Chapter 4

Electrostatic Energy and Capacitors

4.1 Introduction

The solution of many mechanical problems gets very much simplified by means of energy consideration. The study of the mechanical behaviour of an electrical system is also found to be simplified by the use of energy consideration. In this chapter we shall calculate the energy associated with various electrostatic charge distributions. For an electrostatic system no kinetic energy is imparted to the charges and the energy is wholly potential in nature. The work necessary to assemble a system of charges against coulomb forces is stored in the system as a potential energy. This is known as *electrostatic energy*. Here it is assumed that the charges are brought to their positions at rest from their initial positions at rest at infinite distance apart. From a knowledge of this electrostatic energy it is possible to calculate the forces and torques acting on the system.

4.2 Electrostatic Energy of an Assembly of Point Charges

Let us calculate the electrostatic energy of an assembly of point charges. We can do this by calculating the work done in assembling the system by bringing the charges in one by one from positions at infinite distance apart. To place the first charge q_1 at the position $\vec{r}_1$ we require no work ($u_1 = 0$) because there is no interacting coulomb field. Next to bring the charge q_2 to the position $\vec{r}_2$ we require work to be done against the Coulomb repulsion due to q_1. This work equals

$$u_{12} = -\frac{1}{4\pi\epsilon_0}\int_{\infty}^{r_{12}} \frac{q_1 q_2}{r^2} dr = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}}, \tag{4.2-1}$$

where $r_{12} = |\vec{r}_1 - \vec{r}_2|$ is the distance of q_2 from q_1.

Next to bring the charge q_3 to the position $\vec{r}_3$ work is to be done against the field of both q_1 and q_2. So the work required is

$$\frac{1}{4\pi\epsilon_0}\frac{q_1 q_3}{r_{13}} + \frac{1}{4\pi\epsilon_0}\frac{q_2 q_3}{r_{23}} = u_{13} + u_{23}. \tag{4.2-2}$$

Thus, total electrostatic energy of a system of three charges is $u_{12} + u_{13} + u_{23}$, which is equal to the sum of potential energies of various pairs.

Proceeding in this way to bring the charges one after another we find that the total work involved in assembling N point charges is

$$U = u_{12} + u_{13} + u_{23} + \cdots = \sum_{\text{all pairs}} \frac{1}{4\pi\epsilon_0}\frac{q_i q_j}{r_{ij}}, \tag{4.2-3}$$

where q_i and q_j are a pair of charges separated by a distance r_{ij}.

For N point charges the expression for U can also be written as

$$U = \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{1}{4\pi\epsilon_0}\frac{q_i q_j}{r_{ij}}, \tag{4.2-4}$$

where the factor $\frac{1}{2}$ is included to avoid double counting of each pair. Note that the terms with $j = i$ are excluded because it represents self-terms.

The electrostatic energy U can also be written in terms of the electrostatic potential. Thus,

$$U = \frac{1}{2}\sum_{i=1}^{N} q_i \phi_i, \tag{4.2-5}$$

where

$$\phi_i = \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{1}{4\pi\epsilon_0}\frac{q_j}{r_{ij}}, \tag{4.2-6}$$

is the potential at the location of the ith charge due to all other charges excepting q_i.

4.3 Electrostatic Energy of a Continuous Charge Distribution

For a region of space having a continuous charge distribution characterised by a volume density $\rho(\vec{r})$ over a volume V the expression (4.2-5) for U can be written in the form

$$U = \frac{1}{2}\int_V \rho(\vec{r})\,\phi(\vec{r})\,dV. \tag{4.3-1}$$

For a general type of charge distribution characterised by a volume density $\rho(\vec{r})$ over a volume V, surface density $\sigma(\vec{r})$ over a surface S, line density $\lambda(\vec{r})$ over a length l and points charges $q_i (i = 1, 2, \ldots, N)$ total electrostatic energy of the system will be given by

$$U = \frac{1}{2}\int_V \rho(\vec{r})\,\phi(\vec{r})\,dV + \frac{1}{2}\int_S \sigma(\vec{r})\,\phi(\vec{r})\,dS + \frac{1}{2}\int_l \lambda(\vec{r})\,\phi(\vec{r})\,dl + \frac{1}{2}\sum_{i=1}^{N} q_i\phi_i. \quad (4.3\text{-}2)$$

