12.3 PLANE DIFFRACTION GRATING :

(a) Construction :

Plane diffraction grating consists of a number of parallel and equidistant lines ruled on an optically plane and parallel glass plate by a fine diamond point. The number of such ruled lines per mm is of the order of 100. Each ruled line behaves as an opaque line while the transparent portion between two consecutive ruled lines behaves as a slit. If a be the width of a clear space and b be the width of a ruled line, then the distance $(a + b)$ is called *grating element* or *grating constant*. The two points in the consecutive clear spaces whose distance of separation is $(a + b)$, are called *corresponding points*.

Drawing of *exactly* parallel and equidistant lines on a glass plate by a diamond point is an extremely difficult task and hence the grating becomes too costly. To make it comparatively cheaper, a cast of this ruled surface is made with some transparent material and such casts are called *replica gratings*. Cellulose acetate, properly diluted, is poured on the surface of the master grating and then dried to a thin, tough

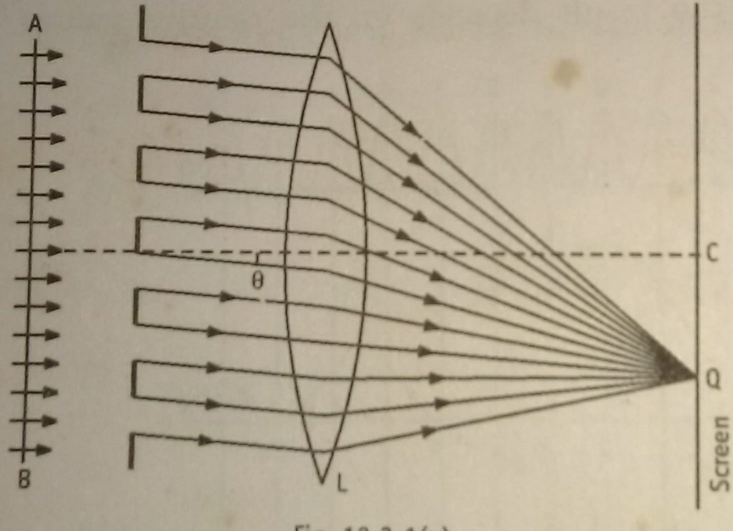

Fig. 12.3-1(a)

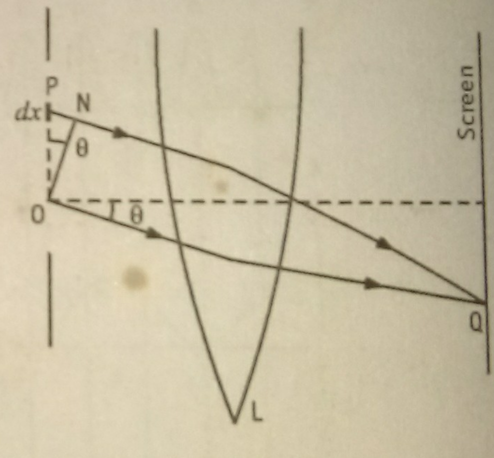

Fig. 12.3-1(b)

film which can be easily separated from the master grating under water. This film can then be mounted on a plane glass plate forming what is known as *replica grating*. Due to distortion and shrinkages of the film, replica seldom functions as well as its master.

(b) Theory :

Let a parallel beam of monochromatic light of wavelength λ be incident normally on a plane diffraction grating consisting of N slits each of width a and with equal opaque space b between two successive slits. According to Huygens-Fresnel principle every point of the incident wavefront in the plane of the slits may be regarded as the origin of secondary spherical wavelets. The secondary wavelets travelling normal to the slits are brought to focus by a convex lens L on the screen at C. The wavelets travelling at an angle θ with the normal are brought to focus at Q. Let us calculate the intensity of light at Q. Let the complex light disturbance at Q due to secondary wavelets from O (the centre of the 1st slit) be represented by $Ae^{i\omega t}$ where A is the amplitude and ω is the circular frequency. The ‡phase difference between the waves at Q [Fig. 12.3-1(b)] coming from O and from a point P at a distance x from O is given by

$$\frac{2\pi}{\lambda}\times PN = \frac{2\pi}{\lambda}\times x\sin\theta = lx \qquad \text{where } l = \frac{2\pi}{\lambda}\cdot\sin\theta \qquad ...(12.3\text{-}1)$$

Hence the disturbance at Q due to secondary waves from P will be proportional to $e^{i(\omega t - lx)}$. The disturbance at Q due to the diffracting element dx can be written as

$$dy = CA\,dx.e^{i(\omega t - lx)}$$

where we assume the amplitude to be proportional to the width dx ; C is a proportionality constant.

