Chapter 1

Partial Differential Equations

Relevant Information on

1. Definition of P.D.E.—Order, Degree, Linearity.
2. Construction of a P.D.E. by Elimination of Arbitrary Constants.
3. Construction of a P.D.E. by Elimination of Arbitrary Functions.

1.1 Definitions

Partial Differential Equations (P.D.E.) are those equations which contain one or more partial derivatives and hence they must involve at least two independent variables.

The **order** of a P.D.E. is the order of the derivative of the highest order in the equation.

The **degree** of a P.D.E. is the greatest exponent of the highest order derivative involved in the equation.

Examples. [In what follows we shall (in almost all cases) take z as the dependent variable and x, y as independent variables. Further, the following symbols are universally used:

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s, \quad \frac{\partial^2 z}{\partial y^2} = t]$$

(i) $xp + yq = z$, i.e., $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$,

is an example of a Partial Differential Equation of first order and of *first degree.*

(ii) $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$ or $r + 3s + t = 0$;

is an example of a Partial Differential Equation (P.D.E.) of *second order* and of *first degree.*

A P.D.E. is called a *Linear Partial Differential Equation* if all the derivatives in it are of the first degree.

The equations

$$\left.\begin{array}{rcl} px + qy & = & z \\ \text{and} \quad px^2 + qy^2 & = & z^3 \end{array}\right\} \text{ are both linear}$$

since p and q are of first degree.

[See that in a linear p.d.e. the dependent variable z may be of higher degree, contrary to what we have seen in ordinary differential equations. Recall that $\frac{dy}{dx} + y^3 = x^2$ is a first order, first degree but not linear ordinary differential equation (since the dependent variable y is of degree 3).]

All partial differential equations of first order *but not linear* are called **Non-linear** Partial Differential Equations of first order; e.g., $p^2 + q^2 = 1$ is a non-linear p.d.e. of first order. What can you say about $p + \log q = z^2$?

A few more Examples of P.D.E.

(iii) $xp + yq = z \to$ p.d.e. of *first order, linear.*

(iv) $r + t = 0 \to$ p.d.e. of *second order, linear.*

(v) $r + 5s + t = 0 \to$ p.d.e. of *second order, linear.*

(vi) $3px + qy = z^2 \to$ p.d.e. of *first order, linear.*

We acquaint our readers with some important P.D.E. used in **Theoretical Physics:**

(vii) **Laplace's Equation:** $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$, written as $\nabla^2 w = 0$ where ∇^2 **(Laplacian)** is an operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, is an example of a p.d.e. of *second order, linear.*

(viii) **Heat Equation:** $\nabla^2 w = \frac{1}{\sigma}\frac{\partial w}{\partial t}$ is an example of a P.D.E. of *second order, linear.*

(ix) **Wave Equation:** $\nabla^2 w = \frac{1}{a^2}\frac{\partial^2 w}{\partial t^2}$ (second order, linear).

[In (viii), (ix), w is the dependent variable and x, y, z, t are four independent variables; t is the time variable, (x, y, z) are the rectangular Cartesian coordinates of a point in space.]

1.2 Construction of Partial Differential Equations by the Process of Elimination of Arbitrary Constants

Let $\phi(x, y, z, a, b) = 0$ be a relation between three variables x, y, z and two arbitrary constants a, b. As usual, z is the dependent variable and x, y two independent variables. In order to eliminate a, b we require two other equations besides the given relation $\phi(x, y, z, a, b) = 0$.

Differentiating the given relation $\phi = 0$ with respect to x we obtain

$$\phi_x + \phi_z \frac{\partial z}{\partial x} = 0. \qquad (1.2.1)$$

Differentiating the given relation $\phi = 0$ with respect to y, we obtain

$$\phi_y + \phi_z \frac{\partial z}{\partial y} = 0. \qquad (1.2.2)$$

Using the given relation $\phi = 0$, (1.2.1), (1.2.2) we can eliminate a, b and obtain

$$F(x, y, z, p, q) = 0, \quad \text{where } p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y}.$$

This is a partial differential equation of **order one.**

Remember. If the number of arbitrary constants = the number of independent variables, then elimination of arbitrary constants will give rise to a p.d.e. of first order.

▶ **Example 1.2.1** *Eliminate a, b from the relation* $z = ax^2 + by^2 + ab$.

Solution: Differentiating first with respect to x and then with respect to y, we get

$$\frac{\partial z}{\partial x} = 2ax; \quad \frac{\partial z}{\partial y} = 2by$$

$$\text{i.e.,} \quad a = \frac{p}{2x}; \quad b = \frac{q}{2y}.$$

Substituting these values in the given relation we obtain

$$z = \frac{p}{2x}x^2 + \frac{q}{2y}y^2 + \frac{pq}{4xy}.$$

Simplifying, $4xyz = 2px^2y + 2qy^2x + pq$, which is the required eliminate, a p.d.e. of first order.

II **Remember.** If the number of arbitrary constants is less than the number of independent variables, then elimination of the arbitrary constants will give rise two distinct partial differential equations of first order, **e.g.**

▶ **Example 1.2.2** *Eliminate the arbitrary constant a from the given relation* $z = a(x + y)$.

Solution: As before,

$$\text{Diff. w.r.t. } x \to \frac{\partial z}{\partial x} = a, \text{ i.e., } p = a$$

$$\text{Diff. w.r.t. } y \to \frac{\partial z}{\partial y} = a, \text{ i.e., } q = a.$$

We thus get two distinct p.d.e. of order one

$$z = p(x + y) \text{ and } z = q(x + y).$$

These two equations are the required eliminant.

III **Remember.** If the number of arbitrary constants is more than the number of independent variables, then on elimination of the constants, a p.d.e (or equations) of order more than one can be obtained, **e.g.**

▶ **Example 1.2.3** *Eliminate the three arbitrary constants a, b, c from the relation* $z = ax + by + cxy$.

Solution: Differentiating the given relation:

$$\left.\begin{array}{l}\text{with respect to } x, \text{ we get } p = a + cy \\ \text{with respect to } y, \text{ we get } q = b + cx\end{array}\right\}$$

With these two equations we cannot eliminate all the three constants a, b, c.

So, we differentiate $p = a + cy$ with respect to x and obtain

$$\frac{\partial p}{\partial x} = \frac{\partial^2 z}{\partial x^2} = 0, \quad \text{i.e., } r = 0.$$

Similarly we obtain,

$$\frac{\partial q}{\partial y} = \frac{\partial^2 z}{\partial y^2} = 0, \quad \text{i.e., } t = 0$$

$$\text{and} \quad \frac{\partial p}{\partial y} = c, \quad \text{i.e., } \frac{\partial^2 z}{\partial y \partial x} = s = c.$$

Thus we have

$$p = a + sy \quad \text{and} \quad q = b + sx$$
$$\text{i.e., } a = p - sy \quad \text{and} \quad b = q - sx.$$

Finally from the given relation $z = ax + by + cxy$, it follows

$$z = (p - sy)x + (q - sx)y + sxy$$
$$\text{or, } z = px + qy - sxy.$$

∴ *Required eliminant is*

$$r = 0, \ t = 0 \text{ and } z = px + qy - sxy.$$

▶ **Example 1.2.4** *Suppose z depends on x, y, t. Eliminate three arbitrary constants a, b, c from*

$$z = a(x + y) + b(x - y) + abt + c.$$

Solution: $\frac{\partial z}{\partial x} = a + b, \frac{\partial z}{\partial y} = a - b, \frac{\partial z}{\partial t} = ab.$

Since $(a + b)^2 = (a - b)^2 + 4ab$. Hence,

$$\left(\frac{\partial z}{\partial x}\right)^2 = \left(\frac{\partial z}{\partial y}\right)^2 + 4\frac{\partial z}{\partial t}.$$

The required eliminant is a partial differential equation of first order (but of degree two).

