SEMESTER-VI

HONOURS

CORE COURSE---C 13T

UNIT-I (MARKS-07) AND UNIT-II(MARKS-14)

UNIT-I **Dr. Pradip Kumar Gain**

Syllabus for Unit-I : Metric spaces: sequences in a metric space, Cauchy sequences, Complete metric spaces, Cantor's theorem.

Sequences in a metric space

DEFINITION: A ***sequence*** in a metric space (X,d) is a function defined on the set of natural numbers $\mathbb{N}$ with values in X and is specified by listing its values as $x_1, x_2, x_3, \ldots\ldots, x_n, \ldots\ldots\ldots$ or as $\{x_n\}_{n=1}^{\infty}$ or as $\{x_n\}$ where x_n is the image of n, $n \in \mathbb{N}$ and is known as the nth term of the sequence.

NOTE : The function stated above is not necessarily one-to-one and therefore, the range set of the sequence may be finite or infinite whereas set of all terms of a sequence is always infinite.

EXAMPLES: The range set of the sequence $\left\{\frac{1}{n}\right\}$ in R is $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots\ldots\}$ is infinite. The set of all terms is $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots\ldots\}$ which also infinite. Again the range set the sequence $\{(-1)^n\}$ is $\{-1, 1\}$ which is finite. But the set of all terms of a sequence is infinite.

DEFINITION: Let $\{x_n\}$ be a sequence in the metric space (X,d). Let $\{n_1, n_2, n_3, \ldots\ldots\ldots, n_k, \ldots\ldots\ldots\}$ be a strictly increasing sequence of natural numbers. Then the sequence $\{x_{n_k}\}$ ie, $\{x_{n_1}, x_{n_2}, x_{n_3}, \ldots\ldots\ldots, x_{n_k}, \ldots\ldots\ldots\}$ is called a ***subsequence*** of the sequence $\{x_n\}$.

EXAMPLES:

DEFINITION: A sequence $\{x_n\}$ in a metric space (X,d) is said to ***converge*** to a point $x \in X$, if for given $\varepsilon > 0$, we can find a positive integer m (depending on $\in$) such that $d(x_n, x) < \varepsilon$, whenever $n \geq m$.

We then write $d(x_n, x) \to 0$ as $n \to \infty$ or, $\lim_{n\to\infty} x_n = x$ or, $x_n \to x$ as $n \to \infty$.

OR

Equivalently, a sequence $\{x_n\}$ in a metric space (X,d) is said to converge to a point $x \in X$, if for given $\varepsilon > 0$, we can find a positive integer m (depending on ε) such that $x_n \in S(x,\varepsilon)$ for all $n \geq m$ where, $S(x,\varepsilon)$ is a sphere of radius ε centred at x.

EXAMPLE: The the sequence $\left\{\frac{1}{n}\right\}$ converge to 0.

DEFINITION (Cauchy Sequence): A ***sequence*** $\{x_n\}$ in a metric space (X,d) is said to be a ***Cauchy sequence*** or ***Fundamental sequence*** *iff* for each $\varepsilon > 0$ there exists a positive integer p such that $d(x_n, x_m) < \varepsilon$ for all $n,m \geq p$. That is, $d(x_n, x_m) \to 0$ as $m,n \to \infty$.

THEOREM 1.1: ***Every convergent sequence is Cauchy sequence. Converse is not necessarily true.***

Proof: Let $\{x_n\}$ be a convergent sequence in the metric space (X,d) and let $x_n \to x$. Hence for given $\varepsilon > 0$, there exists a positive integer p such that $d(x_n, x) < \frac{\varepsilon}{2}$, $d(x_m, x) < \frac{\varepsilon}{2}$ for all $n,m \geq p$. ....(1)

Now $d(x_n, x_m) \leq d(x_n, x) + d(x, x_m)$ [since d is metric . Triangle inequality holds]

$= d(x_n, x) + d(x_m, x)$ [since d is metric . Symmetric property holds]

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ [by (1)]

$= \varepsilon \ \forall n,m \geq p$

Thus $d(x_n, x_m) < \varepsilon$ for all $n,m \geq p$. Hence $\{x_n\}$ is a Cauchy sequence in the metric space (X,d).

To show converse is not true let us consider the space $X =$ (0, 1] of the real line with usual metric. Let us consider the sequence $\{x_n\} = \left\{\frac{1}{n}\right\}, n \in \mathrm{N}$. For a given $\varepsilon > 0$, we choose a positive integer $p(> \frac{2}{\varepsilon})$, $d(x_n, x_m) = |x_n - x_m| \leq |x_n| + |x_m| = \frac{1}{n} + \frac{1}{m}, \forall n,m \geq p$

$$\Rightarrow d(x_n, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \forall n,m \geq p$$

$$\Rightarrow d(x_n, x_m) < \varepsilon \quad \forall n,m \geq p$$

Hence $\{x_n\} = \left\{\frac{1}{n}\right\}, n \in \mathrm{N}$ is a Cauchy sequence in $X =$ (0, 1].

But $\{x_n\} = \left\{\frac{1}{n}\right\}, n \in \mathrm{N}$, converges to 0 which is not a point of $X =$ (0, 1].

Thus $\{x_n\} = \left\{\frac{1}{n}\right\}$ is a Cauchy sequence in X but does not converge to any point in X .

THEOREM 1.2: ***Let $\{x_n\}$ be a Cauchy sequence in the metric space (X,d). If $\{x_n\}$ possesses a convergent subsequence $\{x_{n_k}\}$ converging to x, then the sequence $\{x_{n_k}\}$ also converges to x.***

Proof: Let $\{x_{n_k}\}$ be a convergent subsequence of the Cauchy sequence $\{x_n\}$ converges to $x \in X$. Then for each $\varepsilon > 0$ there exists a positive integer p such that $d(x_{n_k}, x) < \frac{\varepsilon}{2}$ for all $n_k \geq p$ ....(1).

Again as $\{x_n\}$ is a Cauchy sequence, for each $\varepsilon > 0$ there exists a positive integer q such that $d(x_n, x_m) < \frac{\varepsilon}{2}$ for all $n, m \geq q$ ....(2).

Let $r = Max(p,q)$

Then $d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x)$ [since d is metric . Triangle inequality holds]

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for all $n \geq r$.

$\Rightarrow d(x_n, x) < \varepsilon$ for all $n \geq r$. Hence the Cauchy sequence converges to $x \in X$.

THEOREM 1.3: ***A Cauchy sequence $\{x_n\}$ in a metric space (X,d) converges If and only if it has a convergent subsequence $\{x_{n_k}\}$.***

Proof: Let $\{x_n\}$ be a cauchy sequence converges to $x \in X$. Hence for given $\varepsilon > 0$, there exists a positive integer p such that $d(x_n, x) < \varepsilon$ for all $n \geq p$ and hence $d(x_{n_k}, x) < \varepsilon$ for all $n_k \geq p$. Therefore, the subsequence $\{x_{n_k}\}$ of the cauchy sequence $\{x_n\}$ converges to $x \in X$.

Conversely, let $\{x_{n_k}\}$ be a convergent subsequence of the cauchy sequence $\{x_n\}$ converges to $x \in X$. Then by previous theorem, the cauchy sequence $\{x_n\}$ converges to $x \in X$.

