PARTIAL DERIVATIVES

The ordinary derivative of a function of several variables with respect to one of the independent ~~variables with respect to one of the independent~~ variables, keeping all other independent variables constant is called the partial derivative of the function with respect to the variables. Partial derivatives of $f(x,y)$ with respect to x is generally denoted by $\frac{\partial f}{\partial x}$ or f_x or $f_x(x,y)$, while those with respect to y are denoted by $\frac{\partial f}{\partial y}$ or f_y or $f_y(x,y)$.

$$\therefore \frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$$

and $$\frac{\partial f}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}$$

when these limits exist.

The partial derivatives at a particular point (a,b) are often denoted by

$$\left[\frac{\partial f}{\partial x}\right]_{(a,b)}, \quad \frac{\partial f(a,b)}{\partial x}, \quad \text{or} \quad f_x(a,b)$$

and $$\left[\frac{\partial f}{\partial y}\right]_{(a,b)}, \quad \frac{\partial f(a,b)}{\partial y}, \quad \text{or} \quad f_y(a,b)$$

$$\therefore f_x(a,b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a,b)}{h}$$

$$f_y(a,b) = \lim_{k \to 0} \frac{f(a, b+k) - f(a,b)}{k}$$

Second order derivative :-

$$f_{xx}(a,b) = \frac{\partial^2 f(a,b)}{\partial x^2} = \lim_{h \to 0} \frac{f_x(a+h, b) - f_x(a,b)}{h}$$

$$f_{xy}(a,b) = \frac{\partial^2 f(a,b)}{\partial x \partial y} = \lim_{h \to 0} \frac{f_y(a+h, b) - f_y(a,b)}{h}$$

$$f_{yx}(a,b) = \frac{\partial^2 f(a,b)}{\partial y . \partial x} = \lim_{k \to 0} \frac{f_x(a, b+k) - f_x(a,b)}{k}$$

$$f_{yy}(a,b) = \frac{\partial^2 f(a,b)}{\partial y^2} = \lim_{k \to 0} \frac{f_y(a, b+k) - f_y(a,b)}{k}$$

Provided limits exist.

— o —

Ex: 38 If $f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y}, & \text{when } x \neq y \\ 0 & \text{when } x = y \end{cases}$

Show that the partial derivative of f exist at $(0,0)$. Examin whether f is continuous at $(0,0)$ [CH'86]

Soln: Now
$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h}$$
$$= \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$
$$= \lim_{h\to 0} \frac{h^2-0}{h} = \lim_{h\to 0} h = 0.$$

$$f_y(0,0) = \frac{\partial f(0,0)}{\partial y}$$
$$= \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{-k^2-0}{k} = -\lim_{k\to 0} k = 0.$$

Hence the partial derivatives $\dfrac{\partial f(0,0)}{\partial x}$ and $\dfrac{\partial f(0,0)}{\partial y}$ exist.

2nd part: Take $x-y = mx^3$ or $y = x - mx^3$

Then $f(x,y) = f(x, x-mx^3)$
$$= \frac{x^3 + x^3(1-mx^2)^3}{mx^3} \to \frac{2}{m} \text{ as } x \to 0.$$

This is different for different values of m. Thus the simultaneous limit does not exist. Hence the function $f(x,y)$ is not cont. at $(0,0)$

— o —

Ex: 39 Let $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Prove that $f_x(0,0)$ and $f_y(0,0)$ both exist but $f(x,y)$ is not continuous at $(0,0)$ [WH'92]

Soln: Now,
$$\frac{\partial f(0,0)}{\partial x} = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$
$$\frac{\partial f(0,0)}{\partial y} = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0.$$

2nd part:-

Let $(x,y) \to (0,0)$ along the straight line $y=mx$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,mx)\to(0,0)} \frac{mx^2}{x^2+m^2x^2} = \frac{m}{1+m^2}.$$

this is different for different values of m.

Hence the simultaneous limit does not exist.

Therefore the given function f is not continuous at $(0,0)$ all though the partial derivatives $f_x(0,0)$ and $f_y(0,0)$ exists.

— 0 —

Ex: 30 If $f(x,y) = \begin{cases} xy\,\dfrac{x^2-y^2}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

examine whether $f_{xy}(0,0)$ and $f_{yx}(0,0)$ exist or not.