4.4 Electrostatic Energy in Terms of Field Distribution

In some cases it becomes important to express electrostatic energy in terms of field vectors $\vec{E}$ and $\vec{D}$ of the system. Suppose we have a finite region of space V in a dielectric medium of permittivity ϵ, characterised by a volume density ρ of free charges. The electrostatic energy for this system is given by

$$U = \tfrac{1}{2}\int_V \rho(\vec{r})\,\phi(\vec{r})\,dV. \quad (4.4\text{-}1)$$

Now using the differential form of Gauss's law, $\vec{\nabla}\cdot\vec{D} = \rho$, we can write

$$U = \tfrac{1}{2}\int_V \left(\vec{\nabla}\cdot\vec{D}\right)\phi\,dV. \quad (4.4\text{-}2)$$

Using the vector identity

$$\vec{\nabla}\cdot\left(\phi\vec{D}\right) = \vec{\nabla}\phi\cdot\vec{D} + \phi\left(\vec{\nabla}\cdot\vec{D}\right)$$

we get

$$U = \tfrac{1}{2}\int_V \vec{\nabla}\cdot\left(\phi\vec{D}\right)\,dV - \tfrac{1}{2}\int_V \vec{\nabla}\phi\cdot\vec{D}\,dV. \quad (4.4\text{-}3)$$

By using divergence theorem and the relation $\vec{E} = -\vec{\nabla}\phi$ we get

$$U = \tfrac{1}{2}\oint_S \left(\phi\vec{D}\right)\cdot d\vec{S} + \tfrac{1}{2}\int_V \vec{E}\cdot\vec{D}\,dV, \quad (4.4\text{-}4)$$

where S is the surface enclosing the volume V. Let us now allow the region of integration V to expand to infinity. This is permissible since from Eq. (4.4-1) we find that the contribution to the integral will be zero wherever $\rho = 0$. Now at large distances from the charge distribution ϕ falls off as $1/r$, $\vec{D}$ as $1/r^2$ while the surface area S grows like r^2. So, roughly speaking, the surface integral in Eq. (4.4-4) goes down like $1/r$. Thus, as the surface S is expanded to include all of space the surface integral vanishes and we are left with

$$U = \tfrac{1}{2}\int_{\text{all space}} \vec{E}\cdot\vec{D}\,dV. \quad (4.4\text{-}5)$$

This equation suggests that we may consider the electrostatic energy as stored in the electric field with an energy density (i.e., energy per unit volume)

$$u = \tfrac{1}{2}\vec{E}\cdot\vec{D}. \tag{4.4-6}$$

In case of linear isotropic dielectrics $\vec{D} = \epsilon\vec{E}$ and we can write $u = \frac{1}{2}\epsilon E^2$. For charges in vacuum, $u = \frac{1}{2}\epsilon_0 E^2$.

4.5 Where is the Electrostatic Energy Stored?

Equations (4.4-1) and (4.4-5) show that the electrostatic energy can be expressed either as an integral over charge distribution or as an integral over the electrostatic field. Thus, one can assume that the electrostatic energy is stored either in the charge or in the field. Both the ideas lead to the same total electrostatic energy. From the stand point of electrostatics alone it is not possible to tell where the energy is actually stored. However, in non-static case (e.g., in radiation theory) it is found to be more useful to consider that the energy is stored in the field. When a charged particle oscillates it emits electromagnetic wave. The wave travels from one point to another and carries energy with it but there is no charge in the wave. Thus, it seems reasonable to locate energy within the field and not at the charges, which produce the field.

