3. $xyp + y(2x - y)q = 2xz$.

4. $x(y^m - z^m)p + y(z^m - x^m)q = z(x^m - y^m)$.

5. $p - qy \log y = z \log y$.

6. $x^2 p + y^2 q = x + y$.

7. $z(p + q) = z^2 + (x - y)^2$.

8. $(xz + y^2)p + (yz - 2x^2)q + 2xy + z^2 = 0$.

9. $(x^2 + 2y^2)p - xyq = xz$.

Answers

1. $\phi(y-x, e^{-2x}y+x) = 0$ **2.** $\phi\left(xy, xe^{\frac{1}{z-xy}}\right) = 0$. **3.** $\phi(xy-x^2, z/xy) = 0$. **4.** $x^m+y^m+z^m = \phi(xyz)$. **5.** $\phi(yz, e^x \log y) = 0$. **6.** $\phi\left[\frac{1}{y} - \frac{1}{x}, e^{-z}(x-y)\right] = 0$. **7.** $\log\{z^2 + (x-y)^2\} - 2x = \phi(x-y)$. **8.** $\phi(yz + x^2, 2xz - y^2) = 0$. **9.** $\phi(x^2y^2 + y^4, yz) = 0$.

2.4 Integral Surfaces Through a Given Curve

Working Rule:-

Given: $Pp + Qq = R$, a linear p.d.e. of first order. We obtain two independent solutions

$$u(x, y, z) = c_1 \quad \text{and} \quad v(x, y, z) = c_2 \tag{2.4.1}$$

by using Lagrange's auxiliary equations.

Suppose now we wish to obtain the integral surface which passes through a curve whose parametric equations are

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

where t is a parameter.

(2.4.1) is then written as

$$u[x(t), y(t), z(t)] = c_1 \quad \text{and} \quad v[x(t), y(t), z(t)] = c_2$$

We eliminate t and get a relation involving c_1 and c_2. Finally, putting $c_1 = u$ and $c_2 = v$ we get the required integral surface passing through the given curve.

▶ **Example 2.4.1**. *Find the integral surface of the linear partial differential equation*

$$x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$$

which contains the straight line $x+y=0, z=1$. **[I.A.S. 1998]**

Solution: Lagrange's auxiliary equations corresponding to the given p.d.e. are

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z}. \qquad (1)$$

Each ratio of (1)

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2+z^2-(x^2+z)+(x^2-y^2)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}.$$

$\therefore$ $\frac{1}{x}dx + \frac{1}{y}dy + \frac{dz}{z} = 0$, i.e., $\log x + \log y + \log z = \log c_1$

$$\textbf{or,} \quad xyz = c_1. \qquad (2)$$

Each ratio of (1) (Multipliers $x, y, -1$)

$$= \frac{x\,dx + y\,dy - dz}{x^2(y^2+z) - y^2(x^2+z) - (x^2-y^2)z} = \frac{x\,dx + y\,dy - dz}{0}$$

or $\quad x\,dx + y\,dy - dz = 0$

i.e., $\quad x^2+y^2-2z = c_2. \qquad (3)$

Now the straight line in the parametric forms:

$$x = t, \quad y = -t, \quad z = 1.$$

From (2) we then write $-t^2 = c_1$.
From (3) we write $2t^2 - 2 = c_2$.

$\therefore$ The required integral surface can be obtained as $-2c_1 - 2 = c_2$

or, $\quad -2(xyz) - 2 = x^2+y^2-2z$

or, $\quad 2xyz + x^2 + y^2 - 2z + 2 = 0.$

▶ **Example 2.4.2** *Find the equation of the integral surface of the linear differential equation*

$$2y(z-3)p + (2x-z)q = y(2x-3)$$

which passes through the circle $x^2+y^2 = 2x, z = 0$.