The resultant complex disturbance at Q due to all the slits will be

$$y = \int_{-a/2}^{+a/2} CA.e^{i(\omega t - lx)}dx + \int_{d-a/2}^{d+a/2} CA.e^{i(\omega t - lx)}dx + \ldots + \int_{(N-1)d-a/2}^{(N-1)d+a/2} CA.e^{i(\omega t - lx)}dx$$

where $d = a + b$ is the grating element.

$$\therefore\; y = CAe^{i\omega t}\left[\frac{e^{-ilx}}{-il}\right]_{-a/2}^{+a/2} + CAe^{i\omega t}\left[\frac{e^{-ilx}}{-il}\right]_{d-a/2}^{d+a/2} + \ldots + CAe^{i\omega t}\left[\frac{e^{-ilx}}{-il}\right]_{(N-1)d-a/2}^{(N-1)d+a/2}$$

$$= CAa\cdot\frac{\sin\frac{al}{2}}{\frac{al}{2}}\left[1 + e^{-ild} + \ldots + e^{-i(N-1)ld}\right].e^{i\omega t}$$

$$= CAa\frac{\sin\frac{al}{2}}{\frac{al}{2}}\cdot\frac{e^{-iNld}-1}{e^{-ild}-1}\cdot e^{i\omega t} \qquad ...(12.3\text{-}2)$$

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‡ N.B See foot note of page 330

The resultant intensity I at Q is obtained by multiplying y with its complex conjugate y^*. Thus

$$I = yy^* = (CAa)^2 \cdot \frac{\sin^2 \frac{al}{2}}{\left(\frac{al}{2}\right)^2} \cdot \frac{e^{-iNld} - 1}{e^{-ild} - 1} \cdot \frac{e^{+iNld} - 1}{e^{+ild} - 1}$$

$$= (CAa)^2 \cdot \frac{\sin^2 \frac{al}{2}}{\left(\frac{al}{2}\right)^2} \cdot \frac{2 - e^{iNld} - e^{-iNld}}{2 - e^{ild} - e^{-ild}}$$

$$= (CAa)^2 \cdot \frac{\sin^2 \frac{al}{2}}{\left(\frac{al}{2}\right)^2} \cdot \frac{2 - 2\cos Nld}{2 - 2\cos ld}$$

$$\left[\begin{array}{l} \because\ e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos\theta - i\sin\theta \end{array}\right]$$

$$= I_0 \cdot \frac{\sin^2 \alpha}{\alpha^2} \cdot \frac{\sin^2 N\beta}{\sin^2 \beta} \qquad \text{...(12.3-3)}$$

where $I_0 = (CAa)^2$, $\alpha = \frac{al}{2} = \frac{\pi}{\lambda} a \sin\theta$ and $\beta = \frac{\pi}{\lambda} d \sin\theta = \frac{\pi}{\lambda}(a+b)\sin\theta$

...(12.3-4)

The resultant intensity thus depends on two factors : (i) $I_0 \frac{\sin^2 \alpha}{\alpha^2} = I_1$ which gives the diffraction pattern of a single slit and (ii) $\frac{\sin^2 N\beta}{\sin^2 \beta} = I_2$ which gives the interference pattern of the diffracted light beams from N slits.

Note that by putting $N = 2$ you can get the double slit intensity pattern.

