Study Material - Sem. 1 - C2T

- Special Theory of Relativity -

- Dr. T. Kar - Class 4

Mass - Energy Relation

As a final step in the deductions from The Theory of relativity one can arrive at a very important universal relation between mass and energy as follows :—

As in classical mechanics, energy may be defined in terms of work which is force multiplied by displacement. And force, being the rate of change of momentum, is given by —

$$F = \frac{d(mv)}{dt}$$

But since both mass and velocity are variables in the theory of relativity, then—

$$F = m\frac{dv}{dt} + v\frac{dm}{dt}$$

Supposing the force F acting through a distance 'dx' raises the kinetic energy by dE, then —

$$dE = F\,dx$$

$$= m\frac{dv}{dt}dx + v\frac{dm}{dt}dx$$

Since $v = \frac{dx}{dt}$, $dE = mv\,dv + v^2 dm \longrightarrow (1)$

Now, $m = \frac{m_0}{\sqrt{1-v^2/c^2}}$ or, $m^2\left(1-\frac{v^2}{c^2}\right) = m_0^2$

or, $m^2(c^2-v^2) = m_0 c^2$ or, $m^2c^2 - m^2v^2 = m_0^2c^2 \longrightarrow (2)$

Differentiating eq. (2) we get,

$2mc^2dm - 2mv^2dm - m^2 2v\,dv = 0$ [∵ m_0 & c are constants]

or, $mc^2 dm - mv^2 dm - m^2 v\,dv = 0$

or, $c^2 dm - v^2 dm - mv\,dv = 0$

or, $c^2 dm = mv\,dv + v^2 dm = dE$ [from (2)]

$\therefore dE = c^2 dm \longrightarrow (3)$

When a body is accelerated from rest to a velocity v, its mass increases from m_0 to m and the total kinetic energy acquired is obtained by integrating eq. (3).

$$\int_0^E dE = E = \int_{m_0}^{m} c^2 dm = c^2 (m - m_0)$$

$$\therefore E = (m - m_0) c^2 \longrightarrow (4)$$

Eq. (4) indicates that the K.E. of motion is the actuating influence of the increase in mass. It further throws out the hint that what we have called the rest mass m_0 has to be understood as an internal store of energy in the body. Since the total energy W possessed by a moving body is made up of the kinetic energy of motion and the stored up internal energy,

$$W = E + m_0 c^2 = (m - m_0) c^2 + m_0 c^2$$

$$\text{or, } W = mc^2 \longrightarrow (5)$$

Eq. (5) is the famous Einstein's mass-energy relation which states a universal equivalence between mass and energy, unknown in classical dynamics. Energy can thus manifest itself mass, apart from its various forms like kinetic, potential, thermal, electromagnetic etc. Mass and energy should no longer be considered as two independent entities. When one disappears, an equivalent amount of the other comes into being. This relation is of fundamental importance in nuclear physics in explaining the mass loss and generation of energy in nuclear reaction. Nuclear fission, nuclear fusion etc. are easily explain-ed using the above relation.

Relativistic Formula for K. E.

We have, $E = mc^2 - m_0c^2$

$$= \frac{m_0c^2}{\sqrt{1-v^2/c^2}} - m_0c^2$$

$$= m_0c^2 \left[\frac{1}{\sqrt{1-v^2/c^2}} - 1 \right] \rightarrow (1)$$

At low speeds, however, the relativistic expression reduces to the classical one, as is shown below.

Expanding $1/\sqrt{1-v^2/c^2}$ binomially and keeping only the first two terms, since

$c >> v$, we have,

$$E = m_0c^2\left[1+\frac{v^2}{2c^2}-1\right] = \frac{m_0c^2v^2}{2c^2} = \frac{1}{2}m_0v^2$$ (2)

Important Relativistic Formulae

The following relativistic formulae are particularly useful in nuclear and elementary particle physics.