Solution: Lagrange's auxiliary equations corresponding to the given equation are

$$\frac{dx}{2y(z-3)} = \frac{dy}{2x-z} = \frac{dz}{y(2x-3)}. \qquad (1)$$

From first and the last ratio we get by integration

[i.e., by integrating $(2x-3)dx = 2(z-3)dz$]

We get

$$x^2 - 3x - z^2 + 6z = c_1. \qquad (2)$$

Again each ratio of (1)

$$= \frac{\frac{1}{2}dx + y\,dy - dz}{y(z-3) + y(2x-z) - y(2x-3)} \quad \begin{bmatrix}\text{Multipliers} \\ \text{are } \frac{1}{2}, y, -1\end{bmatrix}$$

$$= \frac{\frac{1}{2}dx + y\,dy - dz}{0}.$$

$$\therefore \quad \frac{1}{2}dx + y\,dy - dz = 0 \Rightarrow \frac{x}{2} + \frac{y^2}{2} - z = \text{constant}$$

$$\Rightarrow x + y^2 - 2z = c_2. \qquad (3)$$

Now the given circle in the parametric form is

$$x = t, \quad y = \sqrt{2t - t^2}, \quad z = 0$$

From (2) and (3) we obtain

$$t^2 - 3t = c_1 \quad \text{and} \quad t + (2t - t^2) = c_2$$

Eliminating t, we get $c_1 + c_2 = 0$

$$\therefore \quad x^2 - 3x - z^2 + 6z + x + y^2 - 2z = 0$$

i.e., $x^2 + y^2 - z^2 - 2x + 4z = 0$ **Reqd. Integral Surface**

We need not always write the equation of the given curve in parametric form. See the example given below:

▶ **Example 2.4.3** *Find the integral surface of the linear partial differential equation* $(x-y)p + (y-x-z)q = z$ *through the circle* $x^2+y^2 = 1$, $z = 1$.

Solution: Given equation: $(x-y)p+(y-x-z)q=z$

Lagrange's auxiliary equations are:

$$\frac{dx}{x-y}=\frac{dy}{y-x-z}=\frac{dz}{z}. \qquad (1)$$

Each ratio $=\frac{dx+dy+dz}{(x-y)+(y-x-z)+(z)}=\frac{dx+dy+dz}{0}$

$\therefore\ dx+dy+dz=0$ **or,** $x+y+z=c_1$. (2)

Taking the last two ratios of (1) we get $\frac{dy}{y-x-z}=\frac{dz}{z}$

or, $\dfrac{dy}{y-x-(c_1-x-y)}=\dfrac{dz}{z}$, using (2)

or, $\dfrac{dy}{2y-c_1}=\dfrac{dz}{z}$.

On integration this gives $\frac{1}{2}\log(2y-c_1)=\log z+\text{constant}$

or, $\log(2y-c_1)-2\log z=\log c_2$

or, $\dfrac{2y-c_1}{z^2}=c_2$

or, $\dfrac{2y-(x+y+z)}{z^2}=c_2$ [using (2)]

or, $y-x-z=c_2z^2$. (3)

Given curve: $x^2+y^2=1, z=1$

Using $z=1$, in (2) and (3)

$$x+y=c_1-1,\quad (y-x)=c_2+1$$

$$\therefore\quad (x+y)^2+(y-x)^2=(c_1-1)^2+(c_2+1)^2$$

or, $2(x^2+y^2)=(c_1-1)^2+(c_2+1)^2$

or, $c_1^2+c_2^2-2c_1+2c_2=0$ (using $x^2+y^2=1$ of the curve).

From (2) and (3) it follows

$$(x+y+z)^2+\frac{(y-x-z)^2}{z^4}-2(x+y+z)+2\frac{(y-x-z)}{z^2}=0$$

i.e., the required surface is

$$z^4(x+y+z)^2+(y-x-z)^2-2z^4(x+y+z)+2z^2(y-x-z)=0.$$

Example 2.4.4 *Find the integral surface of*

$$x^2p+y^2q+z^2=0,$$

which passes through the hyperboloid $xy=x+y, z=1$. [I.A.S. 1994]

Solution: Given equation: $x^2p+y^2q+z^2=0$

or, $x^2p+y^2q=-z^2$. (1)