1.3 Construction of Partial Differential Equations by the Process of Elimination of Arbitrary Functions

Suppose that u and v are two functions of x, y, z and suppose that there is a relation between u and v expressed

$$\left.\begin{array}{ll} \textit{either} & \text{as } \phi(u,v)=0 \\ \textit{or} & \text{as } u=f(v) \end{array}\right\} \quad \phi \text{ and } f \text{ are arbitrary.} \qquad (1.3.1)$$

We shall see that a partial differential equation is obtained on elimination of ϕ (or f). In fact we shall obtain a **linear p.d.e.** (i.e., first degree in p and q).

Differentiating (1.3.1) w.r.t. x and then w.r.t. y we get

$$\frac{\partial \phi}{\partial u}\left[\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right]+\frac{\partial \phi}{\partial v}\left[\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right]$$

or,
$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0.$$

Similarly,

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=0.$$

Eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ we easily get

$$\frac{\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}}=\frac{\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}}$$

or,
$$\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)-\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0$$

or,
$$p\left[\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right]+q\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}\right]=\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}.$$

or,
$$pP+qQ=R \qquad (1.3.2)$$

where $P=\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}$, etc.

Thus from (1.3.1) involving an arbitrary function ϕ we obtain a partial differential equation (1.3.2). This P.D.E. does not contain ϕ and it is linear in p and q.

Remember. When the relation between x, y, z contains two arbitrary functions, the partial differential equation derived therefrom will, in general, give a partial differential equation involving partial derivatives of order higher than the second.

▶ **Example 1.3.1** *Eliminate the arbitrary function ϕ from the relation:*

$$z=\phi\left(\frac{y}{x}\right).$$

Solution: Taking partial derivative first w.r.t. x and then w.r.t. y we get

$$p = \frac{\partial z}{\partial x}=\phi'\left(\frac{y}{x}\right)\times\left(-\frac{y}{x^2}\right)$$

$$q = \frac{\partial z}{\partial y}=\phi'\left(\frac{y}{x}\right)\times\left(\frac{1}{x}\right)$$

whence it follows: $\frac{p}{q}=\frac{-y}{x^2}/\frac{1}{x}$ **or,** $xp+yq=0$ **(first order p.d.e.).**

▶ **Example 1.3.2** *Eliminate the arbitrary functions f and ϕ from:* $y=f(x-at)+\phi(x+at)$.

Solution: Here y depends on x and t. So we take partial derivatives of y w.r.t. x and t.

Thus

$$\frac{\partial y}{\partial x} = f'(x-at)+\phi'(x+at)$$

$$\frac{\partial^2 y}{\partial x^2} = f''(x-at)+\phi''(x+at)$$

and
$$\frac{\partial y}{\partial t} = f'(x-at)\{-a\}+\phi'(x+at)\{a\}$$

$$\frac{\partial^2 y}{\partial t^2} = a^2 f''(x-at)+a^2\phi''(x+at)$$

whence, it easily follows:

$$\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2} \quad \textbf{(a p.d.e. of second order).}$$

Remember. $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ is a very useful p.d.e. of second order occurring in the study of a stretched string. It can be easily said that $y = f(x - at) + \phi(x + at)$ is the general solution of the second order p.d.e. $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$.

▶ **Example 1.3.3** *Find the partial differential equation arising from* $\phi\left(\frac{z}{x^3}, \frac{y}{x}\right) = 0$, *where* ϕ *is an arbitrary function of its arguments.*

Solution: Let us write the given relation in the form

$$\phi(u, v) = 0 \quad \left[u = \frac{z}{x^3}, v = \frac{y}{x}\right]$$

Differentiating with respect to x and y, we write

$$\frac{\partial \phi}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v} \cdot \frac{\partial v}{\partial x} = 0$$

$$\text{or,} \quad \frac{\partial \phi}{\partial u} \cdot \left\{-\frac{3z}{x^4} + \frac{1}{x^3} \cdot \frac{\partial z}{\partial x}\right\} + \frac{\partial \phi}{\partial v}\left\{-\frac{y}{x^2}\right\} = 0. \qquad (1)$$

Also,

$$\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v} \cdot \frac{\partial v}{\partial y} = 0$$

$$\text{or,} \quad \frac{\partial \phi}{\partial u} \cdot \left\{\frac{1}{x^3}\frac{\partial z}{\partial y}\right\} + \frac{\partial \phi}{\partial v}\left(\frac{1}{x}\right) = 0. \qquad (2)$$

Eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ from (1) and (2) we get

$$\begin{vmatrix} \frac{p}{x^3} - \frac{3z}{x^4} & -\frac{y}{x^2} \\ \frac{q}{x^3} & \frac{1}{x} \end{vmatrix} = 0 \quad \left[\text{writing } p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}\right]$$

$$\text{or,} \quad \frac{p}{x^4} - \frac{3z}{x^5} + \frac{qy}{x^5} = 0 \quad \text{or,} \quad px + qy = 3z. \quad \textbf{(Reqd. p.d.e.)}$$

Note 1.3.1 *The given relation may not be written as* $\phi\left(\frac{z}{x^3}, \frac{y}{x}\right) = 0$ *but as* $\frac{z}{x^3} = f\left(\frac{y}{x}\right)$ *i.e.,* $z = x^3 f\left(\frac{y}{x}\right)$, *where* f *is an arbitrary function of its argument.*

Let $v = \frac{y}{x}$. Then $z = x^3 f(v)$. Differentiation with respect to x and y yields

$$\begin{aligned} p &= 3x^2 f(v) + x^3 \frac{df}{dv} \cdot \frac{\partial v}{\partial x} \\ &= 3x^2 f(v) + x^3 \frac{df}{dv} \cdot \left(-\frac{y}{x^2}\right) \\ &= 3x^2 f(v) - xyf'(v) \quad \left[\text{writing } \frac{df}{dv} = f'(v)\right] \end{aligned}$$

$$\text{and} \quad q = x^3 f'(v)\frac{\partial v}{\partial y} = x^3 f'(v) \cdot \frac{1}{x} = x^2 f'(v).$$