Complete metric spaces

DEFINITION: A ***metric space*** (X,d) is said to be ***complete*** if every Cauchy sequence in X converges to some point in X .

The ***metric space*** (X,d) is called ***incomplete*** if it is not complete.

EXAMPLES (complete metric spaces):

Ex-1. ***Any set X with discrete metric forms a complete metric space.***

Solution : Let (X,d) be a metric space with discrete metric d such that $d(x,y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$. Let $\{x_n\}$ be a Cauchy sequence in the discrete metric space (X,d).

Then $d(x_n, x_m) = \begin{cases} 0 \text{ if } x_n = x_m \\ 1 \text{ if } x_n \neq x_m \end{cases}$. Since as $\{x_n\}$ is a Cauchy sequence, for each $\varepsilon > 0$ there exists a positive integer p such that $d(x_n, x_m) < \frac{1}{2}$ for all $n, m \geq p$ [taking $\varepsilon = \frac{1}{2}$]. Then by definition of discrete metric space $d(x_n, x_p) = 0 \ \forall \ n \in \mathbb{N}$.

$\Rightarrow x_n \to x_p$ as $n \to \infty$ which shows that every Cauchy sequence converges to a point of X which is also a term of the sequence. Hence discrete metric space X is complete.

Ex-2. ***The real line R is complete.***

Solution : Let $\{x_n\}$ be a cauchy sequence in R. By the definition of Cauchy sequence for each $\varepsilon > 0$ there exists a positive integer p such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq p$. Since R is a metric space with usual metric, we must have $d(x_n, x_m) = |x_n - x_m| < \varepsilon \ \ \forall \ \ n, m \geq p$. But it follows from the Cauchy's general principle of convergence of a sequence of real numbers that the above situation implies the convergence of a sequence $\{x_n\}$ to some point $x \in R$. Hence R is complete.

Ex-3. ***Prove that the space $C[0,1]$ of all continuous real valued functions on $[0,1]$ with the metric d, defined by $d(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}$, is a complete metric space.***

Solution : Clearly, $d(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\} \geq 0$

Also $d(f,g) = 0$ *iff* $\sup\{|f(x) - g(x)| : x \in [0,1]\} = 0$

Iff $f(x) - g(x) = 0 \ \ \forall \ \ x \in [0,1]$

Iff $f(x) = g(x) \qquad \forall \ \ x \in [0,1]$

Iff $f = g$ [non-negative property holds]

Also $d(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}$

$= \sup\{|g(x) - f(x)| : x \in [0,1]\}$

$= d(g,f)$ [symmetric property holds]

Also for any three functions f , g , h , we have, $d(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}$

$= \sup\{|f(x) - h(x) + h(x) - g(x)| : x \in [0,1]\}$

$\leq \sup\{|f(x) - h(x)| : x \in [0,1]\} + \sup\{|h(x) - g(x)| : x \in [0,1]\}$

$= d(f,h) + d(h,g)$

Thus $d(f,g) \leq d(f,h) + d(h,g)$ [Triangle inequality holds]

Hence $\{C[0,1], d\}$ is a metric space.

Let $\{f_n\}$ be a Cauchy sequence in $C[0,1]$. Then for each $\varepsilon > 0$ there exists a positive integer p such that $d(f_n, f_m) < \varepsilon$, for all $n, m \geq p$.

$\Rightarrow \sup\{|f_n(x) - f_m(x)| : x \in [0,1]\} < \varepsilon$, for all $n, m \geq p$.

$\Rightarrow \{|f_n(x) - f_m(x)|\} < \varepsilon$, for all $n, m \geq p$ and for all $x \in [0,1]$. Using Cauchy's condition for convergence, we can say that $\{f_n\}$ converges uniformly on $[0,1]$. If $f_n \to f$ then f is also continuous on $[0,1]$. Therefore, the Cauchy sequence $\{f_n\}$ converges to $f \in C[0,1]$.

Hence $C[0,1]$ is a complete metric space.

EXAMPLES (incomplete metric spaces):

Ex-1. ***The space*** $X = (0, 1]$ ***of the real line with usual metric*** $d(x,y) = |x-y|$**,** $\forall x, y \in X$ ***is not complete.***

Solution : Let us consider the sequence $\{x_n\} = \left\{\frac{1}{n}\right\}, n \in \mathrm{N}$. For a given $\varepsilon > 0$, we choose a positive integer $p(> \frac{2}{\varepsilon})$, $d(x_n, x_m) = |x_n - x_m| \le |x_n| + |x_m| = \frac{1}{n} + \frac{1}{m}, \forall n, m \ge p$

$$\Rightarrow d(x_n, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \forall n, m \ge p$$

$$\Rightarrow d(x_n, x_m) < \varepsilon \ \ \forall n, m \ge p.$$

Hence $\{x_n\} = \left\{\frac{1}{n}\right\}, n \in \mathrm{N}$ is a Cauchy sequence in $X = (0, 1]$.

But $\{x_n\} = \left\{\frac{1}{n}\right\}, n \in \mathrm{N}$, converges to 0 which is not a point of $X = (0, 1]$.

Thus $\{x_n\} = \left\{\frac{1}{n}\right\}$ is a Cauchy sequence in X but does not converge to any point in X.

Hence $X = (0, 1]$ is not complete.

Ex-2. ***The set*** Q ***of all rational numbers with usual metric*** $d(x,y) = |x-y|$**,** $\forall x, y \in Q$ ***is not complete.***

Solution : With usual metric $d(x,y) = |x-y|$, $\forall x, y \in Q$ is metric space. Let us consider the sequence $\{x_n\} = \left\{\frac{1}{3^n}\right\}$. This is a Cauchy sequence in Q. $\{x_n\} = \left\{\frac{1}{3^n}\right\}$ converges to $0 \in Q$.

Again let us consider a sequence $\{x_n\} = \left\{\left[1 + \frac{1}{n}\right]^n\right\}$. This is a Cauchy sequence in Q but this sequence $\{x_n\} = \left\{\left[1 + \frac{1}{n}\right]^n\right\}$ converge to a point $e \notin Q$. So every Cauchy sequence in Q is not convergent. Hence (Q, d) is not complete metric space.

DEFINITION : A ***sequence*** $\{F_n\}$ = $\{F_1, F_2, F_3, \ldots\ldots\ldots\ldots\}$ of sets is said to be ***nested*** if $F_1 \supset F_2 \supset F_3 \supset \ldots\ldots\ldots\ldots \supset F_n \supset \ldots\ldots\ldots$ That is, if $F_n \supset F_{n+1} \ \ \forall n \in \mathrm{N}$

CANTOR INTERSECTION THEOREM : ***If*** $\{F_n\}$ ***is a nested sequence of non-empty closed subsets of metric space*** (X, d) ***such that*** $\delta(F_n) \to 0$ ***as*** $n \to \infty$, ***then*** X ***is complete iff*** $\bigcap_{n=1}^{\infty} F_n$ ***consists of exactly one point, where*** $\delta(F_n)$ ***denotes the diameter of*** F_n.