(a) $W^2 = p^2c^2 + m_0^2c^4$

(b) $p = m_0c\sqrt{\frac{1}{1-\frac{v^2}{c^2}} - 1}$

(c) $1+\frac{E}{m_0c^2} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \sqrt{1+\frac{p^2}{m_0^2c^2}}$

Proof: We have, $m = \frac{m_0}{\sqrt{1-v^2/c^2}}$

Squaring the above equation and multiplying both the sides by $(1-\frac{v^2}{c^2})c^4$, we get,

$$m^2 = \frac{m_0^2}{(1-v^2/c^2)}$$

or, $m^2c^4\left(1-\frac{v^2}{c^2}\right) = m_0^2c^4$

or, $m^2c^4 - m^2v^2c^2 = m_0^2c^4$

or, $W^2 - p^2c^2 = m_0^2c^4$ [since $W = mc^2$ & $p = mv$]

or, $\boxed{W^2 = p^2c^2 + m_0^2c^4} \longrightarrow (1)$

The above relation connecting the total energy and the momentum shows that a particle with no rest mass i.e $m_0 = 0$ can still have momentum given by $p = \frac{W}{c} = mc$ as in the case of a photon or a neutrino.

Now, $m_0c\sqrt{\frac{1}{1-v^2/c^2} - 1} = m_0c\sqrt{\frac{1-1+v^2/c^2}{1-v^2/c^2}}$

$= \frac{m_0c\left(\frac{v}{c}\right)}{\sqrt{1-v^2/c^2}} = \frac{m_0v}{\sqrt{1-v^2/c^2}} = mv = p$

Therefore, $\boxed{p = m_0c\sqrt{\frac{1}{1-\frac{v^2}{c^2}} - 1}} \longrightarrow (2)$

We have, Kinetic energy,

$E = mc^2 - m_0c^2 = \frac{m_0c^2}{\sqrt{1-v^2/c^2}} - m_0c^2$

$= m_0c^2\left[\frac{1}{(1-v^2/c^2)^{1/2}} - 1\right]$

or, $\frac{E}{m_0c^2} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1$

or, $\boxed{1 + \frac{E}{m_0c^2} = \frac{1}{\sqrt{1-v^2/c^2}}} \longrightarrow (3)$

From (2), $\frac{p}{m_0c} = \sqrt{\frac{1}{1-v^2/c^2} - 1}$ or, $\frac{p^2}{m_0^2c^2} = \frac{1}{1-\frac{v^2}{c^2}} - 1$

or, $1 + \frac{p^2}{m_0^2 c^2} = \frac{1}{1 - \frac{v^2}{c^2}}$

or, $$\boxed{\frac{1}{\sqrt{1 - v^2/c^2}} = \sqrt{1 + \frac{p^2}{m_0^2 c^2}}} \longrightarrow (4)$$

∴ Combining (3) & (4), we get,

$$1 + \frac{E}{m_0 c^2} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \sqrt{1 + \frac{p^2}{m_0^2 c^2}}$$

Example 5. Atomic particles in the form of a beam have a velocity of 92% of the speed of light. What is their relativistic mass compared to their rest mass?

Ans. We have $m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$

Here $v = \frac{92}{100} c = 0.92c$

$$\therefore \frac{m}{m_0} = \frac{1}{\sqrt{1 - \frac{(0.92)^2 c^2}{c^2}}} = \frac{1}{\sqrt{1 - (0.92)^2}} = 2.55$$

Example 6. At what speed will an electron move in order to double its rest mass? (Given $c = 3 \times 10^{10}$ m/sec)

Ans. We have $m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$

Here, $\frac{m}{m_0} = 2$, $c = 3\times10^{10}$ cm/sec

$\therefore \frac{m}{m_0} = \frac{1}{\sqrt{1-v^2/c^2}}$ or, $2 = \frac{1}{\sqrt{1-v^2/c^2}}$

or, $4 = \frac{1}{1-v^2/c^2}$ or, $1-\frac{v^2}{c^2} = \frac{1}{4}$ or, $1-\frac{1}{4} = \frac{v^2}{c^2}$

or, $\frac{v^2}{c^2} = \frac{3}{4}$ or, $\frac{v}{c} = \frac{\sqrt{3}}{2}$ or $v = \frac{1.732}{2} \times 3\times10^{10}$ cm/sec

$= 2.598\times10^{10}$ cm/sec

Example 7. A particle of rest mass m_0 moves with a speed $c/\sqrt{2}$. Calculate its mass, momentum, total energy and kinetic energy.