Eliminating $f'(v)$ we easily obtain

$$p = 3x^2 f(v) - xy\left(\frac{q}{x^2}\right)$$

$$\text{or,} \quad xp + yq = 3x^3 f(v) = 3z \quad [\because \ z = x^3 f(v)]$$

i.e., the required p.d.e.is, as before

$$\mathbf{px + qy = 3z.}$$

▶ **Example 1.3.4** *Eliminate the arbitrary function* ϕ *from*

$$z = e^{my}\phi(x - y).$$

Solution: Differentiating z w.r.t. x and y, we obtain

$$p = \frac{\partial z}{\partial x} = e^{my}\phi'(x - y) \cdot 1$$

$$q = \frac{\partial z}{\partial y} = me^{my}\phi(x - y) + e^{my}\phi'(x - y) \cdot \{-1\}.$$

Hence $q = mz - p$, or, $p + q = mz$. **(Reqd. Eliminant)**

1.4 Further Solved Problems

▶ **Example 1.4.1** *Eliminate arbitrary constants* a *and* b *from* [I.A.S. 1997]

$$z = (x^2 + a)(y^2 + b).$$

Solution: Differentiating z w.r.t. x and y, we obtain

$$p = 2x(y^2 + b)$$
$$q = 2y(x^2 + a)$$

$$\therefore \quad pq = 4xy(x^2 + a)(y^2 + b) = 4xyz. \quad \textbf{(Reqd. Eliminant)}$$

$$\text{or,}\quad \frac{1}{(z-c)^2} - \frac{pq(x-a)(y-b)}{(z-c)^4} - \frac{q(y-b)}{(z-c)^3} - \frac{p(x-a)}{(z-c)^3} + \frac{pq(x-a)(y-b)}{(z-c)^4} = 0$$

$$\text{or,}\quad \frac{1}{(z-c)^2} - \frac{1}{(z-c)^3}\{p(x-a)+q(y-b)\} = 0$$

or, $p(x-a)+q(y-b) = z-c$ **is the required p.d.e.**

▶ **Example 1.4.5** *Eliminating the arbitrary functions $f(x)$ and $g(y)$ from $z = yf(x)+xg(y)$ obtain the differential equation*

$$xys = px+qy-z$$

where

$$p = \frac{\partial z}{\partial x},\quad q = \frac{\partial z}{\partial y},\quad \textit{and}\quad s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}.$$

Solution: Differentiating $z = yf(x)+xg(y)$ w.r.t. x and y, we get

$$\left.\begin{aligned} p &= yf'(x)+g(y) \\ q &= f(x)+xg'(y) \end{aligned}\right\} \qquad (1)$$

In order to eliminate f, g, f', g' we find the second order partial derivatives

$$\left.\begin{aligned} r &= \frac{\partial^2 z}{\partial x^2} = \frac{\partial p}{\partial x} = yf''(x) \\ t &= \frac{\partial^2 z}{\partial y^2} = \frac{\partial q}{\partial y} = xg''(y) \\ \text{and}\quad s &= \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial q}{\partial x} = f'(x)+g'(y) \end{aligned}\right\} \qquad (2)$$

From (1) we have $f'(x) = \frac{p-g(y)}{y}$ and $g'(y) = \frac{q-f(x)}{x}$.

∴ From (2) it follows:

$$\begin{aligned} s &= f'(x)+g'(y) = \frac{p-g(y)}{y} + \frac{q-f(x)}{x} \\ \text{or,}\quad xys &= px - xg(y) + qy - yf(x) \\ &= px+qy-[yf(x)+xg(y)] \\ &= px+qy-z. \quad \textbf{(Proved)} \end{aligned}$$

▶ **Example 1.4.6** *Form a partial differential equation by eliminating the arbitrary functions f and ϕ from*

$$z = f(x+iy)+\phi(x-iy) \textit{ where } i^2 = -1.$$

Solution:

$$\begin{aligned} \frac{\partial z}{\partial x} &= f'(x+iy)+\phi'(x-iy) \\ \frac{\partial^2 z}{\partial x^2} &= f''(x+iy)+\phi''(x-iy) \\ \frac{\partial z}{\partial y} &= if'(x+iy)-i\phi'(x-iy) \\ \frac{\partial^2 z}{\partial y^2} &= i^2 f''(x+iy)+i^2\phi''(x-iy) \\ &= -f''(x+iy)-\phi''(x-iy) \end{aligned}$$

whence if follows, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$, or, $\nabla^2 z = 0$. **(Reqd. p.d.e.)**

Examples I

[**Standard Symbols:** For **Partial Differential Equation** one may write **p.d.e**; with respect to ≡ w.r.t.

$$p = \tfrac{\partial z}{\partial x},\quad q = \tfrac{\partial z}{\partial y},\quad r = \tfrac{\partial^2 z}{\partial x^2},\quad s = \tfrac{\partial^2 z}{\partial x \partial y},\quad t = \tfrac{\partial^2 z}{\partial y^2}]$$

1. Eliminate the arbitrary constants a, b and obtain p.d.e.:

   (a) $z = (x+a)(y+b)$;

   (b) $z = ax + a^2y^2 + b$;

   (c) $z = a(x+y)+b$

2. Eliminating the arbitrary constants a, b, obtain from $z = (x-a)^2 + (y-b)^2$ the p.d.e. $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4z$.

3. Eliminate the arbitrary constants a and b from $z = ax+by+a^2+b^2$ and obtain the p.d.e. $z = px+qy+p^2+q^2$.

4. Eliminate the constants k and A to obtain a second order partial differential equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$$

from the given relation $z = Ae^{kt}\sin kx$.

5. Find the differential equation of the set of all right circular cones whose axes coincide with z-axis. [I.A.S. 1998]

Remember. The general equation of the set of all right circular cones whose axes coincide with z-axis having semi-vertical angle α and vertex $(0,0,c)$ is given by $x^2 + y^2 = (z-c)^2 \tan^2\alpha$, c and α are arbitrary constants.

6. Eliminate a, b, c from $z = a(x+y) + b(x-y) + abt + c$

7. Show that the differential equation of all cones which have their vertices at $(0,0,0)$ is $px + qy = z$.

Verify that the surface $yz + zx + xy = 0$ satisfies the p.d.e. [I.A.S. 1979]

Remember. The equation of any cone with the vertex at the origin is $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$.

8. By eliminating three arbitrary constants a, b, c from the family of ellipsoids $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, obtain

$$zx\frac{\partial^2 z}{\partial x^2} + x\left(\frac{\partial z}{\partial x}\right)^2 - z\frac{\partial z}{\partial x} = 0,$$
$$zy\frac{\partial^2 z}{\partial y^2} + y\left(\frac{\partial z}{\partial y}\right)^2 - z\frac{\partial z}{\partial y} = 0,$$
$$\text{or,}\quad zs + pq = 0.$$

9. Eliminate the arbitrary constants a, b, c from

$$ax^2 + by^2 + cz^2 = 1$$

and obtain $pq + zs = 0$.

10. Obtain a p.d.e. of the family of spheres of radius 3 with centres on the plane $y = x$.

11. Form a p.d.e. by eliminating the arbitrary function ϕ from $\phi(x+y+z,\ x^2+y^2-z^2) = 0$. What is the order of this p.d.e.?

12. Eliminate the arbitrary function f and obtain a p.d.e.:

(a) $z = f(x^2 - y^2)$;

(b) $x + y + z = f(x^2 + y^2 + z^2)$;

(c) $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$;

(d) $z = e^{ax+by} f(ax + by)$;

(e) $z = f\left(\frac{xy}{z}\right)$.