Soln: Now, $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0.$

$$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k)-f(0,k)}{h}$$

$$= \lim_{h\to 0} \frac{hk\frac{(h^2-k^2)}{(h^2+k^2)} - 0}{h} = \lim_{h\to 0} k\left(\frac{h^2-k^2}{h^2+k^2}\right)$$

$$= k\left(\frac{0-k^2}{0+k^2}\right) = -k$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0}\frac{0-0}{k} = 0.$$

$$f_y(h,0) = \lim_{k\to 0} \frac{f(h,k)-f(h,0)}{k}$$

$$= \lim_{k\to 0} \frac{hk\left(\frac{h^2-k^2}{h^2+k^2}\right)-0}{k} = \lim_{k\to 0} h\left(\frac{h^2-k^2}{h^2+k^2}\right)$$

$$= \frac{h(h^2-0)}{(h^2+0)} = h.$$

$$\therefore f_{xy}(0,0) = \lim_{h\to 0} \frac{f_y(h,0)-f_y(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h-0}{h} = 1.$$

$$f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(h,0) - f_x(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{-k-0}{k} = -1$$

Hence $f_{xy}(0,0) \neq f_{yx}(0,0)$.

—— o ——

Ex: If $f(x,y) = (x^2+y^2)\tan^{-1}(y/x)$, when $x \neq 0$ and $f(0,y) = \frac{\pi y^2}{2}$, then show that $f_{xy}(0,0) \neq f_{yx}(0,0)$. [UCH'88] [CCH'97(2)]

Solⁿ: Now

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} \quad \text{——— ①}$$

$$f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} \quad \text{——— ②}$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h}$$

$$= \lim_{h\to 0} \frac{(h^2+k^2)\tan^{-1}\left(\frac{k}{h}\right) - \frac{\pi k^2}{2}}{h} \quad \left[\tfrac{0}{0} \text{ form}\right]$$

$$= \lim_{h\to 0} \frac{2h\tan^{-1}\left(\frac{k}{h}\right) + \frac{h^2+k^2}{1+\frac{k^2}{h^2}}\left(-\frac{k}{h^2}\right) - 0}{1}$$

$$= \lim_{h\to 0} \left[2h\tan^{-1}(k/h) - k\right] = 0 - k = -k.$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{\frac{\pi k^2}{2} - 0}{k} = \frac{\pi}{2}\lim_{k\to 0} k = 0.$$

$$f_y(h,0) = \lim_{k\to 0} \frac{f(h,k) - f(h,0)}{k}$$

$$= \lim_{k\to 0} \frac{(h^2+k^2)\tan^{-1}\left(\frac{k}{h}\right) - 0}{k}$$

Scanned by CamScanner

$$= \lim_{k\to 0} \frac{(h^2+k^2)\tan^{-1}(\frac{k}{h}) - 0}{k} \quad \left(= \frac{0}{0} \text{ form}\right)$$

$$= \lim_{k\to 0} \left[2k\tan^{-1}\left(\frac{k}{h}\right) + \frac{k^2+h^2}{1+\frac{k^2}{h^2}} \times \frac{1}{h}\right]$$

$$= 0 + h = h.$$

Then from ① and ②, we have

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{h-0}{h} = 1$$

$$f_{yx}(0,0) = \lim_{k\to 0} \frac{-k-0}{k} = -1$$

$\therefore f_{xy}(0,0) \neq f_{yx}(0,0)$

<u>Ex</u>: 32 Let $f(x,y) = xy\frac{x^2-y^2}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$, Prove that

(i) f, f_x, f_y are continuous in (x,y)

(ii) f_{xy} and f_{yx} exist at every point (x,y) and are continuous except at $(0,0)$

(iii) $f_{xy}(0,0) = 1$ and $f_{yx}(0,0) = -1$

<u>Ex</u>: 33 Let $f(x,y) = \frac{1}{4}(x^2+y^2)\operatorname{Log}(x^2+y^2)$, when $(x,y) \neq (0,0)$ and $f(0,0) = 0$, show that $f_{xy} = f_{yx}$ ~~and $f(0,0)=0$~~ at all points (x,y). Also show that neither of the derivative is continuous in (x,y) at the origin.

<u>Soln</u>: For $x \neq 0, y \neq 0$, we have

$$f_x = \frac{1}{2}x\{1 + \operatorname{Log}(x^2+y^2)\}$$

$$f_y = \frac{1}{2}y\{1 + \operatorname{Log}(x^2+y^2)\}$$

and hence $f_{xy} = f_{yx} = \frac{xy}{x^2+y^2}$.

Scanned by CamScanner

For $x=0, y=0$, it can be easily shown using the definition of partial derivatives that
$$f_x = f_y = f_{xy} = f_{yx} = 0.$$
Hence $f_{xy} = f_{yx}$ at every point.

We now shown that the simultaneous limits $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist. For if we let (x,y) approach $(0,0)$ through the line $y=mx$, we have

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x\cdot m\cdot x}{x^2+m^2x^2} = \frac{m}{1+m^2}.$$

Hence the limit does not exist, since it depends upon 'm',

It following that $f_{xy} = f_{yx}$ is not continuous at the origin.