Ans. Relativistic mass $m = \frac{m_0}{\sqrt{1-v^2/c^2}} = \frac{m_0}{\sqrt{1-\frac{c^2}{2c^2}}}$

$= \frac{m_0}{\sqrt{1-\frac{1}{2}}} = \sqrt{2}\,m_0$

$= 1.414\,m_0$

Momentum $= p = mv = \frac{m_0}{\sqrt{1-v^2/c^2}}v = \sqrt{2}\,m_0\frac{c}{\sqrt{2}} = m_0c$

Total energy $= mc^2 = \sqrt{2}\,m_0c^2 = W$

Kinetic energy $= E = W - m_0c^2$

$= \sqrt{2}\,m_0c^2 - m_0c^2$

$= (1.414-1)\,m_0c^2 = 0.414\,m_0c^2$

Example 8. Total energy of a particle is exactly twice its rest energy. Find the value of its speed.

Ans. By the problem, $mc^2 = 2m_0c^2$ $\therefore m = 2m_0$

or, $\dfrac{m_0}{\sqrt{1-v^2/c^2}} = 2m_0$ or, $1-\dfrac{v^2}{c^2} = 1/4$ or, $v^2 = \dfrac{3}{4}c^2$

or, $v = \dfrac{\sqrt{3}}{2}c = 0.866c$

Example 9. A body whose specific heat is 0.2 is heated through 100°C. Find the percentage increase in mass. Given: $c = 3\times10^{10}$ cm/sec.

Ans. Let m gm be the mass of the body. The heat energy absorbed by it is given by —

$$\Delta E = mst = m\times0.2\times100 \text{ Calories}$$
$$= m\times0.2\times100\times4.2\times10^7 \text{ ergs}$$
$$= 84m\times10^7 \text{ ergs}$$

∴ Increase in its mass $\Delta m = \dfrac{\Delta E}{c^2} = \dfrac{84m\times10^7}{(3\times10^{10})^2}$

$$= 9.33\times10^{-13}m$$

∴ $\dfrac{\Delta m}{m} = 9.33\times10^{-13}$

∴ Percentage increase in mass $= \dfrac{\Delta m}{m}\times100 = 9.33\times10^{-11}$

Example 10. The speed of an electron in an uniform electric field changes from 0.95c to 0.98c. Compute the changes in mass and the work done on the electron to change the velocity.

Ans. The masses m_1 and m_2 of the electron corresponding to velocities $0.95c$ and $0.98c$. ~~compute the change in mass~~ respectively are,

$$m_1 = \frac{m_0}{\sqrt{1-(0.95)^2}} = 3.2\, m_0 \;; \quad m_2 = \frac{m_0}{\sqrt{1-(0.98)^2}} = 5.0\, m_0$$

where m_0 is the rest mass of electron $= 9.1 \times 10^{-31}$ kg

So the change in mass $\Delta m = m_2 - m_1$

$$= (5 - 3.2)\, m_0$$
$$= 1.8 \times 9.1 \times 10^{-31} \text{ kg}$$
$$= 16.4 \times 10^{-31} \text{ kg}$$

The work done on the electron is the change in its kinetic energy and is given by —

$$\Delta E = (\Delta m) c^2 = 1.8\, m_0 c^2 = 1.8 \times 9.1 \times 10^{-31} \times (3 \times 10^8)^2 = 0.92 \text{ MeV.}$$

Four-Dimensional Space

In classical physics, the time coordinate is left unaffected upon transformation from one inertial frame of reference to another. In relativistic mechanics, however, the time coordinate in one inertial frame depends on the time and the space coordinates of another inertial frame. Therefore, in relativity, the time t and the space coordinate x, y, z are treated together in what is known as a space-