(f) $z = f(y+ax) + g(y+bx)$, $a \neq b$.

13. Find the p.d.e. of all surfaces of revolution, having z-axis as the axis of revolution. [I.A.S. 1997]

[The equation of any surface of revolution having z-axis as the axis of rotation may be taken as $z = \phi\left(\sqrt{x^2+y^2}\right)$, where ϕ is an arbitrary function.]

14. Find a p.d.e. by eliminating the arbitrary function F from the relation $F(x^2 + y^2 + z^2,\ z^2 - 2xy) = 0$.

15. Construct a PDE by eliminating a and p from $z = a e^{-p^2 t} \cos px$.

Answers

1. (a) $z = pq$; (b) $q = 2yp^2$; (c) $q = p$. 5. $yp - xq = 0$. 6. $4\frac{\partial z}{\partial t} = p^2 - q^2$. 10. $(x-y)^2(p^2 + q^2 + 1) = 9(p-q)^2$. 11. $(y+z)p - (z+x)q = x - y$. 12. (a) $yp + xq = 0$; (b) $(y-z)p + (z-x)q = x - y$; (c) $x^2p + yq = 2y^2$; (d) $bp + aq = 2abz$; (e) $px - qy = 0$. 13. $yp = xq$. 14. $(p-q)z = y - x$.

Chapter 2

Lagrange's Solution of a Linear Partial Differential Equation Pq + Qq = R

Relevant Information on

1. $Pp+Qq=R$ (P, Q, R are function of x, y, z) is a typical linear partial differential equation of first order.
2. Lagrange's solution of $Pp+Qq=R$ using Lagrange's Auxiliary equations: $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$.
3. Integral surfaces: Orthogonal surfaces.

2.1 The General Solution of a Linear Equation

We refer to **Chapter 1, Art. 1.3:** Given an arbitrary functional relation

$$\phi(u,v)=0. \tag{2.1.1}$$

We can deduce a linear p.d.e. of first order in the form

$$Pp+Qq=R. \tag{2.1.2}$$

If (2.1.1) is deduced from (2.1.2), then we call (2.1.1), the **general solution** of (2.1.2). Since ϕ *is an arbitrary function*, the general solution (2.2.1) is more general than another solution of (2.1.2) that merely contains two arbitrary constants.

As for example, let $z = f(x^2 - y^2)$, where f is arbitrary. Then

$$p = \frac{\partial z}{\partial x} = f'(x^2 - y^2)2x,\quad q = \frac{\partial z}{\partial y} = f'(x^2 - y^2)(-2y)$$

which leads to $py + qx = 0$, a **linear p.d.e.**

We shall call $z = f(x^2 - y^2)$ the general solution of $py + qx = 0$.

See that we could write

$$z = a(x^2 - y^2) + b(x^2 - y^2)^2 \quad (a, b \text{ are arbitrary constants})$$

$$\text{or, } z = a\sin(x^2 - y^2) + b \quad (a, b \text{ are arbitrary constants})$$

as the solution of $py + qx = 0$. But certainly $z = f(x^2 - y^2)$ is a more general solution; all these solutions with arbitrary constants are included in $z = f(x^2 - y^2)$.

2.2 An Equation that is Equivalent to $Pp + Qq = R$

$\therefore$ A general type of a linear p.d.e. in p and q is

$$Pp + Qq = R, \tag{2.2.1}$$

where P, Q, R are functions of x, y, z.

Suppose that $u(x, y, z) = c$ satisfies (2.2.1).

Differentiation with respect to x and y gives

$$\left.\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p &= 0 \\ \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q &= 0 \end{aligned}\right\} \quad p = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial z}};\quad q = -\frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial z}}.$$

Substituting these values of p and q in (2.2.1) we obtain

$$P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z} = 0. \tag{2.2.2}$$

Therefore, if $u = c$ be an integral of (2.2.1), then $u = c$ also satisfies (2.2.2).

Conversely, if $u = c$ be an integral of (2.2.2), it is also an integral of (2.2.1).

Dividing (2.2.2) by $\frac{\partial u}{\partial z}$ and using the values of p and q, (2.2.2) reduces to $Pp + Qq = R$.

So we find equation (2.2.2) is equivalent to equation (2.2.1).

2.3 Lagrange's Method of Solving $Pp + Qq = R$

Lagrange's Rule: Statement: *The general solution of the linear partial differential equation*

$$Pp + Qq = R, \tag{2.3.1}$$

where P, Q, R are functions of x, y, z, is given by

$$\phi(u, v) = 0, \tag{2.3.2}$$

where ϕ is an arbitrary function and

$$\left.\begin{aligned} u(x, y, z) &= c_1 \\ v(x, y, z) &= c_2 \end{aligned}\right\} \tag{2.3.3}$$

are two independent solutions of the Auxiliary Equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{2.3.4}$$

This is known as **Lagrange's solution** of the linear equation.

Proof. Given $\phi(u, v) = 0$.

We consider z as dependent variable, and x and y as independent variables so that

$$\frac{\partial z}{\partial x} = p,\quad \frac{\partial z}{\partial y} = q,\quad \frac{\partial y}{\partial x} = 0,\quad \frac{\partial x}{\partial y} = 0.$$

Differentiating $\phi(u, v) = 0$ with respect to x, we get

$$\frac{\partial \phi}{\partial u}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right] + \frac{\partial \phi}{\partial v}\left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right] = 0$$

$$\frac{\frac{\partial \phi}{\partial u}}{\frac{\partial \phi}{\partial v}} = -\frac{\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}}{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}} = -\frac{\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}}\left(p = \frac{\partial z}{\partial x}\right).$$

Thus we find that $u = c_1, v = c_2$ form a solution of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$. Hence $u = c_1$, $v = c_2$ determine u and v for substitution in $\phi(u, v) = 0$.

This is what we wished to prove.

Note 2.3.1 *Equations* $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ *are known as Lagrange's Auxiliary (or Subsidiary) equations.*

Working Rule
Solution of $Pp + Qq = R$ (Lagrange's Method)

1. Put the given linear equation in the form

$$Pp + Qq = R.$$

2. Write down Lagrange's Auxiliary Equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

3. Taking two of the ratios at a time, and by using method of solving ordinary differential equations, obtain

$$u(x, y, z) = c_1, \quad v(x, y, z) = c_2.$$

as two independent solutions of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

4. The general solution of $Pp+Qq = R$ can be written as $\phi(u, v) = 0$ or in the form $u = f(v)$ or $v = F(u)$.

In Lagrange's Method of solving $Pp+Qq = R$, the most important part is to obtain $u = c_1$, $v = c_2$ from Lagrange's Auxiliary Equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

We ask our readers to remember **four types** of problems:

Solved Problems: (Type I – Type IV)

Type I

▶ **Example 2.3.1** *Solve:* $\frac{y^2 z}{x} p + xzq = y^2$.