—o—

Ex: 34 Prove that $f_{xy}(0,0) \neq f_{yx}(0,0)$ if
$$f(x,y) = \begin{cases} xy\,\dfrac{x^2+k^2y^2}{x^2+y^2}, & x^2+y^2 \neq 0, \\ 0, & x^2+y^2 = 0, \end{cases}$$
Also given $k^2 \neq 1$.

—o—

A sufficient condition for continuity :-

A sufficient condition that a function f be continuous at (a,b) is that one of the partial derivatives exists and be bounded in a neighbourhood of (a,b) and that the other exists at (a,b).

Let f_x exist and be bounded in a neighbourhood of (a,b) and let $f_y(a,b)$ exists, then for any ~~function~~ point $(a+h, b+k)$ of this neighbourhood we have
$$f(a+h, b+k) - f(a,b) = h f_x(a+\theta h, b+k) + k[f_y(a,b) + \eta]$$
where $0<\theta<1$, and $\eta \to 0$ as $k\to 0$

Proceeding to limits as $(h,k)\to(0,0)$ since $f_x(a+\theta h, b+k)$ is bounded, we have

$$\lim_{(h,k)\to(0,0)} f(a+h, b+k) = f(a,b)$$

$\Rightarrow f$ is continuous at (a,b)

Differentiability :-

Let (x,y), $(x+\delta x, y+\delta y)$ be two neighbourhood points in the domain of definition of a function. The changes δf in the function as the point changes from (x,y) to $(x+\delta x, y+\delta y)$ is given by

$$\delta f = A\delta x + B\delta y + \delta x\,\phi(\delta x,\delta y) + \delta y\,\psi(\delta x,\delta y)$$

where A and B are constant independent of δx, δy and ϕ,ψ are functions of $\delta x, \delta y$ tending to zero as $\delta x, \delta y$ tend to zero simultaneous,

Also, $A\delta x + B\delta y$ is ~~then~~ called the differential of 'f' at (x,y) and is denoted by df. thus

$$df = A\delta x + B\delta y$$

From (1) when $(\delta x,\delta y)\to(0,0)$, we get

$$f(x+\delta x, y+\delta y) - f(x,y) \to 0.$$

or, $$f(x+\delta x, y+\delta y) \to f(x,y)$$

$\Rightarrow$ the function is continuous at (x,y)

* thus every differentiable function is continuous

Again, from (1), when $\delta y = 0$.

$$\delta f = A\delta x + \delta x\,\phi(\delta x, 0)$$

Dividing by δx and proceeding to limits as $\delta x\to 0$ we get,

$$\frac{\partial f}{\partial x} = A \quad , \quad \text{similarly } \frac{\partial f}{\partial y} = B.$$

thus the constants A and B are respectively the partial derivatives of 'f' with respect to x and y.

Hence a function which is differentiable at a point possesses the first order partial derivatives there at.

Converse, of course is not true, so that func exist which are continuous and may even possesses partial derivatives at point but are not differentiable there at.

Again the differential of f is given by

$$\delta f = A\delta x + B\delta y = \frac{\partial f}{\partial x}\cdot \delta x + \frac{\partial f}{\partial y}\cdot \delta y$$

Taking $f = x$, we get $dx = \delta x$.

Similarly taking $f = y$; we obtain $dy = \delta y$

Thus the differentiables dx, dy of x, y are respectively δx, and δy and

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y}\cdot dy$$
$$= f_x dx + f_y dy$$

is differentiable of f at (x, y)

Note: If we replace $\delta x, \delta y$ by (h, k) in equⁿ ①, we say that the function is differential at a point (a, b) of the domain of definition if df can be expressed as

$$df = f(a+h, b+k) - f(a, b)$$
$$= Ah + Bk + h\phi(h,k) + k\psi(h,k)$$

where $A = f_x$, $B = f_y$ and ϕ, ψ are functions of h, k tending to zero as h, k tend to zero simultaneously.

CU'96, VH'90

A sufficient condition for Differentiability

Theorem:- If (a, b) be a point of the domain of definition of a function such that

(i) f_x is continuous at (a, b)

(ii) f_y exists at (a, b), then f is differentiable

Proof: The condition (i) implies that f_x exists in a certain neighbourhood $(a-\delta, a+\delta;\ b-\delta, b+\delta)$ of (a, b). Let $(a+h, b+k)$ be a point of this neighbourhood, thus

$$df = f(a+h, b+k) - f(a, b)$$

$$= f(a+h, b+k) - f(a, b+k) + f(a, b+k) - f(a, b) \longrightarrow (1)$$

Since f_x exists in $(a-\delta, a+\delta;\ b-\delta, b+\delta)$, applying Lagrange's mean value theorem, we get

$$f(a+h, b+k) - f(a, b+k)$$

$$= h f_x(a+\theta h, b+k) \longrightarrow (2)$$

where $0<\theta<1$, and depends on h and k.