Proof: **The condition is necessary**

Let (X,d) be a complete metric space and let $\{F_n\}$ = $\{F_1, F_2, F_3, \ldots\ldots\ldots \ldots\}$ be a nested of non-empty closed subsets of X with $\delta(F_n) \to 0$ as $n \to \infty$. We shall show that $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

Since each $F_n(n \in \mathrm{N}) \neq \phi$, we can construct a sequence $\{x_n\}$ by choosing $x_1, x_2, x_3, \ldots\ldots \in F_n$. That is, $x_n \in F_n$, $\forall n = 1,2,3,\ldots\ldots\ldots$. As $\delta(F_n) \to 0$ as $n \to \infty$, for given $\in > 0$ there exists $m \in \mathrm{N}$ such that $\delta(F_n) < \in$ for all $n \geq m$ ....(1). Since $\{F_n\}$ is nested, $F_n \subset F_m$ for all $n \geq m$. Hence $x_n \in F_m$, for all $n \geq m$.

$\Rightarrow d(x_n, x_m) \leq \delta(F_m)$, for all $n \geq m$.

$\Rightarrow d(x_n, x_m) \leq \varepsilon$, for all $n \geq m$. [using (1)].

Hence, $\{x_n\}$ is a Cauchy sequence in F_n. That is, in X. Since, (X,d) is complete metric space (given), the sequence $\{x_n\}$ must be convergent. Let it converges to $x \in X$. That is, $x_n \to x$ as $n \to \infty$. We shall show that $x \in \bigcap_{n=1}^{\infty} F_n$. If possible, let $x \notin \bigcap_{n=1}^{\infty} F_n$. This implies that x should not lie in some of the sets $F_1, F_2, F_3, \ldots\ldots\ldots\ldots$ Let $x \notin F_k$. Since F_k is closed and $x \notin F_k$, $d(x, F_k) = \inf\{d(x,y) : y \in F_k\} > 0$. Let $d(x, F_k) \geq r$. Then $d(x,y) \geq r$ for all $y \in F_k$. Therefore, $S(x, \frac{r}{2})$ and F_k are disjoint. Now $n > k \quad \Rightarrow F_n \subset F_k \quad \Rightarrow \{x_1, x_2, x_3, \ldots\} \subset F_k$ [since $x_n \in F_n \ \forall n = 1,2,3,\ldots\ldots\ldots$]

$\Rightarrow x_n \notin S(x, \frac{r}{2})$ which is not possible since $\{x_n\}$ converges to $x \in X$.

Hence $x \in \bigcap_{n=1}^{\infty} F_n$ showing that $\bigcap_{n=1}^{\infty} F_n$ is non-empty.

In order to prove that $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point, let us suppose in contrary that $\bigcap_{n=1}^{\infty} F_n$ contains two points x and y.

Then $\delta(F_1) > d(x,y)$, $\delta(F_2) > d(x,y)$, $\delta(F_3) > d(x,y), \ldots\ldots\ldots\ldots\ldots\ldots$

Since d is metric and so $d(x,y) > 0$, $\delta(F_n)$ does not tend to 0 which contradicts the fact $\delta(F_n) \to 0$ as $n \to \infty$. Therefore, $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

The condition is sufficient

Let us suppose that the given condition is sufficient. We shall show that X is complete. Let $\{x_n\}$ be a Cauchy sequence in X.

Let us consider for each $n \in \mathrm{N}$, $A_n = \{x_n, x_{n+1}, x_{n+2}, \ldots\ldots\ldots\ldots\}$.

That is, $A_1 = \{x_1, x_2, x_3, \ldots\ldots\ldots\ldots\}$, $A_2 = \{x_2, x_3, x_4, \ldots\ldots\ldots\ldots\}$, $A_3 = \{x_3, x_4, x_5, \ldots\ldots\ldots\ldots\}, \ldots\ldots\ldots$

Obviously, $A_1 \supseteq A_2 \supseteq A_3 \supseteq$ .... and we have, $\overline{A_1} \supseteq \overline{A_2} .. \supseteq \overline{A_3} .. \supseteq$ ....

Since, $\{x_n\}$ is a Cauchy sequence and $\delta(A_n) \to 0$ as $n \to \infty$, we have, $\delta(\overline{A_n}) \to 0$ as $n \to \infty$. So $\{\overline{A_n}\} = \{\overline{A_1}, \overline{A_2}, \overline{A_3}, ...\}$ is a nested sequence of closed and non-empty sets in X, where $\delta(\overline{A_n}) \to 0$ as $n \to \infty$. So by hypothesis there exists an $x \in X$ such that $x \in \bigcap_{n=1}^{\infty} \overline{A_n}$.

Now $x_n \in A_n \subset \overline{A_n} \Rightarrow x_n \in \overline{A_n}$ Also $x \in \overline{A_n}$. Therefore, $d(x, x_n) < \delta(\overline{A_n})$. Since $\delta(\overline{A_n}) \to 0$, $d(x, x_n) \to 0$ as $n \to \infty$. That is, $x_n \to x$ as $n \to \infty$. Hence the Cauchy sequence $\{x_n\}$ converges to $x \in X$. As $\{x_n\}$ is arbitrary it follows that every Cauchy sequence in (X, d) converges. Hence (X, d) is complete metric space.

<u>INSTRUCTION FOR STUDENTS</u> :

NOTE: Definition of Cauchy sequence and theorems related to Cauchy sequence , Definition of Complete metric space and examples related to Complete metric spaces and Incomplete metric spaces are important .

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CORE COURSE--C13T

UNIT-II (MARKS-14)

UNIT-II **Dr. Pradip Kumar Gain**

Syllabus for Unit-II : Continuity: Continuous mappings, Sequential criterion and other characterizations of continuity, Uniform continuity, Connectedness: Connected subsets of R. Compactness: Sequential compactness, Heine-Borel property, totally bounded spaces, finite intersection property(FIP), continuous functions on compact sets. Homeomorphism. Contraction mappings, Banach fixed point theorem and its applications to ordinary differential equations.

Functions/Mappings

DEFINITION : Let X and Y be two non-empty sets. If there is a rule of correspondence f which corresponds each element $x \in X$ a unique element $y \in Y$, then f is said to be a ***function or mapping*** or a map from X to Y or f maps X into Y.

In symbol we write $f : X \to Y$. In such a case the set X is called the Domain of f and the set Y is called Codomain of f. If f relates $x \in X$ with $y \in Y$, we write $y = f(x)$. Here x is called preimage of y under f and y is called image of x under f.

Continuous mappings

DEFINITION : Let (X,d) and (Y,d') be two metric spaces. A ***function*** $f:(X,d) \to (Y,d')$ is said to be ***continuous*** at a point $a \in X$, if and only if for all $\varepsilon > 0$, chosen arbitrarily, there exists a $\partial(>0)$ (depending on ε and a) such that $d(x,a) < \partial \Rightarrow d'(f(x), f(a)) < \varepsilon$. That is, $x \in S_X(a,\partial) \Rightarrow f(x) \in S_Y(f(a),\varepsilon)$.

The function f is said to be continuous on (X,d) if and only if it is continuous at each point of X.

REMARKS : It is clear that a function $f:(X,d) \to (Y,d')$ is continuous at a point $a \in X$, if and only if for all $\varepsilon > 0$, chosen arbitrarily, there exists a $\partial(>0)$ (depending on ε and a) such that $f(S_X(a,\partial)) \subset S_Y(f(a),\varepsilon)$.