Solution: Lagrange's Auxiliary Equations are

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2} \quad \left[\text{cf. } \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}\right]$$

From the **first two ratios** we see that the **variable** z **cancels out** and we are left with

$$\frac{dx}{y^2/x} = \frac{dy}{x} \quad \text{or,} \quad x^2dx - y^2dy = 0.$$

Integrating, we get

$$x^3 - y^3 = c_1. \quad [\text{cf. } u = c_1]$$

Taking **the first** and **last ratio** we see that y^2 **cancels out** and we obtain

$$\frac{dx}{z/x} = \frac{dz}{1} \quad \text{or,} \quad x\,dx - z\,dz = 0.$$

Integrating we get

$$x^2 - z^2 = c_2 \quad [\text{cf. } v = c_2]$$

∴ The **required general solution** of the given equation is

$$\phi(u, v) = 0$$

i.e., $\phi(x^3 - y^3, x^2 - z^2) = 0$

where ϕ is any arbitrary function of its arguments.

▶ **Example 2.3.2** *Solve:* $ap + aq = z$.

Solution: Lagrange's Auxiliary Equations are

$$\frac{dx}{a} = \frac{dy}{a} = \frac{dz}{z} \quad \left(\text{cf. } \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}\right)$$

First two ratios at once give $u = x - y = c_1$.

Again, Second and Third Ratios give $v = y - a\log z = c_2$.

∴ **The required general solution is**

$$\phi(u, v) = 0, \quad \text{i.e.,} \quad \phi(x - y, y - a\log z) = 0.$$

▶ **Example 2.3.3** *Solve:* $y^2p - xyq = x(z - 2y)$.

Solution: Lagrange's Auxiliary Equations are

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z - 2y)}.$$

First two ratios give

$$\frac{dx}{y} = \frac{dy}{-x}, \quad \text{i.e.,} \quad x\,dx + y\,dy = 0 \Rightarrow u \equiv x^2 + y^2 = c_1.$$

Last two ratios give

$$\frac{dy}{-y} = \frac{dz}{z - 2y},$$

i.e., $\dfrac{dz}{dy} + \dfrac{1}{y}z = 2$ $\left(\begin{array}{l}\text{ord. diff. eqn. Linear Form} \\ \text{with I.F.} = e^{\int \frac{1}{y}dy} = e^{\log y} = y\end{array}\right)$

Multiplying by the I.F. y and integrating we get

$$zy = y^2 + c_2, \quad \text{i.e.,} \quad u = zy - y^2 = c_2.$$

∴ The **required general solution** is $\phi(u, v) = 0$

$$\phi(x^2 + y^2, zy - y^2) = 0, \quad \phi \text{ being arbitrary.}$$

A

Try Yourself (*Examples for Practice*)

Solve the following linear equations by using Lagrange's Auxiliary Equations (Lagrange's Method)

1. $2p + 3q = 1$.
2. $p + q = \sin x$.
3. $xzp + yzq = xy$.
4. $p\tan x + q\tan y = \tan z$.
5. $zp + x = 0$.
6. $yzp + zxq = xy$.
7. $x^2p + y^2q + z^2 = 0$.
8. $x^2p + y^2q = z^2$.
9. $xp + yq = z$.

Answers

1. $\phi(x - 2z, y - 3z) = 0$. **2.** $\phi(x - y, z + \cos x) = 0$. **3.** $\phi(z^2 - xy, \frac{y}{x}) = 0$. **4.** $\frac{\sin x}{\sin y} = \phi\left(\frac{\sin y}{\sin z}\right)$. **5.** $x^2 + z^2 = \phi(y)$. **6.** $\phi(x^2 - y^2, x^2 - z^2) = 0$. **7.** $\phi\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{y} + \frac{1}{z}\right) = 0$. **8.** $\phi\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{y} - \frac{1}{z} = 0\right)$. **9.** $\phi\left(\frac{x}{y}, \frac{x}{z}\right) = 0$.

Type II

Suppose that one integral of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ can be obtained easily by taking suitably chosen two ratios (as used in Type I Problems) and suppose that another integral **cannot** be obtained by this same method.

Then one integral known to us is used to find another integral (see the Solved Examples given below).

In the second integral the constant of integration of the first integral should be removed.

▶ **Example 2.3.4** *Solve:* $p + 3q = 5z + \tan(y - 3x)$.

Solution: The Lagrange's Auxiliary Equations are

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y - 3x)}.$$

First see that one integral can be easily obtained from the first two ratios, namely $\frac{dx}{1} = \frac{dy}{3}$. Thus $y - 3x = c_1$ (*first integral*). The second integral cannot be obtained in this manner. However, we take first ratio and the last ratio and obtain

$$\frac{dx}{1} = \frac{dz}{5z + \tan(y - 3x)} = \frac{dz}{5z + \tan c_1} \text{ (using the known integral)}$$

Writing

$$5dx = \frac{5dz}{5z + \tan c_1}$$

and integrating we obtain

$$5x - \log(5z + \tan c_1) = \text{arbitrary constant } c_2 \text{ (say)}.$$

$\therefore$ **The required general solution is**

$$5x - \log[5z + \tan(y - 3x)] = \phi(y - 3x),$$

where ϕ is arbitrary.

▶ **Example 2.3.5** *Solve:* $xyp + y^2 q = xyz - 2x^2$.

Solution: Lagrange's Auxiliary Equations are

$$\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{xyz - 2x^2}$$

Taking the first two ratios (cancelling y), we get $\frac{dx}{x} = \frac{dy}{y}$ and hence, on integration $\frac{x}{y} = c_1$.

From second and third ratios

$$\frac{dy}{y^2} = \frac{dz}{xyz - 2x^2}$$

$$= \frac{dz}{(c_1 y)yz - 2c_1^2 y^2} \quad \left(\because \frac{x}{y} = c_1 \text{ or, } x = c_1 y\right)$$

$$\Rightarrow \frac{dy}{y^2} = \frac{dz}{c_1 y^2 z - 2c_1^2 y^2}$$

$$\Rightarrow c_1 dy = \frac{dz}{(z - 2c_1)}$$

$$\Rightarrow c_1 y - \log(z - 2c_1) = c_2$$

$$\Rightarrow x - \log\left(z - \frac{2x}{y}\right) = c_2.$$

$\therefore$ **The required general solution is**

$$x - \log\left(z - \frac{2x}{y}\right) = \phi\left(\frac{x}{y}\right),$$

ϕ being an arbitrary function.

▶ **Example 2.3.6** *Solve:* $xzp + yzq = xy$.

Solution: The Lagrange subsidiary equations are

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}.$$

Taking the first two ratios (concelling z), we get $\frac{dx}{x} = \frac{dy}{y}$ and this gives, on integration

$$\log x - \log y = \log c_1 \quad \text{or,} \quad \frac{x}{y} = c_1.$$

The second and third ratios are

$$\frac{dy}{yz} = \frac{dz}{xy}$$

or, $\frac{dy}{z} = \frac{dz}{x}$

or, $x dy = z\, dz$

or, $c_1 y\, dy = z\, dz$ (using the known integral $x = c_1 y$)

Hence, on integration we get

$$c_1 \frac{y^2}{2} = \frac{z^2}{2} + \text{constant}$$

or, $c_1 y^2 - z^2 = c_2$

or, $xy - z^2 = c_2$.