Again, since f_x is continuous at (a, b), therefore

$$\lim_{(h,k)\to(0,0)} f_x(a+\theta h, b+k) = f_x(a, b)$$

So that we can write

$$f_x(a+\theta h, b+k) = f_x(a, b) + \phi(h, k) \longrightarrow (3)$$

where $\phi(h, k) \to 0$ as $(h, k) \to (0, 0)$

~~Again~~, since by condition (ii) $f_y(a, b)$ exists, therefore

$$\lim_{k\to 0} \frac{f(a, b+k) - f(a, b)}{k} = f_y(a, b)$$

So that we can write

$$\frac{f(a, b+k) - f(a, b)}{k} = f_y(a, b) + \psi(k) \longrightarrow (4)$$

where $\psi(k) \to 0$ as $k \to 0$

$\therefore$ from (1), (2), (3) & (4) we get

$$df = h f_x(a, b) + k f_y(a, b) + h\phi(h, k) + k\psi(k).$$

$\Rightarrow f$ is differentiable at (a, b).

—o—

Note: In a similar way it can be shown that f is differentiable at (a, b), if f_x exists and f_y is continuous at (a,b).

Young's theorem :-

Statement :- If f_x and f_y are both differentiable at a point (a, b) of the domain of definition of a function f, then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Proof :-

The differentiability of f_x and f_y at (a,b) implies that they exist in a certain neighbourhood of (a, b) and that all the second order partial derivatives f_{xx}, f_{xy}, f_{yx} exist at (a, b).

We prove the theorem by taking equal increment is both for x and y and calculating $\phi(h,h)$ in two different way.

Let $(a+h, b+k)$ be a point of this neighbourhood. Consider,

$$\phi(h,h) = f(a+h, b+h) - f(a+h, b) - f(a, b+h) + f(a,b)$$

$$G(x) = f(x, b+h) - f(x,b)$$

so that,

$$\phi(h,k) = G(a+h) - G(a) \quad \text{——} (1)$$

Since f_x exists in a neighbourhood of (a,b), the function $G(x)$ is derivable in $(a, a+h)$ and therefore by Lagrange's mean value theorem we get from (1)

$$\phi(h,k) = hG'(a+\theta h)$$

$$\phi(h,h) = hG'(a+\theta h), \quad 0 < \theta < 1$$

$$= h\{f_x(a+\theta h, b+h) - f_x(a+\theta h, b)\} \quad \text{——} (2)$$

Again, since f_x is differentiable at (a,b) we have

$$f_x(a+\theta h, b+h) - f_x(a,b) = \theta h f_{xx}(a,b) + h f_{yx}(a,b) + \theta h\phi_1(h,h) + h\psi_1(h,h) \quad \text{——} (3)$$

and
$f_x(a+\theta h, b) - f_x(a,b) = \theta h f_{xx}(a,b) + \theta h \phi_2(h,h)$ —— (4)

where ϕ_1, ψ_1, ϕ_2 all tend to zero as $h \to 0$.

From (2), (3), (4), we get

$$\frac{\phi(h,h)}{h^2} = f_{yx}(a,b) - \theta\phi_1(h,h) + \psi_1(h,h) - \theta\phi_2(h,h) \quad \text{—— (5)}$$

By a similar argument, on considering

$H(y) = f(a+h, y) - f(a,y)$

we can shown that

$$\frac{\phi(h,h)}{h^2} = f_{xy}(a,b) + \phi_3(h,h) + \theta'\psi_2(h,h) - \theta'\psi_3(h,h) \quad \text{—— (6)}$$

where ϕ_3, ψ_2, ψ_3 all tend to zero as $h \to 0$, we obtain on taking the limit as $h \to 0$, we obtain from (5) and (6)

$$\lim_{h\to 0} \frac{\phi(h,h)}{h^2} = f_{xy}(a,b) = f_{yx}(a,b).$$

—o—

Schwarz's theorem:- [CU' 85, 87'; 89, 96]

Statement:- If f_y exists in a certain nbd of a point (a,b) of the domain of definition of function f, and f_{yx} is continuous at (a,b), then $f_{xy}(a,b)$ exists and is equal to $f_{yx}(a,b)$

Proof! Under the given conditions f_x, f_y and f_{yx} all exist in a certain neighbourhood of (a,b). Let $(a+h, b+k)$ be a point of this nbd

Consider,

$\phi(h,k) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a$

So that $\phi(h,k) = G(a+h) - G(a)$ —— (1)