-time geometry or a four-dimensional space. It is convenient to transform the dimension of the time coordinate to that of the space coordinate. This is accomplished by multiplying the time t by the universal constant c, the speed of light in free space and recognise the variable $\omega = ct$ as the fourth coordinate in place of t. The kinematic state of a particle can be expressed by a four-vector whose components are (x, y, z, ω). The quantity $x^2+y^2+z^2-\omega^2$ is invariant under the Lorentz transformation as shown below. Putting $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$, we obtain

$$x'^2+y'^2+z'^2-\omega'^2 = x'^2+y'^2+z'^2-c^2t'^2$$

$$= \gamma^2(x-vt)^2+y^2+z^2-c^2\gamma^2\left(t-\frac{vx}{c^2}\right)^2$$

$$= \gamma^2(x^2-2xvt+v^2t^2)+y^2+z^2-c^2\gamma^2\left(t^2-\frac{2vxt}{c^2}+\frac{v^2x^2}{c^4}\right)$$

$$= \gamma^2x^2-\cancel{2\gamma^2xvt}+\gamma^2v^2t^2+y^2+z^2-c^2\gamma^2t^2+\cancel{2\gamma^2vxt}-\frac{v^2x^2\gamma^2}{c^2}$$

$$= \left(\gamma^2x^2-\frac{v^2x^2\gamma^2}{c^2}\right)+y^2+z^2-\gamma^2t^2(c^2-v^2)$$

$$= \gamma^2x^2\left(1-\frac{v^2}{c^2}\right)+y^2+z^2-\gamma^2t^2c^2\left(1-\frac{v^2}{c^2}\right)$$

$$= \frac{x^2(1-v^2/c^2)}{(1-v^2/c^2)}+y^2+z^2-\frac{c^2t^2\left(1-\frac{v^2}{c^2}\right)}{(1-v^2/c^2)}$$

$$= x^2+y^2+z^2-c^2t^2 = x^2+y^2+z^2-\omega^2$$

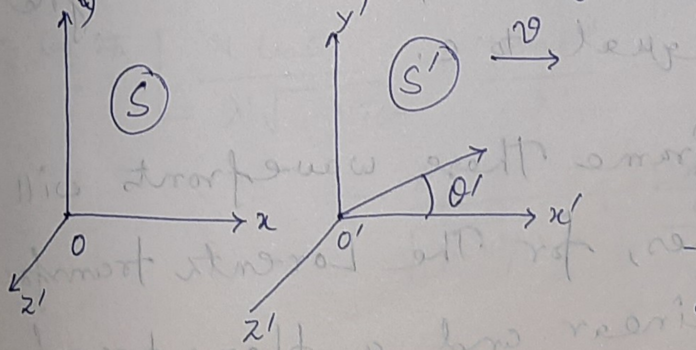
A four-vector is defined to be any ordered set of four real numbers (a_1, a_2, a_3, b) which under the Lorentz transformation leaves the quantity $a_1^2 + a_2^2 + a_3^2 - b^2$ invariant.

A four-vector momentum is one whose components are p_x, p_y, p_z and $\frac{E}{c}$. The fourth component is actually the relativistic energy E divided by c to make its dimensions as the same as that of the momentum. In S-frame,

$$p_x = \frac{m_0 v_x}{\sqrt{1-u^2/c^2}}, \quad p_y = \frac{m_0 v_y}{\sqrt{1-u^2/c^2}}, \quad p_z = \frac{m_0 v_z}{\sqrt{1-u^2/c^2}}$$

and $E = \dfrac{m_0 c^2}{\sqrt{1-u^2/c^2}} \Rightarrow p_x^2 + p_y^2 + p_z^2 - \dfrac{E^2}{c^2}$ = invariant.

Doppler Effect in Light (in Relativity)



Let us consider a train of plane monochromatic light waves of unit amplitude emitted from a source of light at the origin of the S'-frame. The rays or wave-normals are in the $x'y'$ plane and make an angle θ' with the x'-axis. An expression describing the propagation would be of the form —

$$\cos 2\pi \left[\frac{x'\cos\theta' + y'\sin\theta'}{\lambda'} - \nu' t'\right] \longrightarrow ①$$

for this is a single periodic function, amplitude unity, representing a wave moving with velocity $\lambda'\nu' (= c)$ in the θ'-direction. Notice, for example, that for $\theta' = 0$ it reduces to $\cos 2\pi \left[\frac{x'}{\lambda'} - \nu' t'\right]$ and for $\theta' = \frac{\pi}{2}$ it reduces to $\cos 2\pi \left[\frac{y'}{\lambda'} - \nu' t'\right]$, well known expression for propagation along the positive-x' and positive-y' directions respectively, of waves of frequency ν' and wavelength λ'. The alternate forms, $\cos \frac{2\pi}{\lambda'}[x' - \lambda'\nu' t']$ and $\cos \frac{2\pi}{\lambda'}[y' - \lambda'\nu' t']$ show that the wave speed is $\lambda'\nu'$ which, for electromagnetic waves, is equal to c.