∴ The required general solution is $\phi(u, v) = 0$,

or, $\phi\left(\frac{x}{y}, xy - z^2\right) = 0$, ϕ being arbitrary.

▶ **Example 2.3.7** *Solve:* $py + qx = xyz^2(x^2 - y^2)$.

Solution: Lagrange's subsidiary equations are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{xyz^2(x^2 - y^2)}$$

From first two fractions we obtain $x^2 - y^2 = c_1$.

The last two fractions give

$$\frac{dy}{x} = \frac{dz}{xyz^2(x^2 - y^2)}$$

$\Rightarrow \frac{dy}{1} = \frac{dz}{yz^2 c_1}$ $(\because x^2 - y^2 = c_1)$

$\Rightarrow y\, dy = \frac{1}{c_1}\frac{dz}{z^2}$.

On integration,

$$\frac{y^2}{2} = \frac{1}{c_1}\left(-\frac{1}{z}\right) + \text{constant}$$

or, $c_1 y^2 = -\frac{2}{z} + \text{constant}$

or, $(x^2 - y^2)y^2 + \frac{2}{z} = c_2$.

Hence the required general solution is

$$(x^2 - y^2)y^2 + \frac{2}{z} = \phi(x^2 - y^2), \quad \phi \text{ is arbitrary.}$$

B

Try Yourself (*Examples for Practice*)
Solve by Lagrange's Method

1. $z(z^2 + xy)(px - qy) = x^4$.
2. $p - q = \frac{z}{x+y}$.
3. $p - 2q = 3x^2 \sin(y + 2x)$.
4. $(x^2 - y^2 - z^2)p + 2xyq = 2xz$.
5. $z(p - q) = z^2 + (x + y)^2$.

Answers

1. $\phi(xy, x^4 - z^4 - 2xyz^2) = 0$. 2. $x - (x + y)\log z = \phi(x + y)$. 3. $x^3 \sin(y+2x) - z = \phi(y+2x)$. 4. $\frac{x^2+y^2+z^2}{z} = \phi\left(\frac{y}{z}\right)$. 5. $e^{2y}[z^2+(x+y)^2] = \phi(x+y)$.

Type III

We may required to write (using a well-known Rule of Ratio and Proportion)

Lagrange's Subsidiary Equations as

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{P_1 dx + Q_1 dy + R_1 dz}{P_1 P + Q_1 Q + R_1 R}.$$

Multipliers P_1, Q_1, R_1 should be chosen carefully. If $P_1 P + Q_1 Q + R_1 R = 0$, then write $P_1 dx + Q_1 dy + R_1 dz = 0$. Now, on integration of $P_1 dx + Q_1 dy + R_1 dz = 0$ we may obtain an integral $u_1(x, y, z) = c_1$.

With the help of another set of suitably chosen multipliers we may obtain another integral $u_2(x, y, z) = c_2$.

Sometimes only one integral is possible by use of suitable multipliers and the other integral may be obtained by methods of Type I and Type II.

Then the required general solution is written as

$$\phi(u_1, u_2) = 0$$

where ϕ is an arbitrary function.

▶ **Example 2.3.8** *Solve:* $\frac{b-c}{a}yzp + \frac{c-a}{b}zxq = \frac{a-b}{c}xy.$

Solution: The Lagrange's subsidiary equations are

$$\frac{dx}{\left(\frac{b-c}{a}\right)yz} = \frac{dy}{\left(\frac{c-a}{b}\right)zx} = \frac{dz}{\left(\frac{a-b}{c}\right)xy}$$

$$\text{or,} \quad \frac{a\,dx}{(b-c)yz} = \frac{b\,dy}{(c-a)zx} = \frac{c\,dz}{(a-b)xy}. \qquad (1)$$

Multiply both Numerator and Denominator of first ratio by x, second ratio by y and third ratio by z.

Then each ratio of (1)

$$= \frac{ax\,dx + by\,dy + cz\,dz}{xyz(b-c+c-a+a-b)} = \frac{ax\,dx + by\,dy + cz\,dz}{0}.$$

Hence we may write

$$ax\,dx + by\,dy + cz\,dz = 0.$$

On integration we obtain

$$ax^2 + by^2 + cz^2 = \text{constant } c_1 \text{ (say)}. \qquad (2)$$

Take another set of multipliers ax (first ratio), by (second ratio) and cz (third ratio). Then each ratio of (1) becomes

$$= \frac{a^2x\,dx + b^2y\,dy + c^2z\,dz}{xyz[a(b-c)+b(c-a)+c(a-b)]} = \frac{a^2x\,dx + b^2y\,dy + c^2z\,dz}{0}.$$

Hence $a^2x\,dx + b^2y\,dy + c^2z\,dz = 0$. Now integrating we get

$$a^2x^2 + b^2y^2 + c^2z^2 = \text{constant } c_2 \text{ (say)}. \qquad (3)$$

From (2) and (3) we write the general solution of the given equation as

$$\phi(ax^2 + by^2 + cz^2, a^2x^2 + b^2y^2 + c^2z^2) = 0,$$

where ϕ is an arbitrary function.

▶ **Example 2.3.9** *Solve:* $(mz - ny)p + (nx - lz)q = ly - mx.$

Solution: The Lagrange's subsidiary equations corresponding to the given equation are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}. \qquad (1)$$

Using multipliers x for the first ratio, y for the second and z for the third we write each ratio of (1)

$$= \frac{x\,dx + y\,dy + z\,dz}{(mz-ny)x + (nx-lz)y + (ly-mx)z} = \frac{x\,dx + y\,dy + z\,dz}{0}.$$

Hence we can write $x\,dx + y\,dy + z\,dz = 0$. On integration we get

$$x^2 + y^2 + z^2 = \text{constant } c_1 \text{ (say)}. \qquad (2)$$

Next we choose l, m, n as multipliers of first, second and third ratios of (1). Then each ratio of (1) becomes

$$= \frac{ldx + mdy + ndz}{l(mz-ny) + m(nx-lz) + n(ly-mx)} = \frac{ldx + mdy + ndz}{0}.$$

Hence

$$ldx + mdy + ndz = 0$$

$$\Rightarrow \quad lx + my + nz = \text{constant } c_2 \text{ (say)}. \qquad (3)$$

The required general solution from (2) and (3) is

$$\phi(x^2 + y^2 + z^2, lx + my + nz) = 0,$$

ϕ being an arbitrary function.

▶ **Example 2.3.10** *Solve:* $x(y^2 - z^2)(p - y(z^2 + x^2)q = z(x^2 + y^2).$

Solution: The Lagrange's Auxiliary Equations corresponding to the given equations are

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{-y(z^2 + x^2)} = \frac{dz}{z(x^2 + y^2)}. \qquad (1)$$

Using x, y, z as multipliers for first, second and third ratio of (1), we write

$$\text{each ratio of (1)} = \frac{x\,dx + y\,dy + z\,dz}{x^2(y^2 - z^2) - y^2(z^2 + x^2) + z^2(x^2 + y^2)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0} \qquad (2)$$

$$\therefore \quad x\,dx + y\,dy + z\,dz = 0, \text{ whence } x^2 + y^2 + z^2 = c_1.$$

Using $\frac{1}{x}, -\frac{1}{y}, -\frac{1}{z}$ as multipliers of ratios of (1) we again find each ratio of (1) becomes

$$= \frac{\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz}{(y^2 - z^2) + (z^2 + x^2) - (x^2 + y^2)} = \frac{\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz}{0}$$

$\therefore$ $\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz = 0$ which yields on integration,

$$\log x - \log y - \log z = \log c_2, \quad \textbf{or,} \quad \frac{x}{yz} = c_2. \tag{3}$$

$\therefore$ The required general solution [from (2) and (3)] is

$$\phi\left(x^2 + y^2 + z^2, \frac{x}{yz}\right) = 0$$

where ϕ is an arbitrary function.