Given curve:

$$xy=x+y,\quad z=1. \qquad (2)$$

Lagrange's auxiliary equations are

$$\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{-z^2}. \qquad (3)$$

Taking the first and third ratio

$$x^{-2}dx+z^{-2}dz=0$$

Integrating $-\frac{1}{x}-\frac{1}{z}=-c_1$ (say)

or, $\dfrac{1}{x}+\dfrac{1}{z}=c_1$. (4)

Taking the second and third ratio $y^{-2}dy+z^{-2}dz=0$

Integrating $-\frac{1}{y}-\frac{1}{z}=-c_2$ (say)

or, $\dfrac{1}{y}+\dfrac{1}{z}=c^2$. (5)

Adding (4) and (5) $\frac{1}{x}+\frac{1}{y}+\frac{2}{z}=c_1+c_2$

or, $\dfrac{x+y}{xy}+\dfrac{2}{z}=c_1+c_2$

or, $\dfrac{xy}{xy}+\dfrac{2}{1}=c_1+c_2$ [using (2)]

or, $c_1+c_2=3$. (6)

Substituting the values of c_1 and c_2 from (4) and (5) in (6), we obtain

$$\frac{1}{x}+\frac{1}{z}+\frac{1}{y}+\frac{1}{z}=3$$

or, $yz+2xy+xz=3xyz$. **Ans.**

▶ **Example 2.4.9** *Find the family of surfaces orthogonal to the family of surfaces given by the differential equation*

$$(y+z)p+(z+x)q=x+y.$$

Solution: Let

$$P=y+z,\quad Q=z+x,\quad R=x+y. \tag{1}$$

Then the given differential equation becomes

$$Pp+Qq=R. \tag{2}$$

Now the differential equation of the family of surfaces orthogonal to the given family is given by

$$Pdx+Qdy+Rdz=0$$

or $(y+z)dx+(z+x)dy+(x+y)dz=0$

or $(ydx+xdy)+(ydz+zdy)+(zdx+xdz)=0.$

Integrating $xy+yz+zx=c$ (c being a parameter) which is the required family of surfaces.

Examples II

1. Find the equation of the integral surface of the differential equation

$$(x^2-yz)p+(y^2-zx)q=z^2-xy$$

which passes through the line $x=1$, $y=0$.

[Following the solved examples of **art. 2.4** obtain $(x-y)(y-z)=c_1$ and $xy+yz+zx=c_2$. Using the equation of the line $x=1$, $y=0$ obtain $c_1c_2=-1$

∴ Required surface $(x-y)(y-z)(xy+yz+zx)+1=0$.]

2. Find the equation of the integral surface satisfying

$$4yzp+q+2y=0$$

and passing through $y^2+z^2=0, x+z=2$. **[I.A.S. 1997]**

3. Find the general integral of the partial differential equation

$$(2xy-1)p+(z-2x^2)q=2(x-yz)$$

and also the particular integral which passes through the line $x=1, y=0$.

[I.A.S. 1981]

4. Find the equation of the integral surface of the P.D.E.

$$2y(z-3)p+(2x-z)q=y(2x-3)$$

which passes through the circle $z=0, x^2-2x+y^2=0$.

5. Find the general solution of the P.D.E.

$$2x(y+z^2)p+y(2y+z^2)q=z^2$$

and hence prove that $yz(z^2+zz-2y)=x^2$ is a solution.

6. Solve: $xp+yq=z$. Find a solution representing a surface meeting the parabola $y^2=4x, z=1$.

7. Find the surface which is orthogonal to the one-parameter system $z=cxy(x^2+y^2)$ and which passes through the hyperbola $x^2-y^2=a^2$, $z=0$.

Answers

2. $y^2+z^2+x+z-3=0$. **3.** $x^2+y^2+z-xz-y=1$. **4.** $x^2+y^2-2x=z^2-4z$. **6.** General solution $\phi\left(\frac{x}{z},\frac{y}{z}\right)=0$; surface $y^2=4xz$. **7.** $(x^2+y^2+4z^2)(x^2-y^2)^2=a^4(x^2+y^2)$.