EXAMPLES :

Sequential criterion of continuity

THEOREM 2.1: ***Let (X,d) and (Y,d') be two metric spaces. A function $f:(X,d)\to(Y,d')$ is said to be continuous at a point $x\in X$, if and only if for all sequences $\{x_n\}$ of elements of X converging to the point x in (X,d), the sequences $\{f(x_n)\}$ of elements of Y converge to $f(x)$ in (Y,d').***

Proof : **The condition is necessary**

Let us suppose that the function $f:(X,d)\to(Y,d')$ is continuous at a point $x\in X$. We shall show that $x_n\to x\Rightarrow f(x_n)\to f(x)$ as $n\to\infty$. Let $\varepsilon>0$, be arbitrarily chosen. Since f is continuous at the point x, there exists a $\partial(>0)$ such that $d(x_n,x)<\partial\Rightarrow d'(f(x_n),f(x))<\varepsilon$. Since $x_n\to x$ as $n\to\infty$ in (X,d), corresponding to $\partial(>0)$ there exists a natural number m depending on ∂ such that $n>m\Rightarrow d(x_n,x)<\partial$. Combining the two results above we conclude that $n>m\Rightarrow d'(f(x_n),f(x))<\varepsilon$, where m is a natural number depending on ∂ and hence dependent on $\varepsilon>0$. This implies $\{f(x_n)\}$ converges to $f(x)$ in (Y,d').

The condition is sufficient

We shall show that if for all sequences $\{x_n\}$ converging to the point x in (X,d) the corresponding sequences $\{f(x_n)\}$ converge to $f(x)$ in (Y,d'), then f is continuous at the point x. If possible let f is not continuous at the point x. Then there exists atleast one $\varepsilon>0$ such that for all $\partial(>0)$ $d(x',x)<\partial$ but $d'(f(x'),f(x))\geq\varepsilon$ for at least one $x'\in X$. Let us consider a sequence of $\partial's$ given by $\partial=\frac{1}{n}$ for all $n\in\mathrm{N}$. So, corresponding to each natural number n, there exists $x_n\in X$ such that $d(x_n,x)<\frac{1}{n}$ but $d'(f(x'),f(x))\geq\varepsilon$. This implies $f(x_n)$ does not tend to $f(x)$ in (Y,d') although $x_n\to x$ as $n\to\infty$ in (X,d), which is a contradiction to our hypothesis. Hence f must be continuous at the point x.

REMARKS : The above theorem shows that convergence of sequence of points remains preserved under a continuous map.

From the above theorem the following theorem follows:

THEOREM 2.2 : ***Let (X,d) and (Y,d') be two metric spaces. A function $f:(X,d)\to(Y,d')$ is continuous if and only if for any $x\in X$ and for all sequences $\{x_n\}$ X converging to x in (X,d), the sequences $\{f(x_n)\}$ converge to $f(x)$ in (Y,d').***

Other characterizations of continuity

THEOREM 2.3: ***Let (X,d) and (Y,d') be two metric spaces. A function $f:(X,d)\to(Y,d')$ is continuous if and only if for any open set G in (Y,d'), $f^{-1}(G)$ is open in (X,d).***

Proof : Let us assume $f:(X,d)\to(Y,d')$ is continuous and a set G is open in (Y,d'). We shall show that its inverse image $f^{-1}(G)$ is open in (X,d). If $f(X)\cap G=\phi$, then $f^{-1}(G)=\phi$ and remains nothing to prove. Let $f(X)\cap G\neq\phi$, then $f^{-1}(G)\neq\phi$. So, there exists atleast one $x\in f^{-1}(G)$. This implies $f(x)\in G$. Since G is open $f(x)$ is an interior point of the set G. So, we can find an $\varepsilon>0$ such that $S_Y(f(x),\varepsilon)\subset G$. Since f is continuous at the point x, there exists a $\partial(>0)$ such that $d(x,x')<\partial\Rightarrow d'(f(x),f(x'))<\varepsilon$. That is, $x'\in S_X(x,\partial)\Rightarrow f(x')\in S_Y(f(x),\varepsilon)$. That is, $f(S_X(x,\partial))\subset S_Y(f(x),\varepsilon)\subset G$. That is, $x\in S_X(x,\partial)\subset f^{-1}(G)$. Thus x is an interior point of the set $f^{-1}(G)$ in (X,d). Since $x\in f^{-1}(G)$ is arbitrarily chosen it follows that $f^{-1}(G)$ is open in (X,d).

Conversely, we assume that the inverse image of every open set G in (Y,d') is open in (X,d). We shall show that f is continuous. We choose any $x\in X$, then $f(x)$ is uniquely determined. For $\varepsilon>0$ chosen arbitrarily, $S_Y(f(x),\varepsilon)$ is an open set in (Y,d'). By proposition $f^{-1}(S_Y(f(x),\varepsilon))$ is open in (X,d). Now, $x\in f^{-1}(S_Y(f(x),\varepsilon))$. So, there exists a $\partial(>0)$ such that $x\in S_X(x,\partial)\subset f^{-1}(S_Y(f(x),\varepsilon))$. This implies, $f(S_X(x,\partial))\subset(S_Y(f(x),\varepsilon))$. That is, $d(x',x)<\partial\Rightarrow d'(f(x'),f(x))<\varepsilon$ where $\partial(>0)$ depends on $\varepsilon>0$. Consequently, f is continuous at x in (X,d). Since x is chosen arbitrarily , f is continuous.

THEOREM 2.4: ***Let (X,d) and (Y,d') be two metric spaces. A function $f:(X,d)\to(Y,d')$ is continuous if and only if for any closed set F in (Y,d'), $f^{-1}(F)$ is closed in (X,d).***

Proof : Let us assume $f:(X,d)\to(Y,d')$ is continuous and a set F is closed in (Y,d'). So, $Y\setminus F$ is open in (Y,d') and therefore, $f^{-1}(Y\setminus F)$ is open in (X,d) (since f is continuous). Now, $f^{-1}(F)=X\setminus f^{-1}(Y\setminus F)$

$\Rightarrow X\setminus f^{-1}(F)=f^{-1}(Y\setminus F)$. So, $\Rightarrow X\setminus f^{-1}(F)$ is open in (X,d). Consequently, $f^{-1}(F)$ is closed in (X,d).

Conversely, we assume that for all sets F closed in (Y,d'), $f^{-1}(F)$ is closed in (X,d). We shall that f is continuous. Let G be any open set in (Y,d'). $Y\setminus G$ is closed in (Y,d') hence $f^{-1}(Y\setminus G)$ is closed in (X,d). Since $f^{-1}(Y\setminus G)=X\setminus f^{-1}(G)$, it follows that $X\setminus f^{-1}(G)$ is

Closed in (X,d). That is, $f^{-1}(G)$ is open in (X,d). Therefore, f is continuous.

<u>THEOREM</u> 2.5: ***Let*** (X,d) ***and*** (Y,d') ***be two metric spaces. A function*** $f:(X,d)\to(Y,d')$ ***is continuous if and only if for any set*** $A\subset X$, $f(ClA)\subset Cl(f(A))$.