Principal maxima :

If the slit width a is very small and observation is confined to the neighbourhood of the central pattern the variation of the factor $\frac{\sin^2 \alpha}{\alpha^2}$ is small and under this condition the maxima will be solely controlled by the factor $I_2 = \frac{\sin^2 N\beta}{\sin^2\beta}$. This factor is maximum when $\beta = m\pi$: $m = 0, \pm 1, \pm 2, \ldots$

or,
$$(a + b)\sin\theta = m\lambda \qquad \ldots(12.3\text{-}5)$$

These are known as *principal maxima.*

For $\beta = m\pi$, $I_2 = \frac{0}{0}$ and hence indeterminate. But in limit $\beta \to m\pi$ we get the maximum value of I_2.

$$\lim_{\beta\to n\pi}\frac{\sin N\beta}{\sin\beta} = \lim_{\beta\to n\pi}\frac{N\cos N\beta}{\cos\beta} = N \quad \text{[using L Hospital's rule]}$$

This gives $I_2 = N^2$ and $I = I_{pm}(say) = I_0\frac{\sin^2\alpha}{\alpha^2}\times N^2 = I_1 \times N^2$

$$\ldots(12.3\text{-}6)$$

Thus the intensity of principal maxima increases as the number of slits (N) increases, but due to the presence of the factor $\frac{\sin^2\alpha}{\alpha^2}$, whose value decreases with the increase of the angle of diffraction (θ), the intensity of principal maxima decreases with the increase in the order number of bands.

Conditions for secondary minima and maxima :

The factor $I_2 = \frac{\sin^2 N\beta}{\sin^2\beta}$ depends on β and for maxima or minima we must have $\frac{dI_2}{d\beta} = 0$.

Now
$$\frac{dI_2}{d\beta} = \frac{2N\sin N\beta.\cos N\beta}{\sin^2\beta} - \frac{2\sin^2 N\beta.\cos\beta}{\sin^3\beta}$$

$$= 2\frac{\sin^2 N\beta}{\sin^2\beta}(N\cot N\beta - \cot\beta) \qquad \ldots(12.3\text{-}7)$$

Hence for maxima or minima, either $\frac{\sin N\beta}{\sin\beta} = 0$ or, $N\cot N\beta = \cot\beta$.

(i) Secondary minima :

When $\sin N\beta = 0$ but $\sin\beta \neq 0$ then $\dfrac{\sin N\beta}{\sin\beta} = 0$ and hence intensity becomes zero (i.e., minimum). Thus for minima

$$N\beta = \pm s\pi$$

or,
$$(a+b)\sin\theta = \pm\frac{s}{N}\lambda \qquad ...(12.3\text{-}8)$$

where s has integral values excepting 0, N, $2N$, $3N$, etc. as for these values of s, $\sin\beta = 0$ and we obtain principal maxima. Thus it is evident from Eqs. (12.3-5) and (12.3-8) that between two consecutive principal maxima there are $(N-1)$ minima. Hence there will be $N-2$ other maxima known as *secondary maxima* between any two adjacent principal maxima.

(ii) Secondary maxima :

The condition $N\cot N\beta = \cot\beta$ makes $\dfrac{dI_2}{d\beta} = 0$. Also, it can be shown that this condition makes $\dfrac{d^2I_2}{d\beta^2}$ negative. Thus the values of β which satisfy the conditon $N\cot N\beta = \cot\beta$ will give the positions of secondary maxima, excepting $\beta = m\pi$ which gives principal maxima.

Now to find the intensity of secondary maxima we note that

$$N\cot N\beta = \cot\beta$$

or,
$$N^2\frac{\cos^2 N\beta}{\cos^2\beta} = \frac{\sin^2 N\beta}{\sin^2\beta}$$

or,
$$\frac{N^2(1-\sin^2 N\beta)}{1-\sin^2\beta} = \frac{N^2\sin^2 N\beta}{N^2\sin^2\beta} = \frac{N^2}{1+(N^2-1)\sin^2\beta}$$

$$\left[\because \frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}\right]$$

Hence the intensity of secondary maxima is given by,

$$I_{sm} = I_1 \times \frac{N^2}{1+(N^2-1)\sin^2\beta} = \frac{I_{pm}}{1+(N^2-1)\sin^2\beta}$$

or,
$$\frac{I_{sm}}{I_{pm}} = \frac{1}{1+(N^2-1)\sin^2\beta} \qquad ...(12.3\text{-}9)$$

This equation shows that as N increases the intensity of secondary maxima relative to the principal maxima decreases. When N is very large the secondary maxima become very weak. This is why secondary maxima are not generally observed with a grating having large N.

The existence of secondary maxima can be clearly understood by making a plot of $\frac{\sin^2 N\beta}{\sin^2 \beta}$. Fig 12.3-2 shows such a plot with $N = 6$.