3. **General solution:** If, in the complete solution (3.1.2) $\phi(x, y, z, a, b) = 0$ one of the constants, say b, is replaced by a known function of the other, say $b = \theta(a)$, then

$$\phi(x, y, z, a, \theta(a)) = 0$$

is a one-parameter family of the surfaces of (3.1.1).

If this family has an envelope, its equation may be found as usual by eliminating a from

$$\phi(x, y, z, a, \theta(a)) = 0 \quad \text{and} \quad \frac{\partial}{\partial a}\phi(x, y, z, a, \phi(a)) = 0$$

and determining that part of the result which satisfies (3.1.1).

▶ **Example 3.1.2** *Let* $b = \theta(a) = a$ *in the complete solution of* Example (3.1.1), *given above.*

3.2 General Method of Solving the P.D.E. of First Order in Two Independent Variables x and y: Charpit's Method

Let
$$F(x, y, z, p, q) = 0 \tag{3.2.1}$$

be a given P.D.E. of first order in two independent variables x and y; z is a function of x and y; $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$. Since z depends on x and y, therefore
$$dz = pdx + qdy. \tag{3.2.2}$$

Now, if another relation can be found between x, y, z, p, q such as
$$f(x, y, z, p, q) = 0 \tag{3.2.3}$$

then p and q can be eliminated:

The values of p and q deduced from (3.2.1) and (3.2.3) can be substituted in (3.2.2) and the elimination of p and q is then possible.

The integral of the O.D.E. thus formed involving x, y, z will satisfy the given equation (3.2.1).

The problem thus reduces to find a relation of the form (3.2.3) which together with (3.2.1) will determine p and q that will render (3.2.2) integrable.

This is a linear equation of first order, which the auxiliary function f of equation (3.2.3) must satisfy. Its integrals are integrals of the auxiliary equations:

$$\frac{dp}{\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial z}}=\frac{dq}{\frac{\partial F}{\partial y}+q\frac{\partial F}{\partial z}}=\frac{dz}{-p\frac{\partial F}{\partial p}-q\frac{\partial F}{\partial q}}=\frac{dx}{-\frac{\partial F}{\partial p}}=\frac{dy}{-\frac{\partial F}{\partial q}}=\frac{df}{0}. \quad (3.2.5)$$

The equations (3.2.5) are known as **Charpit's auxiliary equations.** Any integrals of these equations involving p or q or both can be traken for the required second relation (3.2.3). Actually the simplest relation involving p or q or both is taken as relation (3.2.3). After obtaining the relation (3.2.3) p and q are obtained from (3.2.1) and (3.2.3) and these values are substituted in (3.2.2). On integrating it we get the required complete solution of the given equation.

▶ **Example 3.2.1** *Find a complete integral of $px + qy = pq$.*

Solution: Here the given equation is

$$F(x, y, z, p, q) = px + qy - pq = 0. \quad (1)$$

∴ Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial z}}=\frac{dq}{\frac{\partial F}{\partial y}+q\frac{\partial F}{\partial z}}=\frac{dz}{-p\frac{\partial F}{\partial p}-q\frac{\partial F}{\partial q}}=\frac{dx}{-\frac{\partial F}{\partial p}}=\frac{dy}{-\frac{\partial F}{\partial q}}$$

or,
$$\frac{dp}{p+p\cdot 0}=\frac{dq}{q+q\cdot 0}=\frac{dz}{-p(x-q)-q(y-p)}=\frac{dx}{-(x-q)}=\frac{dy}{-(y-p)}. \quad (2)$$

Taking the first two fractions

$$\frac{dp}{p}=\frac{dq}{q} \Rightarrow \log p = \log q + \log a$$

$$\Rightarrow p = aq \quad (a \text{ is an arb. constant.}) \quad (3)$$

Substituting this value of p in (1) we get

$$aqx + qy - aq^2 = 0 \quad (4)$$

or, $$aq = ax + y \quad (q \neq 0).$$

From (3) and (4), (5)

$$q = \frac{ax+y}{a}, \quad p = ax + y.$$

Putting these values of p and q in

$$dz = pdx + qdy$$

we get

$$dz = (ax+y)dx + \frac{ax+y}{a}dy$$

or, $$adz = a(ax+y)dx + (ax+y)dy$$

or, $$adz = (ax+y)d(ax+y).$$

Integrating

$$az = \frac{(ax+y)^2}{2} + b \quad (b \text{ is an arb. constant})$$

which is a complete integral, a, b being arbitrary constants.