Since f_x exist in a nbd of (a,b), the functio $G(x)$ is derivable in $(a, a+h)$ and therefore by Lagrange's mean value theorem, we get fro

$$\varphi(h,k) = hG'(a+\theta h), \quad 0<\theta<1$$
$$= h\{f_x(a+\theta h, b+k) - f_x(a+\theta h, b)\} \quad (2)$$

~~Again~~, since f_{yx} exists in a nbd of (a,b) the function f_x is derivable with respect to y in $(b, b+k)$, and therefore by Lagrange mean value theorem, we get from (2)

$$\varphi(h,k) = hk f_{yx}(a+\theta h, b+\theta' k), \quad 0<\theta<1.$$

or, $\frac{1}{h}\left\{\frac{f(a+h, b+k) - f(a+h, b)}{k} - \frac{f(a, b+k) - f(a,b)}{k}\right\}$

$$= f_{yx}(a+\theta h, b+\theta' k)$$

Proceeding to limits when $k\to 0$ since f_y and f_{yx} exist in a nbd of (a,b), we get

$$\frac{f_y(a+h, b) - f_y(a,b)}{h} = \lim_{k\to 0} f_{yx}(a+\theta h, b+\theta' k)$$

~~Again~~, taking limits as $h\to 0$, since f_{yx} is continuous at $f(a,b)$, we get

$$f_{xy}(a,b) = \lim_{h\to 0}\lim_{k\to 0} f_{yx}(a+\theta h, b+\theta' k) = f_{yx}(a,b)$$

—o—

Note: If f_{xy} and f_{yx} are both continuous at (a,b), then $f_{xy}(a,b) = f_{yx}(a,b)$ for the assumption of continuity of both these derivatives is a wider assumption.

—o—

Ex: 35 Show that for the function

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$f_{xy}(0,0) = f_{yx}(0,0)$, even though the conditions of Schwarz's theorem and also of Young's theorem are not satisfied.

Scanned by CamScanner

Soln: Now,
$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

Similarly, for $(x,y) \neq (0,0)$

$$f_x(x,y) = \frac{(x^2+y^2)(2xy^2) - (2x)x^2y^2}{(x^2+y^2)^2}$$

$$= \frac{2xy^4}{(x^2+y^2)^2}$$

and $f_y(x,y) = \dfrac{2x^4y}{(x^2+y^2)^2}$

Again $f_{yx}(0,0) = \lim_{k\to 0} \dfrac{f_x(0,k) - f_x(0,0)}{k}$

$$= \lim \frac{0-0}{k} = 0.$$

and $f_{xy}(0,0) = 0$.

So that $f_{xy}(0,0) = f_{yx}(0,0) = 0$.

For $(x,y) \neq (0,0)$, we have

$$f_{yx}(x,y) = \frac{8xy^3(x^2+y^2)^2 - 2xy^4(x^2+y^2).4y}{(x^2+y^2)^4}$$

$$= \frac{8x^3y^3}{(x^2+y^2)^3}$$

Now, we consider the point (x,y) approaches to $(0,0)$ along $y=mx$.

then $f_{yx}(x,y) \longrightarrow \dfrac{8m^3}{(1+m^2)^3}$ as $x\to 0$.

$\therefore$ the simultaneous limit $\lim_{(x,y)\to(0,0)} f_{yx}(x,y)$ does not exist.

$\therefore f_{yx}(x,y)$ is not continuous at $(0,0)$.

Thus the condition of the Schwarch theorem is not satisfied.

Now f_x is defferentable at $(0,0)$

$$f_x(h,k) = f_x(0,0) + f_{xx}(0,0).h + f_{yx}(0,0).k + h\phi + k\psi.$$

or, $\dfrac{2hk^4}{(h^2+k^2)^2} = h\varphi + k\psi$, where φ, ψ tends to zero as $(h,k)\to(0,0)$

Putting $h = \rho\cos\theta$, $k = \rho\sin\theta$. and dividing by ρ, we get

$$2\cos\theta \sin^4\theta = \cos\theta \cdot \varphi + \sin\theta \cdot \psi$$

and $(h,k)\to(0,0)$ is same.

then as $\rho\to 0$ and θ is arbitrary, thus proceeding the limits, we get

$$2\cos\theta \cdot \sin^4\theta = 0$$

which is impossible for arbitrary θ.

$\Rightarrow$ f_x is not differentiable at $(0,0)$

Similarly, it may be shown that f_y is not differentiable at $(0,0)$.

Hence, the conditions of Young's theorem are also not satisfied, but

$$f_{xy}(0,0) = f_{yx}(0,0)$$

VH

Ex: 36 Let $f(x,y) = \begin{cases} (x^2+y^2)\log(x^2+y^2) & , (x,y)\neq(0,0) \\ 0 & , (x,y)=(0,0) \end{cases}$

Show that both f_{xy} and f_{yx} are not continuous at $(0,0)$ but

$$f_{xy}(0,0) = f_{yx}(0,0)$$

Hence interpret whether Schwarz's theorem gives a necessary or sufficient conditions for the equality of derivatives.