<u>THEOREM</u> 2.6: ***Let*** (X,d) ***and*** (Y,d') ***be two metric spaces. A function*** $f:(X,d)\to(Y,d')$ ***is continuous if and only if for any set*** $A\subset X$, $Cl\{f^{-1}(B)\}\subset f^{-1}(ClB)$.

<u>Uniform continuity</u>

<u>DEFINITION</u> : *Let* (X,d) *and* (Y,d') *be two metric spaces. A function* $f:(X,d)\to(Y,d')$ *is said to be* ***uniformly continuous*** *on* (X,d) *if and only if corresponding to* $\varepsilon>0$, *chosen arbitrarily, there exists a* $\partial(>0)$ *(depending on* $\in$ *alone) such that* $d(x_1,x_2)<\partial \Rightarrow d'(f(x_1),f(x_2))<\in \ \forall x_1,x_2\in X$

<u>THEOREM</u> 2.7: ***Let*** (X,d) ***and*** (Y,d') ***be two metric spaces and a function*** $f:(X,d)\to(Y,d')$ ***is a uniformly continuous function. If*** $\{x_n\}$ ***is a Cauchy sequence in*** (X,d) ***then*** $\{f(x_n)\}$ ***is a Cauchy sequence in*** (Y,d').

Proof : Let $\varepsilon>0$, be arbitrarily chosen. Since , f is uniformly continuous in (X,d), there exists a $\partial(>0)$ (depending on $\in$ alone) such that $d(x_1,x_2)<\partial \Rightarrow d'(f(x_1),f(x_2))<\varepsilon$ $\forall x_1,x_2\in X$ ………………..(1). Since $\{x_n\}$ is a Cauchy sequence in (X,d), corresponding to $\partial(>0)$ there exists a positive integer $m=m(\partial)$ such that $n>m \Rightarrow d(x_n,x_{n+p})<\partial$, for all $p\in \mathrm{N}$ ………………………(2). Combining (1) &(2) we get, $n>m\Rightarrow$ for all $p\in \mathrm{N}, d(x_n,x_{n+p})<\partial$ $\Rightarrow d'(f(x_n),f(x_{n+p}))<\varepsilon$. This implies that $\{f(x_n)\}$ is a Cauchy sequence in (Y,d').

<u>Examples of continuous functions:</u>

<u>EXAMPLES</u> 2.1: *Show that the function* $f(x)=\frac{1}{x}$ *mapping the real line into itself is continuous everywhere on the real line except at the origin.*

<u>EXAMPLES</u> 2.2: *Show that the function* $f(x)=\frac{1}{x}$ *mapping the real line into itself given by* $f(x)=\begin{cases} x, x\in Q \\ 1-x, otherwise\end{cases}$ *continuous only at the point* $\frac{1}{2}$.

<u>EXAMPLES</u> 2.3: Let (X,d) be a metric space and A and B are two non-empty disjoint closed sets in X. Prove that there exists a continuous function $f:X\to R$ such that $f(x)=\begin{cases} 1, x\in A \\ -1, x\in B\end{cases}$.

Connectedness:

DEFINITION : Let (X,d) be a metric space and A and B are two subsets of X. The sets A and B are said to be ***separated*** in if and only if neither has a point in common with the closure of another. That is, $A \cap Cl(B) = \phi$, $Cl(A) \cap B = \phi$.

These two conditions can be expressed by $\{A \cap Cl(B)\} \cup \{Cl(A) \cap B\} = \phi$. This is known as "***Hausdorff-Lennes condition***".

NOTE : Two sets A and B are may be separated in one metric space but not in other.

For example, let us consider the set R of all real numbers along usual metric. Then let us consider two sets $\{0\}$ and $(0,1)$. Now, $\{0\} \cap Cl((0,1)) = \{0\} \cap [0,1] = \{0\} \neq \phi$.

Again, let us consider the set R of all real numbers along with discrete metric. Let us consider two sets $\{0\}$ and $(0,1)$. The open sphere $S\left(0, \frac{1}{2}\right)$ with centre 0 and radius $\frac{1}{2}$ contains no point of the set $(0,1)$. Therefore, in this metric space $Cl(\{0\}) \cap (0,1) = \{0\} \cap (0,1) = \phi$, $\{0\} \cap Cl((0,1)) = \{0\} \cap (0,1) = \phi$ and hence the sets $\{0\}$ and $(0,1)$ are separated in this metric space.

THEOREM 2.8: ***Let a set G is open in a metric space (X,d). If G is expressed as the union of two non-empty separated sets A and B. Then both the sets A and B are open in (X,d).***

Disconnected Spaces and Disconnected Sets:

DEFINITION : A ***metric space*** (X,d) is said to be ***disconnected*** if and only if it can be expressed as the union of two non-empty separated sets. That is, $X = A \cup B$ where $A \neq \phi, B \neq \phi$ and $A \cap Cl(B) = \phi$, $Cl(A) \cap B = \phi$.

By theorem 2.8 , both the sets A and B are open in (X,d).

RESULT : A metric space (X,d) is disconnected if and only if it can be expressed as the union of two non-empty disjoint open sets.

RESULT : A metric space (X,d) is disconnected if and only if it can be expressed as the union of two non-empty disjoint closed sets.

DEFINITION : A ***non-empty subset*** A of a metric space (X,d) is ***disconnected*** if and only if it can be expressed as the union of two non-empty separated sets. That is, $A = A_1 \cup A_2$ where $A_1 \neq \phi, A_2 \neq \phi$ and $A_1 \cap Cl(A_2) = \phi$, $Cl(A_1) \cap A_2 = \phi$.

THEOREM 2.8: ***Let (X,d) be a metric space. Then the following conditions are equivalent :***

(i) (X,d) is disconnected.

(ii) X can be expressed as the union of two non-empty disjoint closed sets in (X,d).

(iii) X ***can be expressed as the union of two non-empty disjoint open sets in*** (X,d).

(iv) there exists a non-empty proper subset of X ***, which is both open and closed in the metric space*** (X,d).

Connected Spaces and Connected Sets:

DEFINITION : A ***metric space*** (X,d) is said to be ***connected*** if and only if X is not expressible as the union of two non-empty separated sets in (X,d). In other words, (X,d) is connected if and only if X is not disconnected.

DEFINITION : A ***non-empty subset*** A of a metric space (X,d) is ***connected*** if and only if it cannot be expressed as the union of two non-empty separated sets.

THEOREM 2.9: ***A metric space*** (X,d) ***is connected if and only if*** X ***and*** ϕ ***are the only sets which are both open and closed in*** (X,d).

Proof : Let X and ϕ are the only sets which are both open and closed in (X,d). That is, X is the only non-empty set which is both open and closed in (X,d). We shall prove that (X,d) is connected. If possible, let X is disconnected. The there exists a disconnection (A,B) of (X,d). Obviously, both the sets A and B are non-empty. Since X is open, we find both the sets A and B are open. Similarly, both the sets A and B are closed. Thus there exists a non-empty proper subset A of X which is both open and closed in (X,d). This is a contradiction to our hypothesis that X is the only non-empty set which is both open and closed in (X,d). Therefore, (X,d) is connected.