In Fig. 12.3-2 curves are drawn by plotting $\sin^2 N\beta$ against $N\beta$ [Fig. 12.3-2(a)], $\sin^2 \beta$ against β [Fig. 12.3-2(b)] and also by plotting the product of their ratio and $(1/N^2)$ against s [Fig. 12.3-2(c)], when $N = 6$. The intensity of principal maxima is proportional to $6^2 (= 36)$. The intensities of secondary maxima are also shown in Fig. 12.3-2(c). It is

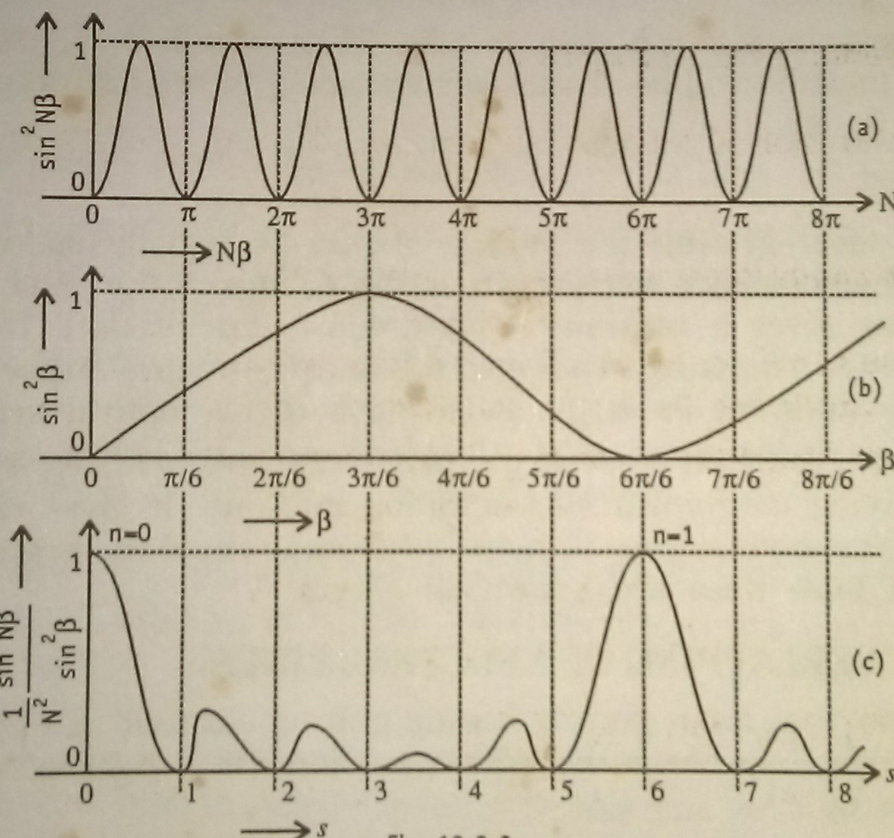

Fig. 12.3-2

evident from this figure that the intensities of secondary maxima fall off as we proceed towards the middle region between two consecutive principal maxima. These secondary maxima are in general unequally spaced and are not quite symmetrical [see Fig. 12.3-2(c)]. This lack of symmetry is greatest immediately adjacent to principal maxima and the secondary peaks are shifted a little towards the adjacent principal maxima.

12.4 ABSENT SPECTRA AND GHOSTS IN A DIFFRACTION GRATING :

For the mth order principal maximum in the direction θ, we have the condition

$$(a + b) \sin \theta = m\lambda \qquad ...(12.4\text{-}1)$$

Suppose the value of a is such that sth order diffraction minimum occurs in the same direction θ then

$$a \sin \theta = s\lambda \qquad ...(12.4\text{-}2)$$

If these two conditions are satisfied simultaneously then mth order principal maximum will be absent from the resulting spectra.

From Eqs. (12.4-1) and (12.4-2) we get

$$\frac{a+b}{a} = \frac{m}{s}$$

For example, if $a = b$ then $m = 2s$. Then $m = 2, 4, 6$, etc. order of principal maxima will be absent corresponding to the diffraction minima $s = 1, 2, 3$, etc.

Ghosts :

In an ideal grating the rulings should be equally spaced. But in practice there remains some errors in the rulings. If the error is *random* the grating gives a continuous background illumination. If the error is *progressive* in nature the spectral lines become sharper in planes which are different from the focal plane of the optical system.