Sol^n Now, $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h}$

$$= \lim_{h\to 0} \frac{h^2\log h^2 - 0}{h} = 0.$$

$$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h}$$

$$= \lim_{h\to 0} \frac{(h^2+k^2)\log(h^2+k^2) - k^2 \log k^2}{h}, \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim \frac{2h - 0}{1} = 0.$$

$$\therefore f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k}$$

$$= \lim \frac{0-0}{k} = 0$$

Similarly, $f_{xy}(0,0) = 0$

Now, for $(x,y) \neq 0$

$$f_x(x,y) = 2x\log(x^2+y^2) + 2x$$

$$= 2x[\log(x^2+y^2) + 1]$$

$$f_{yx}(x,y) = \frac{4xy}{x^2+y^2}$$

Putting $y = mx$, we get,

$$\lim_{(x,y)\to(0,0)} f_{yx}(x,y) = \frac{4m}{1+m^2} \neq 0 = f_{yx}(0,0)$$

$\therefore$ f_{yx} is not continuous and f_{yx} is not continuous at $(0,0)$ but $f_{yx}(0,0) = f_{xy}(0,0)$

This implies that the condition of Schwarz's theorem are sufficient but not necessary.

Ex: 37 Show that the function

$$f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is continuous and possesses partial derivatives at the point $(0,0)$, but is not differentiable at $(0,0)$.

[CU '89
CH '98]

Soln.

Continuity: Put $x = r\cos\theta$, $y = r\sin\theta$

$$\left|\frac{x^3-y^3}{x^2+y^2}\right| = \left|\frac{r^3(\cos^3\theta - \sin^3\theta)}{r^2}\right| = r\left|\cos^3\theta - \sin^3\theta\right|$$

$$\le r(|\cos^3\theta| + |-\sin^3\theta|)$$

$$\le 2r = 2\sqrt{x^2+y^2} < \epsilon$$

if $x^2 < \epsilon^2/4$, and $y^2 < \epsilon^2/4$.

ie, $\left|\frac{x^3-y^3}{x^2+y^2}\right| < \epsilon$, if $|x| < \epsilon/2 = \delta$
$|y| < \epsilon/2 = \delta$.

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = 0.$$

ie, $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

Hence the function is continuous at $(0,0)$

Differentiability:-

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{h^3-0}{h^3} = 1$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0}\frac{-k-0}{k} = -1.$$

Thus the function possesses partial ~~deri~~ at $(0,0)$. If the function is differentiable at $(0,0)$, then by the definition

$$df = f(h,k) - f(0,0)$$
$$= Ah + Bk + h\phi + k\psi \quad \text{--- (1)}$$

where A and B are constants.
ie, $A = f_x(0,0) = 1$, $B = f_y(0,0) = -1$ and ϕ, ψ tends to zero as $(h,k)\to(0,0)$.

Putting $h = \rho\cos\theta$, $k = \rho\sin\theta$ and dividing by ρ, we get

$$\cos^3\theta - \sin^3\theta = \cos\theta - \sin\theta + \phi\cos\theta - \psi\sin\theta \quad \text{--- (2)}$$

For arbitrary θ, $\rho\to 0$ implies that $(h,k)\to(0,0)$
thus we get the limit

$$\cos^3\theta - \sin^3\theta = \cos\theta - \sin\theta$$
$$\sin\theta\cos\theta(\cos\theta - \sin\theta) = 0$$

which is impossible for arbitrary θ. Hence the function is not differentiable at the origin.

—o—

Ex. 38 Show that the function f, where

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2 \neq 0 \\ 0, & x=y=0. \end{cases}$$

is continuous, possesses the partial derivatives, but is not differentiable at the origin.

[LH'92, CU'84], 2008

Soln: Continuity :-

Differentiability :-

If the function f is differentiable at $(0,0)$ then by the definition

$$df = Ah + Bk + h\phi + k\psi \quad \text{——— (1)}$$

where $A = f_x(0,0) = 0$, $B = f_y(0,0) = 0$.

and ϕ, ψ tends to zero as $(h,k) \to (0,0)$

$$\therefore \frac{hk}{\sqrt{h^2+k^2}} = h\phi + k\psi$$

Put $h = \rho\cos\theta$, $k = \rho\sin\theta$.

Then,

$$\sin\theta\cos\theta = 0 + 0 + \phi\cos\theta + \psi\sin\theta \quad \text{——— (2)}$$

For arbitrary θ, $\rho \to 0$ implies $(h,k) \to (0,0)$ and then we have

$$\sin\theta \cdot \cos\theta = 0$$

$$\Rightarrow \sin 2\theta = 0$$

which is not possible for arbitrary θ.