Conversely, let (X,d) is connected. We shall show that X is the only non-empty set which is both open and closed in (X,d). If possible, let there exists a non-empty proper subset A of X which is both open and closed in (X,d). Then its complement $A^C = X \setminus A$ is non-empty. Since A is both open and closed A^C is both closed and open. Therefore, (X,d) is disconnected with a disconnection (A, A^C), which contradicts our assumption. Therefore, X is the only non-empty set which is both open and closed in (X,d).

THEOREM 2.10: ***If two connected sets are not separated, their union is connected.***

THEOREM 2.11: ***In a metric space the union of two non-disjoint connected sets is connected.***

THEOREM 2.12: ***If every two points of a set*** A ***in a metric space*** (X,d) ***are contained in some connected subset of*** A***, then*** A ***is connected set.***

DEFINITION : Let (X,d) be a ***metric space***. If corresponding to every pair a, b of distinct points of X, there exist separated sets A and B in (X,d) with $a \in A$ and $b \in B$, then the space (X,d) is said to be ***totally disconnected***.

CONNECTED SETS IN THE REAL LINE

It is clear that like other spaces, the null set ϕ and singleton sets are connected in the real line.

THEOREM 2.13: A set $A \subset R$ with atleast two points is connected in the real line if and only if A is an interval.

Proof : Let us assume that A is an interval. We shall show that A is connected. Let us assume, if possible, A is disconnected. Then there exist two non-empty sets B and C both open and closed in the subspace A such that $A = B \cup C$. Since B and C are non-empty disjoint sets we choose any $b \in B$ and $c \in C$. Since the sets B and C are disjoint, the points b and c are distinct. That is, $b \neq c$. Let $b < c$. Since A I s an interval and $b, c \in A$ it follows that $b < x < c \Rightarrow x \in A$. So, $[b, c] \subset A = B \cup C$. Also, $y \in [b, c] \Rightarrow$ either $y \in B$ or $y \in C$ but not both. Let $E = [b, c] \cap B$. Now $b \in E$. Since E is non-empty and bounded above E has a finite supremum. Let $u = \sup E$. Then $b \leq u \leq c$. Since $u = \sup E$, no real number less than u can be an upper bound of the set E. Consequently, corresponding to each $\varepsilon (> 0)$, however small, there exists a $v \in E$ such that $u - \varepsilon < v \leq u$. Thus every neighbourhood $S(u, \varepsilon)$ of u in the real line contains a point of E. Since $E \subset B$, we conclude that every neighbourhood of u contains a point of B different from u. So u is a point of accumulation of the set B. Since B is closed, we must have $u \in B$. Also $u \notin C$ (Since the sets B and C are disjoint). Hence $u \neq c$. As $b \leq u \leq c$, it follows that $u < c$. Again for each $\varepsilon (> 0)$, however small, $u + \varepsilon \in C$, if $u + \varepsilon \leq c$. This implies every neighbourhood $S(u, \varepsilon)$ of the point u in the real line contains some point of C different from u. Therefore, u is a point of accumulation of the set C in the real line. Since the set C is closed, we also have $u \in C$. Thus $u \in B \cap C$, which contradicts that the sets B and C are disjoint. Therefore, A must be connected.

Conversely, if possible, let A is a connected subset of R containing at least two points but A is not an interval. Then there exist three points x, y, z such that $x, z \in A$, $y \notin A$ where $x < y < z$. Now, the sets $A_1 = (-\infty, y)$ and $A_2 = (y, \infty)$ re separated open sets in Euclidean line. Let $B_1 = A_1 \cap A$, $B_2 = A_2 \cap A$. Then $B_1 \subset A_1$, $B_2 \subset A_2$ and consequently B_1 and B_2 are separated. As $x \in B_1$, $z \in B_2$ both B_1 and B_2 are non-empty. Also $A = B_1 \cup B_2$ Hence A has a disconnection (B_1, B_2). This is a contradiction to the fact that A is a connected set. Thus A is an interval.

EXAMPLE : Show that the set R of all real numbers is connected in the real line.

SOLUTION : If possible, let the set R is disconnected in the real line and (A, B) is a disconnection of R. Then A, B are non-empty separated stes in thr real line which are both open and closed. Since A, B are non-empty, there exists at least one $a_1 \in A$ and $b_1 \in B$. Since A and B are disjoint, $a_1 \neq b_1$. So either $a_1 < b_1$ or $a_1 > b_1$. Without loss of generality, let $a_1 < b_1$. Let $I_1 = [a_1, b_1]$. Then $|I_1| = b_1 - a_1 > 0$. Now, $\frac{a_1 + b_1}{2} \in R$. Since

$R = A \cup B$, $\frac{a_1 + b_1}{2}$ belongs either to A or to B or belong to both. Also since, $A \cap B = \phi$, $\frac{a_1 + b_1}{2}$ can't belong to both the sets A and B. If $\frac{a_1 + b_1}{2} \in A$, we shall consider the interval $\left[\frac{a_1 + b_1}{2}, b_1\right]$. Let $[a_2, b_2] = \left[\frac{a_1 + b_1}{2}, b_1\right]$. That is, $a_2 = \frac{a_1 + b_1}{2}$ and $b_2 = b_1$.

If $\frac{a_1 + b_1}{2} \in B$, we shall consider the interval $\left[a_1, \frac{a_1 + b_1}{2}\right]$. Let $[a_2, b_2] = \left[a_1, \frac{a_1 + b_1}{2}\right]$. That is, $a_2 = a_1$ and $b_2 = \frac{a_1 + b_1}{2}$. Let $I_2 = [a_2, b_2]$. Then clearly, $I_2 \subset I_1$ and $|I_2| = b_2 - a_2 = \frac{(b_1 - a_1)}{2}$. If we repeat this process, we must find intervals $I_3, I_4, I_5, \ldots$ and so on. In every case, we select the end points a_n and b_n such that $a_n \in A$ and $b_n \in B$. Thus we get a sequence $\{I_n\}$ of bounded closed intervals, where $I_n = [a_n, b_n]$. Also $I_{n+1} \subset I_n$ for all $n \in \mathrm{N}$. And, $|I_n| = (b_n - a_n) = \frac{1}{2^{n-1}}(b_1 - a_1) \to 0$ as $n \to \infty$. Thus $\{I_n\}$ forms a nest of closed intervals with diameter tending towards zero in the real line. Nested Interval Theorem says that there exists one and only one point $c \in \cap\{I_n : n \in \mathrm{N}\}$. It can easily be seen that both the sequences $\{a_n\}$ and $\{b_n\}$ are convergent in the real line and both converges to c. Since $\{a_n\} \subset A$, c is a point of accumulation of the set A in the real line. As A is closed, $c \in A$. Similarly, $c \in B$. Thus $A \cap B \neq \phi$, which contradicts the disconnection of the real line. Therefore, the set R of all real numbers is connected in the real line.

EXAMPLE : Let (X, d) be a connected metric space and $f : (X, d) \to (Y, d')$. Prove that $f(X)$ is a connected subset of Y.

SOLUTION : If possible, let the set $f(X)$ is not connected in the metric space (Y, d'). Then we can find a non-empty proper subset H of $f(X)$ which is both open and closed in the subspace $f(X)$. Since H is open and f is continuous, $f^{-1}(H)$ is open in (X, d). Again since H is closed and f is continuous, $f^{-1}(H)$ is closed in (X, d).