4.6 Self-Energy of a Point Charge

Let us examine whether the energy expressions (4.2-5) and (4.4-5) are equivalent or not. According to Eq. (4.4-5) U is always positive (since $\vec{E}\cdot\vec{D} = \epsilon E^2$ is always positive) whereas according to Eq. (4.2-5) U can take up positive as well as negative values. This means that there is some intriguing difference between the two equations. In Eq. (4.2-5) we do not include the work required to fabricate the point charges. Here we start with ready made point charges and take into account only the work done in bringing them to their desired positions. Equation (4.4-5), on the other hand, gives the total energy stored in a charge distribution including the energy required to fabricate finite point charges from infinitesimal parts. Energy given by the Eq. (4.4-5) is sometimes called the *self-energy* of the charge distribution. Let us calculate the self-energy U of a point charge q placed in vacuum by applying Eq. (4.4-5). Thus,

$$U = \tfrac{1}{2}\epsilon_0 \int_{\text{all space}} E^2\, dV = \tfrac{1}{2}\epsilon_0 \int_{r=0}^{\infty} \left(\frac{1}{4\pi\epsilon_0}\cdot\frac{q}{r^2}\right)^2 \cdot 4\pi r^2 dr = -\frac{q^2}{8\pi\epsilon_0}\left[\frac{1}{r}\right]_0^{\infty}.$$

Obviously the result comes out to be infinite, which is physically absurd. One way out of the difficulty is to consider finite radius of point charges. This leads to the idea of the *classical* radius of an electron.

Discussions

We have derived Eq. (4.4-5) by using Eq. (4.2-5). Then naturally the question arises *why do these equati:-ns describe two different situations?*. In Eq. (4.2-5) ϕ_i is the potential due to all other charges excepting q_i. But when we go over to continuous distribution of charge in Eq. (4.3-1) $\phi(\vec{r})$ represents the total potential at the point $\vec{r}$ because for a continuous distribution the charge right at the point can be taken to be vanishingly small.

4.7 Electrostatic Self-Energy of a Uniformly Charged Sphere

Method I (Direct-method)

Suppose the sphere is built up by assembling a succession of thin spherical shells of infinitesimal thickness. Let at any stage the radius is r. Now to increase the radius from r to $r+dr$ we require to bring an amount of charge $dq = \rho \cdot 4\pi r^2 dr$ from infinity, where ρ is the density of charge. When the radius of the sphere is r, its charge will be $q = \frac{4}{3}\pi r^3 \rho$ and the potential at its surface will be $q/4\pi\epsilon_0 r$. To add the dq amount of charge the work necessary will be

$$dU = \frac{q}{4\pi\epsilon_0 r} \times dq = \frac{\frac{4}{3}\pi r^3 \rho}{4\pi\epsilon_0 r} \times \rho \cdot 4\pi r^2 dr = \frac{4\pi\rho^2}{3\epsilon_0} \cdot r^4 dr.$$

Therefore, total work required to build up the sphere up to a radius a is

$$U = \frac{4\pi\rho^2}{3\epsilon_0} \int_0^a r^4 dr = \frac{4\pi\rho^2}{3\epsilon_0} \cdot \frac{a^5}{5} = \frac{1}{4\pi\epsilon_0} \cdot \frac{3Q^2}{5a}, \quad (4.7\text{-}1)$$

where $Q = \frac{4}{3}\pi a^3 \rho$ is the total charge on the sphere.

Method II [using Eq. (4.4-5)]

$$U = \frac{1}{2}\epsilon_0 \int_{\text{inside}} E^2\, dV + \frac{1}{2}\epsilon_0 \int_{\text{outside}} E^2\, dV$$

Now

$$E = \frac{1}{4\pi\epsilon_0} \cdot \frac{Q_r}{r^2} = \frac{1}{4\pi\epsilon_0} \frac{\frac{4}{3}\pi r^3 \rho}{r^2} = \frac{1}{4\pi\epsilon_0} \cdot \frac{Q \cdot r}{a^3} \text{ for } r < a.$$

$$E = \frac{1}{4\pi\epsilon_0} \cdot \frac{Q}{r^2} \text{ for } r > a$$

$$\therefore \quad U = \frac{\epsilon_0}{2}\int_0^a \left(\frac{Qr}{4\pi\epsilon_0 a^3}\right)^2 \cdot 4\pi r^2 dr + \frac{\epsilon_0}{2}\int_a^\infty \left(\frac{Q}{4\pi\epsilon_0 r^2}\right)^2 \cdot 4\pi r^2 dr$$

$$= \frac{Q^2}{2} \cdot \frac{1}{4\pi\epsilon_0}\left(\frac{1}{5a} + \frac{1}{a}\right) = \frac{1}{4\pi\epsilon_0} \cdot \frac{3Q^2}{5a}$$

Method III [using Eq. (4.3-1)]

From Eq. (1.8-11) the potential inside a uniformly charged sphere of radius a is given by

$$\phi(r) = \frac{Q}{4\pi\epsilon_0 a}\left(\frac{3}{2} - \frac{r^2}{2a^2}\right),$$

where Q is the total charge.