Hence this function is not differentiable at $(0,0)$

—o—

Ex. 39 Prove that the function $f(x,y) = \sqrt{|xy|}$ is not differentiable at the point $(0,0)$ but that f_x and f_y exist both at the origin and have the value 0. Show that f_x and f_y are continuous except $(0,0)$.

[LH'90]

Scanned by CamScanner

Soln: Now, $f_x(0,0) = \lim_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h}$

$$= \lim_{h\to 0} \frac{\sqrt{0}-0}{h} = 0$$

and $f_y(0,0) = \lim_{k\to 0} \dfrac{f(0,k)-f(0,0)}{k}$

$$= \lim_{k\to 0} \frac{0-0}{k} = 0.$$

If the function is differentiable at $(0,0)$, then by definition

$$df = f(h,k)-f(0,0) = Ah + Bk + h\phi + k\psi$$

when $A = f_x(0,0) = 0$ and

$B = f_y(0,0) = 0$ and ϕ, ψ tend to zero as $(h,k)\to(0,0)$.

Put $h = \rho\cos\theta$, $k = \rho\sin\theta$.

$$\therefore |\cos\theta\sin\theta|^{1/2} = 0 + 0 + \cos\theta\cdot\phi + \psi\sin\theta.$$

or, $\sqrt{|\sin\theta\cos\theta|} = \phi\cos\theta + \psi\sin\theta$.

For arbitrary θ, $\rho\to 0 \Rightarrow (h,k)\to(0,0)$

Taking limit as $\rho\to 0$, we have

$$\sqrt{|\sin\theta\cos\theta|} = 0$$

which is impossible for arbitrary θ, Hence the function is not differentiable at $(0,0)$.

—o—

2nd part:

$f_x(x,y) = \lim_{h\to 0} \dfrac{f(x+h,y)-f(x,y)}{h}$

$$= \lim_{h\to 0} \frac{\sqrt{|(x+h)y|} - \sqrt{|x||y|}}{h}$$

$$= \lim_{h\to 0} \sqrt{|y|}\, \frac{|x+h|-|x|}{h\{\sqrt{|x+h|}+\sqrt{|x|}\}}$$

Now as $h\to 0$, we can take $x+h>0$ and $x+h<0$,

i.e. $|x+h| = x+h$ when $x>0$

$|x+h| = -(x+h)$, when $x<0$.

$\therefore \lim f_x(x,y) = \lim_{h\to 0} \frac{\{(x+h)-x\}\sqrt{|y|}}{h\{\sqrt{|x+h|}+\sqrt{|x|}\}}$

$= \frac{1}{2}\sqrt{\left|\frac{y}{x}\right|}, \quad x>0.$

$= \lim_{h\to 0} \frac{\{-(x+h)+x\}\sqrt{|y|}}{h(\sqrt{|x+h|}+\sqrt{|x|})}$

$= -\frac{1}{2}\sqrt{\frac{|y|}{|x|}}, \quad x<0.$

i.e. $f_x(x,y) = \begin{cases} \frac{1}{2}\sqrt{\left|\frac{y}{x}\right|}, & x>0 \\ -\frac{1}{2}\sqrt{\left|\frac{y}{x}\right|}, & x<0. \end{cases}$

similarly, $f_y(x,y) = \begin{cases} \frac{1}{2}\sqrt{\left|\frac{x}{y}\right|}, & y>0 \\ -\frac{1}{2}\sqrt{\left|\frac{x}{y}\right|}, & y<0. \end{cases}$

Obviously $f_x(x,y)$ and $f_x(x,y)$ are not continuous at $(x,y) \neq (0,0)$.

Ex: 41 Let $f(x,y) = \begin{cases} xy & \text{if } |x| \geq |y| \\ -xy & \text{if } |x| < |y| \end{cases}$

is $\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$ for this function at the origin? Given reason for your answer. [CH'98,96] [CH'2000]

soln: Now,

$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{h\cdot 0-0}{h} = 0$

$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k)-f(0,k)}{h} = \frac{-hk+0}{h} = -k$

$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim \frac{0-0}{k} = 0.$

$f_y(h,0) = \lim_{k\to 0} \frac{f(h,k)-f(h,0)}{k} = \lim_{k\to 0} \frac{hk-0}{k} = h.$

Now, $f_{xy}(0,0) = \frac{\partial^2 f(0,0)}{\partial x \partial y} = \lim_{h\to 0} \frac{f_y(h,0)-f_y(0,0)}{h}$

$f_{yx}(0,0) = \frac{\partial^2 f(0,0)}{\partial y \partial x} = \lim_{k\to 0} \frac{f_x(0,k)-f_x(0,0)}{k}$