H is proper subset of $f(X) \Rightarrow f(X) \setminus H \neq \phi \Rightarrow f^{-1}(f(X) \setminus H) \neq \phi \Rightarrow f^{-1}(H) \neq \phi$. Thus $f^{-1}(H)$ is a non-empty proper subset of X which is both open and closed in (X, d). So (X, d) not connected, a contradiction. Hence $f(X)$ must be connected in the metric space (Y, d').

NOTE : ***In case*** $f : (X, d) \to (Y, d')$ ***is an onto continuous map and*** X ***is connected, then*** $Y = f(X)$ ***is connected.***

Compactness:

DEFINITION : Let X be a non-empty set. A ***family*** $A=\{A_\alpha : \alpha \in \Lambda\}$ of subsets of X is said to be a ***cover*** of X if and only if $X = \cup\{A_\alpha : \alpha \in \Lambda\}$, where Λ is an index set.

* In such a case, we say that the family $A=\{A_\alpha : \alpha \in \Lambda\}$ covers X .

DEFINITION : Let Y be a non-empty subset of the set X . A family $B=\{B_\alpha : \alpha \in \Lambda\}$ of subsets of X is said to be a cover of Y if and only if $Y \subset \cup\{B_\alpha : \alpha \in \Lambda\}$, where Λ is an index set.

* In such a case, we say that the family $B=\{B_\alpha : \alpha \in \Lambda\}$ covers Y .

* If there exists a subfamily B' of B which also covers Y , we say that B' is a subcover of B .

*A cover is said to be a finite cover(respt. Countable) if it contains finite (respt. Countable) number of sets.

* If set in the family $A=\{A_\alpha : \alpha \in \Lambda\}$ are all open sets in a metric space (X,d), A is said to be an open cover of X in the metric space (X,d).

NOTE : When a family of subsets of X in a metric space (X,d) covers X , the metric d plays no role. But in order to be an open cover of X for a family of subsets of X , d must have role because, openness of a set depends on the underlying metric.

EXAMPLE : Show that the family $A=\{A_n = (-n,n) : n \in \mathrm{N}\}$, of bounded open intervals, is an open cover of R .

EXAMPLE : Show that each one of the following families is an open cover of the real line:

(i) $A_1 = \{(-|x|,|x|) : x \in R\}$

(ii) $A_2 = \{(x, x+1) : x \in \mathrm{N}\}$

(iii) $A_3 = \{(n-1, n+1) : n \in \mathrm{Z}\}$

DEFINITION : A ***metric space*** (X,d) is said to be a ***Lindelöf*** space if and only if every open cover of X in the metric space (X,d) admits of a countable subcover.

THEOREM 2.14: (Lindelöf Covering Theorem)

In the real line every open cover of a set has a countable subcover.

DEFINITION(Compact Space) : A ***metric space*** (X,d) is said to be a ***compact metric*** space if and only if every open cover of X in the metric space (X,d) admits of a finite subcover.

DEFINITION(Heine-Borel Property) : A ***metric space*** (X,d) is said to satisfy ***Heine-Borel Property*** if and only if every open cover of X in the metric space (X,d) admits of a finite subcover.

DEFINITION(Compact Set) : Let Y be a ***non-empty set*** in a metric space (X,d). Then Y is said to be a ***compact set*** if and only if every open cover of Y in the metric space (X,d) has a finite subcover.

NOTE : It is to be noted that by means of open sets, we consider those sets which are open in the metric space (X,d).

PROPERTIES OF COMPACT SPACES AND COMPACT SETS :

THEOREM 2.15: ***Every closed subset of a compact metric space is compact***

NOTE : If in any metric space we can find at least one closed set which is not compact, we can assert that the space is not compact.

EXAMPLE : Show that the real line is not compact.

SOLUTION : Let us consider the set Z of integers. We know that in the real line Z is closed. The family $A = \{(-n,n) : n \in \mathrm{N}\}$ is an open cover of Z in the real line since $Z \subset R = \cup\{(-n,n) : \alpha \in \Lambda\}$. Since $A = \{(-n,n) : n \in \mathrm{N}\}$ (cover of Z) has no finite subcover, the set Z is not compact. So Z is closed but not compact in the real line. Consequently, the real line is not compact.

THEOREM 2.16: ***In any metric space (X,d) every compact set is closed .***

Combining the theorem 2.15 & theorem 2.16, we get the following theorem:

THEOREM 2.17: ***A subset F of a compact metric space (X,d) is compact if and only if it is closed.***

THEOREM 2.18: ***Every compact subset of a metric space is bounded.***

Heine Borel Theorem

THEOREM 2.19: (Heine Borel Theorem) ***Every closed and bounded set in the real line is compact.***

Converse of Heine Borel Theorem : ***Every compact set in the real line is both closed and bounded.***

Finite Intersection Property

A family $A = \{A_\alpha : \alpha \in \Lambda\}$ of non-empty sets is said to posses finite intersection property if and only if every finite subfamily of $A = \{A_\alpha : \alpha \in \Lambda\}$ has non-empty intersection.

That is, for any arbitrary finite collection $\{A_{\alpha_1}, A_{\alpha_2}, A_{\alpha_3}, \ldots\ldots\ldots, A_{\alpha_n}\}$ of members of $A = \{A_\alpha : \alpha \in \Lambda\}$ we have, $\cap\{A_{\alpha_i} : i = 1,2,3,\ldots\ldots, n\} \neq \phi$

EXAMPLE : The collection $A = \{(-n,n) : n \in \mathrm{N}\}$ of open intervals satisfy finite intersection property. If we consider any finite collection $\{(-n_1,n_1),(-n_2,n_2),(-n_3,n_3),\ldots\ldots,(-n_p,n_p)\}$ of open intervals in R then $\cap\{(-n_r,n_r) : r = 1,2,3,\ldots\ldots, p\} = (-n_\alpha, n_\alpha) \neq \phi$ where $n_\alpha = \min\{n_1, n_2, \ldots\ldots\ldots, n_p\}$.

EXAMPLE : The collection $B = \{(n-1, n+1) : n \in Z\}$ of open intervals does not satisfy finite intersection property. If we consider the finite collection $\{(1,3),(3,5)\}$ of B and we find $(1,3) \cap (3,5) = \phi$

THEOREM 2.19: ***A metric space (X,d) is compact if and only if every infinite family of non-empty closed sets in (X,d) with finite intersection property has non-empty intersection.***

Proof :

CONTINUITY AND COMPACTNESS :

THEOREM 2.20: ***Continuous image of a compact metric space is compact.***

Proof : Let (X,d) be a compact metric space and f is a continuous mapping from (X,d) into another metric space (Y,d'). If $Y' = f(X) \subset Y$, then we are to prove that the set Y' is a compact subset of (Y,d'). Let $A = \{A_\alpha : \alpha \in \Lambda\}$ be any open cover of Y' in (Y,d'). We are to show that it has a finite subcover. By proposition $Y' = \cup\{A_\alpha : \alpha \in \Lambda\}$. Hence we get,