In the S-frame these wavefronts will still be planes, for the Lorentz transformation is linear and a plane transforms into a plane. Hence in the unprimed or S-frame the expression describing the propagation will have the same form:

$$\cos 2\pi \left[\frac{x\cos\theta + y\sin\theta}{\lambda} - \nu t\right] \longrightarrow ②$$

Here, λ and ν are the wavelength and frequency, respectively, measured in the S-frame, and θ is the angle a ray makes with the x-axis. Now, $\lambda\nu = \lambda'\nu' = c$ because c is the velocity of electromagnetic waves, the same for each observer.

Now, let us apply the Lorentz transformation equations directly to eq. (1), putting,

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \quad \text{and} \quad t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - v^2/c^2}},$$ we obtain

$$\cos 2\pi \left[\frac{\frac{x - vt}{\sqrt{1-v^2/c^2}} \cos\theta' + y\sin\theta'}{\lambda'} - \frac{\nu'\left(t - \frac{vx}{c^2}\right)}{\sqrt{1 - v^2/c^2}} \right]$$

$$= \cos 2\pi \left[\frac{(x - vt)}{\lambda'\sqrt{1-v^2/c^2}} \cos\theta' + \frac{y\sin\theta'}{\lambda'} - \frac{\nu'\left(t - \frac{vx}{c^2}\right)}{\sqrt{1-v^2/c^2}} \right]$$

$$= \cos 2\pi \left[\frac{x\cos\theta'}{\lambda'\sqrt{1-v^2/c^2}} + \frac{y\sin\theta'}{\lambda'} - \frac{vt\cos\theta'}{\lambda'\sqrt{1-v^2/c^2}} - \frac{\nu' t}{\sqrt{1-v^2/c^2}} + \frac{\frac{vx\nu'}{c^2}}{\sqrt{1-v^2/c^2}} \right]$$

$$= \cos 2\pi \left[\frac{x\cos\theta'}{\lambda'\sqrt{1-v^2/c^2}} + \frac{y\sin\theta'}{\lambda'} - \frac{vt\cos\theta'\left(\frac{\nu'}{c}\right)}{\sqrt{1-v^2/c^2}} - \frac{\nu' t}{\sqrt{1-v^2/c^2}} + \frac{\frac{vxc}{\lambda' c^2}}{\sqrt{1-v^2/c^2}} \right]$$

$$= \cos 2\pi \left[\frac{x \cos\theta'}{\lambda'\sqrt{1-v^2/c^2}} + \frac{y \sin\theta'}{\lambda'} - \frac{\{(\frac{v}{c})\cos\theta'\}\nu' t}{\sqrt{1-v^2/c^2}} \right.$$

$$\left. - \frac{\nu' t}{\sqrt{1-v^2/c^2}} + \frac{(\frac{v}{c})x}{\lambda'\sqrt{1-v^2/c^2}} \right]$$

$$= \cos 2\pi \left[\frac{(\cos\theta' + \frac{v}{c})x}{\lambda'\sqrt{1-v^2/c^2}} + \frac{y \sin\theta'}{\lambda'} - \frac{\{(\frac{v}{c})\cos\theta' + 1\}\nu' t}{\sqrt{1-v^2/c^2}} \right] \longrightarrow (3)$$

As expected, This has The form of a plane wave in The S-frame and must be identical to eq. (2), which represents The same Thing. Hence, The coefficients of x, y, & t in each equation must be equated, giving us,

$$\frac{\cos\theta}{\lambda} = \frac{\cos\theta' + \frac{v}{c}}{\lambda'\sqrt{1-v^2/c^2}} \longrightarrow (4)$$

$$\frac{\sin\theta}{\lambda} = \frac{\sin\theta'}{\lambda'} \longrightarrow (5)$$

$$\nu = \frac{\nu'(1 + \frac{v}{c}\cos\theta')}{\sqrt{1-v^2/c^2}} \longrightarrow (6)$$

Eq. (6) gives The relativistic equation for The Doppler effect in light.