$\therefore \frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$. (Proved)

Scanned by CamScanner

Ex:42 Show that $f(x,y)=|x|+|y|$ is continuous but not differentiable at $(0,0)$. [VH'2010]

Sol^n:- Now,

$$|f(x,y)-f(0,0)| = ||x|+|y|-0|$$

this for $\epsilon>0$, $\exists$ $\delta>0$ s.t $\quad = |x|+|y| < \epsilon$, if $|x|<\epsilon/2$, $|y|<\epsilon/2$

$|f(x,y)-f(0,0)|<\epsilon$, whenever $|x|<\delta$, $|y|<\delta$,

$\therefore \lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$ where $\delta=\epsilon/2$.

Hence $f(x,y)$ is continuous at $(0,0)$

Now, if the function is differentiable at $(0,0)$ by the definition

$$df = f(h,k)-f(0,0)$$
$$= Ah + Bk + \phi h + \psi k \longrightarrow (1)$$

where $A=f_x(0,0)$, $B=f_y(0,0)$ and ϕ,ψ are function of h and k and tends to zero as $(h,k)\to(0,0)$

Now, $f_x(0,0)=\lim\limits_{h\to0}\dfrac{f(h,0)-f(0,0)}{h} = \lim\limits_{h\to0}\dfrac{|h|}{h}=1$ or -1

ie, $f_x(0,0)$ does not exist and similarly $f_y(0,0)$ does not exist. Hence df is not exist therefore f is not differentiable at $(0,0)$.

CH'2010

Ex:43 Let $f(x,y)=\begin{cases} x^2\sin\frac{1}{x}+y^2\sin\frac{1}{y}, & x\neq0, y\neq0. \\ x^2\sin 1/x, & x\neq0, y=0 \\ y^2\sin 1/y, & x=0, y\neq0 \\ & x=0, y=0. \end{cases}$

Show that $f_x(x,y)$ and $f_y(x,y)$ are discontinuous at $(0,0)$ but that $f(x,y)$ is differentiable at $(0,0)$ [CH'2000]

Sol^n:- $f_x(x,y)=\begin{cases} 2x\sin\frac{1}{x}-\cos\frac{1}{x}, & x\neq0, y\neq0, \\ 0 & x=0, y\neq0 \end{cases}$

ie, $f_y(x,y)=\begin{cases} 2y\sin\frac{1}{y}-\cos\frac{1}{y}, & x\neq0, y\neq0 \\ 0 & x\neq0, y=0. \end{cases}$

$|x+h|$

so neither f_x nor f_y is continuous at $(0,0)$.

Again we have

$f(h,k) - f(0,0) = 0.h + 0.k + h\varphi + k.\psi$

or, $h^2 \sin\frac{1}{h} + k^2 \sin\frac{1}{k} = h\varphi + k.\psi$

Putting $h = \rho\cos\theta$, $k = \rho\sin\theta$ for arbitrary θ.

Now $\rho \to 0$ as $(h,k) \to (0,0)$

$\therefore \rho^2\cos^2\theta \sin\left(\frac{1}{\rho\cos\theta}\right) + \rho^2\sin^2\theta . \sin\left(\frac{1}{\rho\sin\theta}\right)$
$= \rho\cos\theta.\varphi + \psi\, \rho\sin\theta.$

or, $\rho\cos\theta \sin\left(\frac{1}{\rho\cos\theta}\right) + \rho\sin\theta \sin\left(\frac{1}{\rho\sin\theta}\right)$
$= \psi\sin\theta + \varphi\cos\theta.$

For arbitrary θ, as $\rho \to 0$.
L.H.S = R.H.S,
$\therefore$ f is differentiable at $(0,0)$.

Theorem: If $\lim_{(x,y)\to(a,b)} f(x,y) = l$, show that

$\lim_{x\to a} f(x,b) = l = \lim_{y\to b} f(a,y)$ [CH'91, CH'95].
① — ② — 91,92

Proof:-
From definition of limit it following that for every $\epsilon > 0$, $\exists$ $\delta > 0$ such that
$|f(x,y) - l| < \epsilon$, when $|x-a| < \delta$ and $|y-b| < \delta$.

Now, let y be kept fixed at $y = b$.
Therefore, this definition it following that
$|f(x,b) - l| < \epsilon$, when $|x-a| < \delta$, $|y-b| < \delta$.
ie, $|f(x,b) - l| < \epsilon$, whenever $|x-a| < \delta$.
$\Rightarrow \lim_{x\to a} f(x,b) = l$

Similarly, $\lim_{y\to b} f(a,y) = l$

$\therefore \lim_{x\to a} f(x,b) = l = \lim_{y\to b} f(a,y)$.

(Proved)