▶ **Example 2.3.11** *Solve:* $x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$.

Solution: The Lagrange's subsidiary equations corresponding to the given equation are

$$\frac{dx}{x(y^2 + z)} = \frac{dy}{-y(x^2 + z)} = \frac{dz}{z(x^2 - y^2)}. \tag{1}$$

Choosing $x, y, -1$ as multipliers of ratios of (1)

$$\text{each ratio of (1)} = \frac{x\,dx + y\,dy - dz}{x^2(y^2 + z) - y^2(x^2 + z) - z(x^2 - y^2)} = \frac{x\,dx + y\,dy - dz}{0}.$$

Hence $x\,dx + y\,dy - dz = 0$ which, on integration, gives

$$x^2 + y^2 - 2z = c_1. \tag{2}$$

Again, choosing $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers of ratios of (1) we obtain, each ratio of (1) becomes

$$= \frac{dx/x + dy/y + dz/z}{y^2 + z - (x^2 + z) + x^2 - y^2} = \frac{dx/x + dy/y + dz/z}{0}.$$

Hence $\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$ or, $\log xyz = \log c_2$

i.e., $xyz = c_2$. (3)

$\therefore$ From (2) and (3) the required general solution is

$$\phi(x^2 + y^2 - 2z, xyz) = 0,$$

ϕ being an arbitrary function.

▶ **Example 2.3.12** *Solve:* $(y^2 + z^2)p - xyq = -zx$. [I.A.S. 1990]

Solution: The Lagrange's auxiliary equations corresponding to the given equation are

$$\frac{dx}{y^2 + z^2} = \frac{dy}{-xy} = \frac{dz}{-zx}. \tag{1}$$

Using multipliers of the ratios of (1) as x, y, z respectively, we obtain

$$\text{each ratio of (1)} = \frac{x\,dx + y\,dy + z\,dz}{x(y^2 + z^2) - xy^2 - xz^2} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\therefore$ $x\,dx + y\,dy + z\,dz = 0$ which gives, on integration

$$x^2 + y^2 + z^2 = c_1. \tag{2}$$

Again, from last two ratios of (1) we get

$$\frac{dy}{-xy} = \frac{dz}{-xz} \quad \text{or,} \quad \frac{dy}{y} = \frac{dz}{z}.$$

On integration this gives

$$\log y - \log z = \log c_2$$

$$\text{or,} \quad \frac{y}{z} = c_2. \tag{3}$$

$\therefore$ The required general solution [from (2) and (3)] is

$$\phi\left(x^2 + y^2 + z^2, \frac{y}{z}\right) = 0$$

where ϕ is an arbitrary function.

Type IV

As in Type III, with one set of multipliers P_1, Q_1, R_1 we write

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{P_1 dx + Q_1 dy + R_1 dz}{P_1 P + Q_1 Q + R_1 R}.$$

Now, if we see $P_1 dx + Q_1 dy + R_1 dz$ is an exact differential of the denominator $P_1 P + Q_1 Q + R_1 R$, then

$$\frac{P_1 dx + Q_1 dy + R_1 dz}{P_1 P + Q_1 Q + R_1 R}$$

may be combined with a **suitable ratio** of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ and this will give one integral.

Choose another set of multipliers, say P_2, Q_2, R_2, then

$$\text{each ratio} = \frac{P_2 dx + Q_2 dy + R_2 dz}{P_2 P + Q_2 Q + R_2 R}$$

Suppose that the numerator is again an exact differential of $P_2 P + Q_2 Q + R_2 R$. The two ratios

$$\frac{P_1 dx + Q_1 dy + R_1 dz}{P_1 P + Q_1 Q + R_1 R} \text{ and } \frac{P_2 dx + Q_2 dy + R_2 dz}{P_2 P + Q_2 Q + R_2 R}$$

are then combined to give a second integral.

See the Worked Examples given below:

▶ **Example 2.3.13** *Solve:* $(y+z)p + (z+x)q = (x+y)$. **[I.A.S. 1997]**

Solution: The Lagrange's auxiliary equations corresponding to the given equation are

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}. \qquad (1)$$

Choose multipliers $1, -1, 0$ for the first, second and third ratio of (1). Then, each ratio of (1)

$$= \frac{dx - dy}{y + z - z - x} = \frac{dx - dy}{y - x} = \frac{d(x-y)}{-(x-y)}.$$

[See that multipliers are so chosen that the numerator $dx - dy$ is an exact differential, namely $d(x-y)$]

Again, choosing multipliers as $1, 1, 1$, each ratio of (1)

$$= \frac{dx + dy + dz}{2(x+y+z)} = \frac{1}{2}\frac{d(x+y+z)}{x+y+z}.$$

Lastly choose $0, 1, -1$ as multipliers, then each ratio of (1)

$$= \frac{dy - dz}{z + x - (x+y)} = \frac{d(y-z)}{-(y-z)}$$

So we may write

$$-\frac{d(x-y)}{(x-y)} = \frac{1}{2}\frac{d(x+y+z)}{x+y+z} = -\frac{d(y-z)}{y-z}. \qquad (2)$$

First two ratio of (2) give, on integration

$$-\log(x-y) = \frac{1}{2}\log(x+y+z) + \text{constant}$$

i.e., $\quad 2\log(x-y) + \log(x+y+z) = \text{constant}$

or, $\quad (x-y)^2(x+y+z) = c_1.$

Taking first and last ratio of (2), and integrating we easily obtain

$$\frac{x-y}{y-z} = c_2$$

∴ The required general solution is

$$\phi\left((x-y)^2(x+y+z), \frac{x-y}{y-z}\right) = 0$$

where ϕ is an arbitrary function.

▶ **Example 2.3.14** *Solve:* $y^2(x-y)p + x^2(y-x)q = z(x^2+y^2)$. **[I.A.S. 1996]**

Solution: Lagrange's auxiliary equations corresponding to the given equation are

$$\frac{dx}{y^2(x-y)} = \frac{dy}{x^2(y-x)} = \frac{dz}{z(x^2+y^2)}. \qquad (1)$$

From the first two ratios we at once get $x^3 + y^3 = c_1$. Choosing $1, -1, 0$ as multipliers each ratio of (1)

$$= \frac{dx - dy}{y^2(x-y) - x^2(y-x)} = \frac{dx - dy}{(x-y)(x^2+y^2)}$$

Combining it with the third ratio of (1) we write

$$\frac{dx - dy}{(x-y)(x^2+y^2)} = \frac{dz}{z(x^2+y^2)}$$

$$\text{or,} \quad \frac{d(x-y)}{x-y} - \frac{dz}{z} = 0.$$

On integration we get $\log(x-y) - \log z = \log c_2$ (say)

$$\text{or,} \quad \frac{x-y}{z} = c_2$$

∴ The required general solution is

$$\phi\left(x^3 + y^3, \frac{x-y}{z}\right) = 0.$$

where ϕ is any arbitrary function.