Now

$$U = \frac{1}{2}\int \rho\phi(r)\, dV = \frac{Q\rho}{8\pi\epsilon_0 a}\int_0^a \left(\frac{3}{2} - \frac{r^2}{2a^2}\right) \cdot 4\pi r^2 dr$$

$$= \frac{Q\rho}{2\epsilon_0 a}\left[\frac{3}{2} \cdot \frac{r^3}{3} - \frac{r^5}{10a^2}\right]_0^a = \frac{Q}{2\epsilon_0 a} \cdot \frac{Q}{\frac{4}{3}\pi a^3} \cdot \frac{2a^3}{5}$$

$$= \frac{1}{4\pi\epsilon_0} \cdot \frac{3Q^2}{5a}.$$

Classical radius of an electron

Suppose we consider the electron as a uniformly charged sphere of radius r_0 containing a total charge $-e$. The energy required to assemble this sphere of charge is

$$U = \frac{1}{4\pi\epsilon_0} \cdot \frac{3e^2}{5r_0}.$$

Experiments on the phenomenon of pair production show that it takes about 0.51 MeV energy to create an electron. This is equal to the rest mass energy, mc^2, of the electron. Let us assume that the electron radius can be estimated by equating its electrostatic self-energy to the rest mass energy, i.e.,

$$\frac{1}{4\pi\epsilon_0} \cdot \frac{3e^2}{5r_0} = mc^2 \quad \text{or} \quad r_0 = \frac{3e^2}{20\pi\epsilon_0 mc^2}. \tag{4.7-2}$$

Substituting the values of different quantities we get $r_0 \approx 1.7 \times 10^{-15}$ m. The actual radius of the electron is found to be much smaller then r_0. This indicates that the actual structure of the electron is not so simple as described above. It is rather complicated.

4.8 Electrostatic Energy of a System of Charged Conductors; Coefficient of Potential and Capacitance

Let us consider a system of N conductors where all charges reside on the surface of the conductors. In terms surface density of charge σ and the potential ϕ total electrostatic energy of the system can be calculated as

$$U = \frac{1}{2}\int_S \sigma(\vec{r})\,\phi(\vec{r})\,dS. \tag{4.8-1}$$

Since each conductor surface is an equipotential surface the above integration can be written as

$$U = \frac{1}{2}\sum_{i=1}^{N}\phi_i \oint_{S_i}\sigma\,dS = \frac{1}{2}\sum_{i=1}^{N}\phi_i Q_i, \tag{4.8-2}$$

where Q_i is the charge on the ith conductor and ϕ_i is its potential. It is found that a linear relationship exists between the potentials and charges on the various conductors in the system. In fact for a system of N conductors we can write

$$\phi_i = \sum_{j=1}^{N} p_{ij} Q_j \tag{4.8-3}$$

$$\text{and} \quad Q_i = \sum_{j=1}^{N} c_{ij}\phi_j, \tag{4.8-4}$$

where p_{ij} are called the *coefficients of potential*, c_{ii} are the coefficients of capacitance and $c_{ij}(i \neq j)$ are the coefficients of inductance. p_{ij} and c_{ij} depend only on the geometry.

