\hat{k}$

Now,
$$\bar{\nabla}\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (2x\cos y + z\sin y) & (xz\cos y - x^2\sin y) & (x\sin y) \end{vmatrix}$$

$$= \hat{i}(x\cos y - x\cos y) + \hat{j}(\sin y - \sin y) + \hat{k}(z\cos y - 2x\sin y + 2x\sin y - z\cos y)$$
$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = \bar{0}$$

$\therefore$ The given is an exact differential.

So, there exists a scalar function $\phi(x,y,z)$ s.t. $(2x\cos y + z\sin y)dx + (xz\cos y - x^2\sin y)dy + x\sin y\,dz$
$$= d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

Since x, y and z are independent variables,
$$\frac{\partial\phi}{\partial x} = 2x\cos y + z\sin y,\quad \frac{\partial\phi}{\partial y} = xz\cos y - x^2\sin y,\quad \frac{\partial\phi}{\partial z} = x\sin y$$

Integrating partially w.r.t. x, y & z respectively

$$\left.\begin{aligned} \phi &= x^2\cos y + xz\sin y + f(y,z) \\ \phi &= xz\sin y + x^2\cos y + g(x,z) \end{aligned}\right\} f, g \ \&\ h \text{ are arbitrary functions}$$
$$\phi = xz\sin y + h(x,y)$$

Let us choose, $f(y,z) = c_1$, $g(x,z) = c_2$, $h(x,y) = x^2\cos y + c_3$.

$\therefore \phi = x^2\cos y + xz\sin y + c$; $c = c_1 + c_2 + c_3$, an arbitrary constant.

$\therefore$ The corresponding differential equation is given by: $(2x\cos y + z\sin y)dx + (xz\cos y - x^2\sin y)dy + x\sin y\,dz = 0$

$\Rightarrow d\phi = 0$, on integration, we have $\phi = $ constant.

i.e, the g.s. is given by $\boxed{x^2\cos y + xz\sin y = C}$

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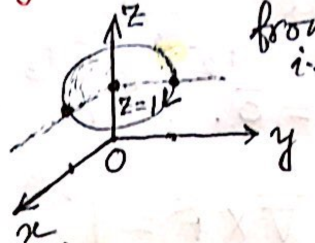
(51). Evaluate $\int_C \bar{A}\cdot d\bar{r}$ along the curve $x^2+y^2=1,\ z=1$ in the positive direction from $(0,1,1)$ to $(1,0,1)$ if $\bar{A} = (yz+2x)\hat{i} + xz\hat{j} + (xy+2z)\hat{k}$.

$x^2+y^2=1,\ z=1$

Let $x=\cos t,\ y=\sin t;\ z=1.$



from $(0,1,1)$ to $(1,0,1)$
i.e., $x: 0\to 1$
$y: 1\to 0$
then $t: \pi/2 \to 0$

Then $\bar{A}\cdot d\bar{r}$

$= (yz+2x)dx + xz\,dy + (xy+2z)dz$

$= (\sin t + 2\cos t)\,d(\cos t) + \cos t\, d(\sin t) + (\cos t\sin t + 2).d(1)$

$= [-\sin^2 t - \sin 2t + \cos^2 t]\,dt + 0$

$= (\cos 2t - \sin 2t)\,dt$

$\therefore \int \bar{A}\cdot d\bar{r} = \int_{t=\pi/2}^{0} (\cos 2t - \sin 2t)\,dt = \frac{1}{2}\left[\sin 2t + \cos 2t\right]_{t=\pi/2}^{0}$

$= \frac{1}{2}[1-\cos\pi] = \frac{1}{2}(1+1) = \underline{1}$. (Ans.)

(50) Show that the work done on a particle in moving it from A to B equals its change in kinetic energy at these points whether the force field is conservative or not.

Let us suppose a particle of constant mass m to move in this force field from A to B in space.

Now, $\bar{F} = m\bar{a} = m\frac{d^2\bar{r}}{dt^2}$.

Then $\bar{F}\cdot\frac{d\bar{r}}{dt} = m\frac{d^2\bar{r}}{dt^2}\cdot\frac{d\bar{r}}{dt} = \frac{m}{2}\frac{d}{dt}\left(\frac{d\bar{r}}{dt}\right)^2$.

Integrating w.r.t. t, we get

$$\int_A^B \bar{F}\cdot\frac{d\bar{r}}{dt}\,dt = \frac{m}{2}\int_A^B \frac{d}{dt}\left(\frac{d\bar{r}}{dt}\right)^2 dt$$

<u>Work done</u> $= \int_A^B \bar{F}\cdot d\bar{r} = \left[\frac{m}{2}\left(\frac{d\bar{r}}{dt}\right)^2\right]_A^B = \frac{m}{2}v^2\Big]_A^B = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2$
$=$ <u>change in K.E.</u>

Where v_A & v_B are the magnitudes of the velocities of the particle at A & B respectively.

~~If $\bar{F}$ be conservative, then $\bar{F} = -\bar{\nabla}\phi$; ϕ is a scalar.~~

~~$\int_A^B \bar{F}\cdot d\bar{r} = -\int_A^B \bar{\nabla}\phi\cdot d\bar{r} = -\int_A^B d\phi = \phi(A)-\phi(B)$~~

$\therefore$ The result follows whether F is conservative or not.