▶ **Example 3.2.2** *Find a complete integral of* $q = 3p^2$.

Solution: Here the given equation is

$$f(x,y,z,p,q) = 3p^2 - q = 0. \tag{1}$$

∴ Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}} = \frac{dz}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}}$$

or, $$\frac{dp}{0+p\cdot 0} = \frac{dq}{0+q\cdot 0} = \frac{dz}{-6p^2+q} = \frac{dx}{-6p} = \frac{dy}{1}. \tag{2}$$

Taking the first fraction of (2), $dp = 0$ so that

$$p = a. \tag{3}$$

Substituting this value of p in (1) we get

$$q = 3a^2. \tag{4}$$

Putting these values of p and q in $dz = pdx + qdy$, we obtain

$$dz = adx + 3a^2dy$$

so that

$$z = ax + 3a^2y + b \quad (a, b \text{ are arbitrary constants}).$$

This is a complete integral.

▶ **Example 3.2.3** *Verify that a complete integral of* $z = pq$ *(by using Charpit's Method) is* $2\sqrt{z} = x\sqrt{a} + \left(\frac{1}{\sqrt{a}}\right)y + b$, *($a, b$ are arbitrary constants).*

Try yourself.

▶ **Example 3.2.4** *Given* $F(x,y,z,p,q) = zpq - p - q = 0$, *verify that Charpit's auxiliary equations are*

$$\frac{dp}{p^2q} = \frac{dq}{pq^2} = \frac{dz}{-p(qz-1) - q(pz-1)} = \frac{dx}{-(qz-1)} = \frac{dy}{-(pz-1)}.$$

Now take first two fractions, obtain $p = \frac{1+a}{z}$ and $q = \frac{1+a}{az}$ and obtain a complete integral

$$z^2 = 2(1+a)\left[x + \frac{1}{a}y\right] + b$$

▶ **Example 3.2.5** *Verify that a complete integral of* $p^2 - y^2q = y^2 - x^2$ *is*

$$z = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \frac{a^2}{y} - y + b.$$

[Take the fractions, $\frac{dp}{2x} = \frac{dx}{-2p}$ and proceed as above.]

▶ **Example 3.2.6** *Find a complete integral of* $z^2(p^2z^2 + q^2) = 1$. **[I.A.S. 1997]**

Here $F(x,y,z,p,q) = p^2z^4 + q^2z^2 - 1 = 0.$

Charpit's auxiliary equations are

$$\frac{dp}{p(4p^2z^3 + 2zq^2)} = \frac{dq}{q(4p^2z^3 + 2zq^2)} = \frac{dz}{-2p^2z^4 - 2q^2z^2}$$
$$= \frac{dx}{-2pz^4} = \frac{dy}{-2qz^2}.$$

Taking the first two fractions, $\frac{dp}{p} = \frac{dq}{q}$ so that $p = aq$.

Solving for p and q,

$$p = \frac{a}{z\sqrt{a^2z^2+1}}, \quad q = \frac{1}{z\sqrt{a^2z^2+1}}$$

so that $dz = pdx + qdy \Rightarrow adx + dy = z\sqrt{a^2z^2+1}dz$.

Integrating,

$$ax + y = \int (a^2z^2+1)^{1/2} \cdot z\, dz$$

$$ax + y + b = \frac{1}{3a^2}(a^2z^2+1)^{3/2}, \quad \text{(putting } a^2z^2+1 = t^2\text{)}$$

or, $\quad 9a^4(ax+y+b)^2 = (a^2z^2+1)^3$

which is a complete integral, a and b being arbitrary constants.