▶ **Example 2.3.15** *Solve:* $(x^2 - y^2 - z^2)p + 2xyq = 2xz$.

[I.A.S. 1973, W.B.C.S. 2001]

Solution: Lagrange's subsidiary equations corresponding to the given equation are

$$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}. \qquad (1)$$

Taking the last two ratios we easily get, on integration $y/z = c_1$.

Choosing x, y, z as multipliers of the ratios,

$$\text{each ratio of (1)} = \frac{x\,dx + y\,dy + z\,dz}{x(x^2 - y^2 - z^2) + 2xy^2 + 2xz^2}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{x(x^2 + y^2 + z^2)}.$$

Combining this with third ratio of (1), we write

$$\frac{x\,dx + y\,dy + z\,dz}{x(x^2 + y^2 + z^2)} = \frac{dz}{2xz}$$

$$\text{or,} \quad \frac{d(x^2 + y^2 + z^2)}{x^2 + y^2 + z^2} = \frac{dz}{z}.$$

On integration we get $\log(x^2 + y^2 + z^2) - \log z = \log c_2$

$$\text{or,} \quad \frac{x^2 + y^2 + z^2}{z} = c_2.$$

Hence the general solution is

$$\phi\left(\frac{y}{z}, \frac{x^2 + y^2 + z^2}{z}\right) = 0$$

where ϕ is an arbitrary function.

▶ **Example 2.3.16** *Solve:* $\cos(x+y)p + \sin(x+y)q = z$.

Solution: Lagrange's auxiliary equations corresponding to the given equation are

$$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z}. \qquad (1)$$

Choosing $1, 1, 0$ as multipliers of the ratios of (1), each ratio of (1)

$$= \frac{d(x+y)}{\cos(x+y) + \sin(x+y)}. \qquad (2)$$

Choosing $1, -1, 0$ as multipliers of the ratios of (1), each ratio of (1)

$$= \frac{d(x-y)}{\cos(x+y) - \sin(x+y)}. \qquad (3)$$

From (1), (2) and (3) we write

$$\frac{dz}{z} = \frac{d(x+y)}{\cos(x+y) + \sin(x+y)} = \frac{d(x-y)}{\cos(x+y) - \sin(x+y)}. \qquad (4)$$

Taking the first two fractions of (4)

$$\frac{dz}{z} = \frac{d(x+y)}{\cos(x+y) + \sin(x+y)}.$$

Putting $x + y = t$ on the right side we write

$$\frac{dz}{z} = \frac{dt}{\cos t + \sin t} = \frac{dt}{\sqrt{2}\left\{\frac{1}{\sqrt{2}}\cos t + \frac{1}{\sqrt{2}}\sin t\right\}}$$

$$= \frac{dt}{\sqrt{2}\sin\left(\frac{\pi}{4} + t\right)}$$

$$\text{or,} \quad \sqrt{2}\frac{dz}{z} = \operatorname{cosec}\left(\frac{\pi}{4} + t\right) dt.$$

Hence on integration,

$$\log z^{\sqrt{2}} = \log\left\{\tan\left(\frac{\pi}{8}+\frac{t}{2}\right)\right\} + \log c_1$$

$\therefore\ z^{\sqrt{2}} = c_1 \tan\left(\frac{\pi}{8}+\frac{t}{2}\right)$, where $t = x+y$

$$\text{or,}\quad z^{\sqrt{2}} \cot\left(\frac{\pi}{8}+\frac{t}{2}\right) = c_1. \qquad (5)$$

Taking the last two fractions of (4) we get

$$d(x-y) = \frac{\cos(x+y)-\sin(x+y)}{\cos(x+y)+\sin(x+y)} d(x+y)$$

On the right side put $x+y=t$ and then integrate

$$x-y = \int \frac{\cos t - \sin t}{\cos t + \sin t} dt + \text{constant}$$

or, $x-y = \log(\sin t + \cos t) + \log c_2' = \log c_2'(\sin t + \cos t)$

or, $e^{x-y} = c_2'[\sin(x+y)+\cos(x+y)]$

or, $[\sin(x+y)+\cos(x+y)]e^{y-x} = \frac{1}{c_2'} = c_2$ (say).

Then the general solution can be written as

$$\phi\left(z^{\sqrt{2}}\cot\left(\frac{\pi}{8}+\frac{x+y}{2}\right), e^{y-x}\{\sin(x+y)+\cos(x+y)\}\right) = 0$$

where ϕ is an arbitrary function.

▶ **Example 2.3.17** *Solve:*

$$px(x+y) - qy(x+y) + (x-y)(2x+2y+z) = 0.$$

Solution: Given equation may be written as

$$x(x+y)p - y(x+y)q = -(x-y)((2x+2y+z)$$

Lagrange's subsidiary equations are

$$\frac{dx}{x(x+y)} = \frac{dy}{-y(x+y)} = \frac{dz}{-(x-y)(2x+2y+z)}. \qquad (1)$$

The first two ratios of (1) at once give, on integration $xy = c_1$.

Again each fraction of (1) becomes

$$= \frac{dx+dy}{x^2+y^2} = \frac{dx+dy+dz}{x(x+y)-y(x+y)-(x-y)(2x+2y+z)}$$

$$\text{or,}\quad \frac{d(x+y)}{(x-y)(x+y)} = \frac{dx+dy+dz}{(x-y)[x+y-2x-2y-z]} = \frac{d(x+y+z)}{-(x+y+z)}.$$

Hence, on integration, we get

$$(x+y)(x+y+z) = c_2$$

$\therefore$ The required general solution is

$$\phi(xy, (x+y)(x+y+z)) = 0$$

where ϕ is an arbitrary function.

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Try Yourself (*Examples for practice*)

Solve the following linear partial differential equations

1. $(1+y)p + (1+x)q = z$.
2. $xzp + yzq = xy$.
3. $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$.
4. $(x^2 - y^2 - yz)p + (x^2 - y^2 - zx)q = z(x-y)$.
5. $xp + yq = z - a\sqrt{x^2+y^2+z^2}$.
6. $(x^3 + 3xy^2)p + (y^3 + 3x^2y)q = 2z(x^2+y^2)$. [I.A.S. 1993]
7. $p + q = x + y + z$.
8. $(2x^2 + y^2 + z^2 - 2yz - zx - xy)p + (x^2 + 2y^2 + z^2 - yz - 2zx - xy)q = x^2 + y^2 + 2z^2 - yz - zx - 2xy$. [I.A.S. 1992]
9. $(x^2 + y^2 + yz)p + (x^2 + y^2 - xz)q = z(x+y)$.
10. $\cos(x+y)p + \sin(x+y)q = z + \frac{1}{z}$.