$X = f^{-1}(Y') = f^{-1}[\cup\{A_\alpha : \alpha \in \Lambda\}] = \cup\{f^{-1}(A_\alpha) : \alpha \in \Lambda\}$..(1)

Since for all $\alpha \in \Lambda$, A_α is open in (Y,d') and f is continuous, it follows that $f^{-1}(A_\alpha)$ is open in (X,d), for all $\alpha \in \Lambda$. Then from (1) it follows that $\{f^{-1}(A_\alpha) : \alpha \in \Lambda\}$ is an open cover of (X,d). Since (X,d) is compact, it has a finite subcover, say, $\{f^{-1}(A_{\alpha_i}) : i = 1,2,3,....., n : \alpha_i \in \Lambda\}$. We show shall that $\{A_{\alpha_i} : i = 1,2,3,....., n : \alpha_i \in \Lambda\}$ is an open cover Y' in (Y,d'). Let $y \in Y'$ be arbitrarily chosen. Then there exists at least one $x \in X$ such that $f(x) = y$. Since $\{f^{-1}(A_{\alpha_i}) : i = 1,2,3,....., n : \alpha_i \in \Lambda\}$ is an open cover of (X,d), for some integer $i'(1 \le i' \le n)$, $x \in f^{-1}(A_{\alpha_i})$. Hence $y = f(x) \in A_{\alpha_i}$. Therefore, $Y' = \cup\{f^{-1}(A_{\alpha_i}) : i = 1,2,3,....., n : \alpha_i \in \Lambda\}$. Consequently, Y' is a compact set in (Y,d'). Therefore, continuous image of a compact metric space is compact.

NOTE : ***In case*** $f : (X,d) \to (Y,d')$ ***is an onto continuous map and*** (X,d) ***is compact, then*** (Y,d') ***is also compact.***

NOTE : ***If*** $f : (X,d) \to (Y,d')$ ***is continuous map and*** $A \subset X$ ***is a compact set in*** (X,d)***, then*** $f(A) \subset Y$ ***is also compact in*** (Y,d')***.***

SEQUENTIALLY COMPACT SPACE

DEFINITION : A **metric space** (X,d) is said to be ***sequentially compact*** if and only if every sequence in X has a convergent subsequence.

DEFINITION : A **non-empty set** $A \subset X$ is said to be ***sequentially compact*** if and only if every sequence in A has a convergent subsequence.

EXAMPLE : In the real line the set R of all real numbers is not sequentially compact.

SOLUTION : Let us consider the sequence $\{x_n\}$ in R defined by $x_n = n$ for all $n \in \mathrm{N}$. Clearly, $\{x_n\}$ has no convergent subsequence. Hence R is not sequentially compact.

EXAMPLE : In the metric space R with usual metric, the set $(0,1) \subset R$ is not sequentially compact.

SOLUTION : Let us consider the sequence $\{x_n\}$ in R defined by $x_n = \frac{1}{n}$ for all $n \in \mathrm{N}$. $\{x_n\}$ has no subsequence which converges to any point in $(0,1)$. Hence $(0,1)$ is not sequentially compact set.

NOTE : However, in the real line the closed interval $[0,1]$ is sequentially compact set.

PROPERTIES OF SEQUENTIALLY COMPACT SETS

THEOREM 2.21: ***In a metric space a sequentially compact set is both bounded and closed.***

THEOREM 2.22: ***A sequentially compact metric space is complete.***

Proof : Let (X,d) be a sequentially compact metric space. In order to prove the theorem it is sufficient to show that any Cauchy sequence $\{x_n\}$ in (X,d) converges in X. Since $\{x_n\}$ is a Cauchy sequence in (X,d), corresponding to $\varepsilon(>0)$, chosen arbitrarily, there exists a positive integer N_1, depending on $\varepsilon(>0)$, such that $d(x_{n+p}, x_n) < \varepsilon$, for all $p \in \mathrm{N}$ and $n > N_1$. Since the metric space (X,d) is sequentially compact, the sequence $\{x_n\}$ must have a subsequence which converges in X. Let the subsequence be $\{x_{n_k}\}$ and $x_{n_k} \to x \in X$ as $n_k \to \infty$. So, there exists a positive integer N_2, depending on $\varepsilon(>0)$, such that $d(x_{n_k}, x) < \varepsilon$, whenever $n_k > N_2$. Let $N = \max\{N_1, N_2\}$. Then for all $n_k > n > N$ we get, $d(x_n, x) \le d(x_n, x_{n_k}) + d(x_{n_k}, x) < \varepsilon + \varepsilon = 2\varepsilon$. Therefore, the sequence $\{x_n\}$ converges to $x \in X$. Consequently, the metric space (X,d) is complete.

THEOREM 2.23: ***Every compact metric space is sequentially compact.***

COMPACTNESS AND TOTAL BOUNDEDNESS

DEFINITION : Let (X,d) be a metric space and ε be an arbitrarily chosen positive quantity. A non-empty subset A of X is said to be an $\varepsilon - net$ of (X,d), if the set of all open spheres of radius ε with centres in A covers X.

This implies for any $x \in X$, we can find at least one $a \in A$, such that $x \in S(a,\varepsilon)$, that is, $d(a,x) < \varepsilon$.

EXAMPLE : In the real line the set Z of all integers is an $1 - net$ but not a $\frac{1}{2} - net$.

DEFINITION : Let (X,d) be a metric space. A non-empty subset A of X is said to be totally bounded if and only if for every $\varepsilon(>0)$, the set A has a finite $\varepsilon - net$.

This implies for any $\varepsilon(>0)$, a finite collection of open spheres of radius ε covers A.

THEOREM 2.24: ***A metric space (X,d) is totally bounded if and only if every sequence in X has a Cauchy subsequence.***

THEOREM 2.25: ***A metric space (X,d) is sequentially compact if and only if it is complete and totally bounded.***

Proof : Let the metric space (X,d) is complete and totally bounded. Since (X,d) is totally bounded, every sequence $\{x_n\}$ in (X,d) has a Cauchy subsequence $\{y_n\}$. Since (X,d) is complete, any Cauchy sequence $\{y_n\}$ is convergent. So every sequence $\{x_n\}$ in (X,d) has a convergent subsequence. Therefore, the metric space (X,d) is sequentially compact.

Conversely, let the metric space (X,d) is sequentially compact. Then every sequence $\{x_n\}$ in (X,d) has a convergent subsequence $\{y_n\}$. Since the sequence $\{y_n\}$ satisfies Cauchy property, by theorem 2.24, it follows that (X,d) is totally bounded.

Moreover, as the metric space (X,d) is sequentially compact, every sequence $\{x_n\}$ in (X,d) has a convergent subsequence. Specifically, every Cauchy sequence $\{x_n\}$ in (X,d) has a convergent subsequence. We know that a Cauchy sequence is convergent if and only if it has a convergent subsequence. Thus every Cauchy sequence in (X,d) is convergent. Hence the metric space (X,d) is complete.

COROLLARY : Let (X,d) be a complete metric space. Then a non-empty subset A of X is compact if and only if A is totally bounded in (X,d).

THEOREM 2.26: ***Every sequentially compact metric space is compact.***

INSTRUCTION FOR STUDENTS :

All definitions and examples are to be followed. All red marked theorems are important for 6th semester of the year 2020.