▶ **Example 3.2.7** *Find a complete integral of*

(i) $q = (z+px)^2$; (ii) $p = (z+qy)^2$.

(i) Here the given equation is $F(x,y,z,p,q) = (z+px)^2 - q = 0$
Charpit's auxiliary equations are

$$\frac{dp}{2p(z+px)+2p(z+px)} = \frac{dq}{2q(z+px)} = \frac{dz}{-2px(z+px)+q} = \frac{dx}{-2x(z+px)} = \frac{dy}{0}.$$

Taking the second and fourth fractions, $\frac{dq}{q} = -\frac{dx}{x}$.

Integrating $\log q = \log a - \log x$ so that $q = a/x$.

Hence from the given equation $p = \frac{\sqrt{a}}{x\sqrt{x}} - \frac{z}{x}$.

$$\therefore \quad dz = pdx + qdy = \left(\frac{\sqrt{a}}{x\sqrt{x}} - \frac{z}{x}\right)dx + \frac{a}{x}dy$$

or, $\quad xdz + zdx = \sqrt{a}\frac{dx}{\sqrt{x}} + ady$

or, $\quad d(xz) = \sqrt{a}x^{-\frac{1}{2}}dx + ady.$

Integrating $xz = 2\sqrt{a}\sqrt{x} + ay + b$ (a, b being arbitrary constants)

(ii) Similar method: A complete integral: $yz = ax + \sqrt{ay} + b$.

▶ **Example 3.2.8** *Solve for a complete integral of* $yzp^2 - q = 0$.

Try yourself.

Solution: $z^2(a^2 - y^2) = (x+b)^2$ or $z^2 = 2ax + a^2y^2 + b$.

▶ **Example 3.2.9** *Find a complete integral, a singular solution and general solution of* $(p^2+q^2)y = qz$. **[Delhi B.Sc. Hons 1989]**

Solution: Here the given equation is

$$F(x,y,z,p,q) = (p^2+q^2)y - qz = 0. \tag{1}$$

$\therefore$ Charpit's auxiliary equations are

$$\frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dz}{-2p^2y+qz-2q^2y} = \frac{dx}{-2py} = \frac{dy}{-2q+z}. \tag{2}$$

Taking the first two fractions, we get $pdp + qdq = 0$ and hence (on integrating)

$$p^2 + q^2 = a^2 \tag{3}$$

(3) and (1) give $q = \frac{a^2y}{z}$ and $p = \frac{a}{z}\sqrt{z^2 - a^2y^2}$.

Putting these values of p and q in $dz = pdx + qdy$, we get

$$dz = \frac{a}{z}\sqrt{z^2 - a^2y^2}dx + \frac{a^2y}{z}dz$$

or, $$\frac{z\,dz - a^2y\,dy}{\sqrt{(z^2 - a^2y^2)}} = adx.$$

Integrating, $(z^2 - a^2y^2)^{1/2} = ax + b$

$$z^2 - a^2y^2 = (ax+b)^2 \tag{4}$$

which is a required **complete integral**.

To find **singular integral** we differentiate this complete integral partially w.r.t. a and b, and obtain

$$0 = 2ay^2 + 2(ax+b)\cdot x \tag{5}$$
$$0 = 2(ax+b). \tag{6}$$

Eliminating a and b between (4), (5) and (6) we get $z = 0$ which clearly satisfies (1) ($\because p = 0, q = 0$) and hence it is the **singular integral**.

General integral: Replacing b by some function of a, say $b = \phi(a)$ in (4) we get

$$z^2 - a^2y^2 = [ax + \phi(a)]^2. \tag{7}$$

Differentiating (7) partially w.r.t. a we get

$$-2ay^2 = 2[ax + \phi(a)][x + \phi'(a)]. \tag{8}$$

The general integral is obtained by eliminating from a (7) and (8).