We can also write eq. (6) inversely as —

$$\nu' = \frac{\nu\left(1 - \frac{v}{c}\cos\theta\right)}{\sqrt{1 - v^2/c^2}} \longrightarrow (7)$$

Let us first check that the relativistic formula reduces to the classical one. That is, for $v \ll c$, we can neglect terms higher that first order in $\frac{v}{c}$. From eq. (7), we get,

$$\nu\left(1 - \frac{v}{c}\cos\theta\right) = \nu'\sqrt{1 - v^2/c^2}$$

$$\therefore \nu = \frac{\nu'\sqrt{1 - v^2/c^2}}{1 - \left(\frac{v}{c}\right)\cos\theta} \simeq \nu'\left[1 + \left(\frac{v}{c}\right)\cos\theta\right] \longrightarrow (8)$$

which is the classical result. This becomes clear on consideration of the more familiar special cases —

Case I. If $\theta = 0°$ ⇒ The observer in S frame sees the source moves towards him or he is moving towards the source.

Then, $\nu = \nu'\left(1 + \frac{v}{c}\right)$ ⇒ observed frequency ν is greater than the proper frequency ν'!

Case II. If $\theta = 180°$ ⇒ The observer in S-frame sees the source moves away from him or he is moving away from the source. Then,

$\nu = \nu'(1-\frac{v}{c}) \Rightarrow$ observed frequency ν is less than the proper frequency ν'.

Case IV: If $\theta = 90° \Rightarrow$ The line of observation is perpendicular to the relative motion.

Then there is no Doppler effect classically i.e, $\nu = \nu'$. All these first order results are classical effects.

Now if v is not small compared to c, we get the relativistic effects. It is convenient to think of these effects separately as a longitudinal one and a transverse one. Thus for the longitudinal Doppler effect in relativity, we have from eq. 7,

$$\nu' = \frac{\nu(1-\frac{v}{c}\cos\theta)}{\sqrt{1-v^2/c^2}}$$

or, $$\nu = \frac{\nu'\sqrt{1-v^2/c^2}}{1-(\frac{v}{c})\cos\theta}$$

Case I. If $\theta = 0 \Rightarrow$ The source and The observer are moving towards one another.

$$\nu = \frac{\nu' \sqrt{1-u^2/c^2}}{\left(1-\frac{u}{c}\right)} = \nu' \frac{\sqrt{(1)^2-(u/c)^2}}{\sqrt{1-u/c}\,\sqrt{1-u/c}}$$

$$= \frac{\nu' \sqrt{\left(1+\frac{u}{c}\right)(1-u/c)}}{\sqrt{(1-u/c)}\,\sqrt{1-u/c}} = \nu' \sqrt{\frac{1+u/c}{1-u/c}}$$

$$= \nu' \sqrt{\frac{c+u}{c-u}} \longrightarrow (9)$$

Case II. $\theta = 180^\circ \Rightarrow$ The source and The observer are moving away from one another.

$$\nu = \frac{\nu' \sqrt{1-u^2/c^2}}{\left(1+\frac{u}{c}\right)} = \nu' \frac{\sqrt{\left(1+\frac{u}{c}\right)(1-u/c)}}{\sqrt{1+u/c}\,\sqrt{1+u/c}}$$

$$= \nu' \sqrt{\frac{1-u/c}{1+u/c}} = \nu' \sqrt{\frac{c-u}{c+u}} \longrightarrow (10)$$

Case. III.

More striking, however, is The fact That The relativistic formula predicts a transverse Doppler effect, an effect That is purely relativistic, for There is no transverse Doppler effect in classical physics at all. If we set $\theta = 90^\circ$, Then

$$\nu = \nu' \sqrt{1 - v^2/c^2}.$$

If our line of sight is 90° to the relative motion, then we should observe a frequency ν which is lower than the proper frequency ν' of the source.