The most common error is periodic in nature. It arises from defects in the driving mechanism of the ruling machine. It gives rise to false lines accompanying the principal maxima of ideal grating. These additional false lines are known as *ghosts*.

12.5 OVERLAPPING OF SPECTRAL LINES :

We know that for a grating having grating element $(a + b)$ the angle of diffraction θ in the mth order spectrum for a light of wavelength λ is given by

$$(a + b) \sin \theta = m\lambda \qquad ...(12.5\text{-}1)$$

For a given grating $a + b$ is constant and hence θ will be the same when $m\lambda$ is constant. Thus if the light incident on the grating surface consists of a large range of wavelengths, the spectral lines of longer wavelength and of lower order may overlap on the spectral lines of shorter wavelength and of higher order.

For example, the 3rd order maximum of light of wavelength $\lambda = 700$ nm and 4th order maximum of light of wavelength 525 nm will be formed in the same direction because

$$m\lambda = 3 \times 700 \text{ nm} = 4 \times 525 \text{ nm} = \text{constant}$$

12.6 ANGULAR DISPERSIVE POWER OF A GRATING :

The angular dispersive power of a grating is defined as the rate of change of the angle of diffraction (θ) with the change in wavelength. Thus angular dispersive power equals $\frac{d\theta}{d\lambda}$.

If θ is the angle of diffraction for rays of wavelength λ which form the *m*th order bright band then we have

$$(a + b)\ \sin\theta = m\lambda \qquad \text{...(12.6-1)}$$

Differentiating with respect to λ we get,

$$\frac{d\theta}{d\lambda} = \frac{m}{(a+b)\cos\theta} \qquad \text{...(12.6-2)}$$

Thus $\frac{d\theta}{d\lambda}$ is directly proportional to the order number *m* of the spectrum. It is also directly proportional to $\frac{1}{a+b}$ i.e., the number of rulings per unit length. Also we note that $\frac{d\theta}{d\lambda}$ is large for large values of θ. Since in a given order θ increases with increase in λ [Eq. (12.6-1)], we conclude that the grating spectra are spread much more at red end than at the blue end of the spectrum.

When θ is small (not exceeding 6°) cos θ remains almost constant. Under this situation *d*θ is proportional to *d*λ i.e., angular separation between two spectral lines is proportional to the wavelength difference. Such a spectrum is known as *normal spectrum*.

5·15. DIFFERENCE BETWEEN PRISM AND DIFFRACTION GRATING SPECTRA

The following points give the difference between grating and prism spectra :

(1) The prism spectrum is produced by dispersion inside the prism which is due to different velocities of light through the prism for different wavelengths; while grating spectrum is produced by diffraction, the angle of diffraction being different for different wavelengths.

(2) A prism gives only one spectrum; but a grating gives a number of spectra of different orders on each side of the central maximum.

(3) The prism spectrum is brighter than grating spectrum. This is due to the fact that in the case of prism the transmitted light is distributed in a single spectrum; while in the case of grating most of the light is concentrated in the central maximum where no spectrum is formed and the rest is distributed in the spectra of different orders.

(4) In the case of a prism, the deviation is least for red and greatest for violet colour; while in the case of a grating, the deviation is least for violet and greatest for the red. Thus the order of colours in the two spectra are opposite.

(5) The prism spectrum depends upon the material of the prism, while grating spectrum is independent of the material of the grating.

(6) The dispersive power of a grating is given by

$$\frac{d\theta}{d\lambda} = \frac{n}{(a+b)\cos\theta}$$

which is constant for a particular order. Thus the spectrum formed by one grating is similar to that obtained by the other and the spectrum formed by the grating is said to be *rational*.

$$\text{The dispersive power of a prism} = \frac{d\mu}{\mu - 1}$$

As the refractive index of a material of the prism is different for different wavelengths. It changes more rapidly at the violet than at the red end of the spectrum. Thus the dispersive power is higher in the violet region than in the red region of the spectrum. Hence there is more spreading of spectral lines towards the violet. Moreover the relative dispersion of any two colours is different with prism of different materials. The spectrum obtained with a prism is called *irrational spectrum*.

(7) The resolving power of diffraction grating is much greater than that of a prism.

(8) In the prism spectrum spectral lines are slightly curved, while in the grating spectra the spectral lines are almost straight.