▶ **Example 3.2.10** *Find a complete and singular integrals of*

$$2xz - px^2 - 2qxy + pq = 0.$$

[Delhi Hons. 1998, 2000]

Solution: Here the given equation is

$$F = 2xz - px^2 - 2qxy + pq = 0. \tag{1}$$

∴ Charpit's auxiliary equations are

$$\frac{dp}{2z - 2qy} = \frac{dq}{0} = \frac{dz}{px^2 + 2xyq - 2pq} = \frac{dx}{x^2 - q} = \frac{dy}{2xy - p}$$

The second fraction gives $dq = 0$ **or** $q = a$.

Putting $q = a$ in (1) we get $p = \frac{2x(z-ay)}{x^2-a}$.

Hence from $dz = pdx + qdy$ we deduce

$$dz = \frac{2x(z-ay)}{x^2-a}dx + ady \quad \textbf{or,} \quad \frac{dz - ady}{z - ay} = \frac{2x\,dx}{x^2 - a}.$$

Hence, on integration,

$$\log(z - ay) = \log(x^2 - a) + \log b$$

$$\textbf{or,} \quad z - ay = b(x^2 - a) \quad \text{or,} \quad z = ay + b(x^2 - a) \tag{2}$$

is the **required complete integral**, a, b being arbitrary constants.

Differentiating the complete integral w.r.t. a and b we get

$$0 = y - b \text{ and } 0 = x^2 - a, \text{ i.e., } a = x^2, b = y$$

substituting these values of a and b in (2) we get $z = x^2y$ which is the **required singular integral**.

Examples III

(Exercises based on Charpit's Method)

Find a complete integral of the following partial differential equations: (Using Charpit's auxiliary equations)

1. $q = px + p^2$.
2. $pxy + pq + qy = yz$.
3. $p^2 + q^2 - 2px - 2qy + 1 = 0$. **[I.A.S. 1999]**
4. $z = px + qy + p^2 + q^2$. **[I.A.S. 1996]**
5. $p^2 + q^2 - 2px - 2qy + 2xy = 0$.
6. $p^2x + q^2y = z$.
7. $2x(q^2z^2 + 1) = pz$. **[I.A.S. 1998]**
8. $2z + p^2 + qy + 2y^2 = 0$.
9. $(p + q)(px + qy) = 1$.
10. $2(z + px + qy) = yp^2$.
11. $p - 3x^2 = q^2 - y$.
12. $p(q^2 + 1) + (b - z)q = 0$.

Answers

1. $z = \frac{x^2}{4} \pm \frac{1}{2}\left[\frac{x}{2}\sqrt{x^2+4a} + 2a\log\{x + \sqrt{x^2 - 4a}\}\right] + ay + b$. **2.** $(z - ax)(y + a)^a = be^y$. **3.** $(a^2 + 1)z = \frac{1}{2}v^2 \pm \left\{\frac{1}{2}v\sqrt{v^2 - (a^2+1)}\right\} - \frac{1}{2}(a^2 + 1)\log\left\{v + \sqrt{v^2 - (a^2+1)}\right\} + b$ where $v = ax + y$. **4.** $z = ax + by + a^2 + b^2$. **5.** $2z = x^2 + y^2 + ax + ay \pm \frac{1}{\sqrt{2}}\left\{(x - y)\sqrt{(x-y)^2 - \frac{a^2}{2}}\right\} - \frac{a^2}{2}\log\left[(x - y) + \sqrt{(x+y)^2 - \frac{a^2}{2}}\right]$. **6.** $(1 + a)^{1/2}\sqrt{z} = \sqrt{a}\sqrt{x} + \sqrt{y} + b$. **7.** $z^2 + 2(a^2 + 1)x^2 + 2ay + b$. **8.** $2y^2z + y^2(x - a)^2 + y^4 = b$. **9.** $z(a + 1)^{1/2} = 2(ax + b)^{1/2} + b$. **10.** $yz - a(x/y) + (a^2/4y^2) = b$. **11.** $z = x^2 + ax \pm \frac{2}{3}(y + a)^{3/2} + b$. **12.** $2\sqrt{a(z - b) - 1} = x + ay + b$.