The electrostatic energy of N conductors can now be written in terms of p_{ij} or c_{ij} :

$$U = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} p_{ij} Q_i Q_j \tag{4.8-5}$$

$$U = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij}\phi_i\phi_j. \tag{4.8-6}$$

Some of the important properties of p_{ij} and c_{ij} are the following:

$$p_{ij} = p_{ji}, \quad p_{ii} > 0 \quad \text{and} \quad p_{ii} \geq p_{ij}$$
$$c_{ij} = c_{ji}, \quad c_{ii} > 0 \quad \text{and} \quad c_{ij} \leq 0 \quad \text{for } i \neq j.$$

4.9 Forces and Torques from Electrostatic Energy

From a knowledge of the electrostatic energy it is possible to calculate the forces and torques acting on the system. Suppose we have an isolated system of charges whose electrostatic energy depends on the particular charge configuration. Now we allow one part of the system to undergo a small displacement $d\vec{r}$ under the influence of the electrical forces $\vec{F}$ acting upon it. The work done by $\vec{F}$ on the system under these circumstances is

$$dW = \vec{F} \cdot d\vec{r}. \tag{4.9-1}$$

As the system is isolated the work is done at the expense of the electrostatic energy U. Thus,

$$dW = -dU = -\vec{\nabla}U \cdot d\vec{r}. \tag{4.9-2}$$

Therefore,

$$\vec{F} = -\vec{\nabla}U \tag{4.9-3}$$

or in components form,

$$F_x = -\left.\frac{\partial U}{\partial x}\right|_Q, \quad F_y = -\left.\frac{\partial U}{\partial y}\right|_Q \quad \text{and} \quad F_z = -\left.\frac{\partial U}{\partial z}\right|_Q, \tag{4.9-4}$$

where a subscript Q has been added to emphasize that the system is isolated and its total charge remains constant during the displacement.

If the part of the system under consideration is constrained to rotate about an axis and it undergoes a small rotation $d\vec{\theta}$ under the action of a torque $\vec{\tau}$ then,

$$dW = \vec{\tau} \cdot d\vec{\theta} = -dU = -\vec{\nabla}_\theta U \cdot d\vec{\theta},$$

where $\vec{\nabla}_\theta$ represents derivatives with respect to θ_x, θ_y and θ_z. Therefore,

$$\vec{\tau} = -\vec{\nabla}_\theta U \tag{4.9-5}$$

and in components form,

$$\tau_x = -\left.\frac{\partial U}{\partial \theta_x}\right|_Q, \quad \tau_y = -\left.\frac{\partial U}{\partial \theta_y}\right|_Q \quad \text{and} \quad \tau_z = -\left.\frac{\partial U}{\partial \theta_z}\right|_Q. \tag{4.9-6}$$

There is an important class of problems in which Q is not maintained constant. All charges reside on the surfaces of conductors and the potentials of all conductors are kept constant during displacement by some external sources of energy such as batteries. If we allow a part of such a system to undergo a small displacement $d\vec{r}$ the work done by the electrical forces $\vec{F}$ acting on it will be $dW = \vec{F} \cdot d\vec{r}$. Additional work dW_B must also be done by the battery to maintain all the conductors at a constant potential. So for the conservation of energy,

$$\begin{aligned} \text{Work done on the system} &= \text{(Increase in potential energy of the system)} \\ &\quad +\text{(work done by the system)}. \\ \therefore \quad dW_B &= dU + dW. \end{aligned}$$

If the change in charge of the jth conductor is dQ_j then

$$dW_B = \sum_j \phi_j dQ_j. \tag{4.9-7}$$

The electrostatic energy U of the system of charged conductors is given by

$$U = \tfrac{1}{2}\sum_j \phi_j Q_j. \tag{4.9-8}$$

So at constant potential

$$dU = \tfrac{1}{2}\sum_j \phi_j dQ_j.$$

Therefore, $dW_B = 2dU$, i.e., energy supplied by the battery to maintain the potential constant is equal to double the increase in the electrical energy of the system.

Using this result we get from Eq. (4.9-7),

$$dW = dU\big|_\phi \quad \text{or} \quad \vec{F}\cdot d\vec{r} = \vec{\nabla}U\cdot d\vec{r}\big|_\phi.$$

Thus, $\vec{F} = \vec{\nabla}U\big|_\phi$. In component forms,

$$F_x = \frac{\partial U}{\partial x}\bigg|_\phi, \quad F_y = \frac{\partial U}{\partial y}\bigg|_\phi \quad \text{and} \quad F_z = \frac{\partial U}{\partial z}\bigg|_\phi. \tag{4.9-9}$$

In a similar way we can show that

$$\tau_x = \frac{\partial U}{\partial \theta_x}\bigg|_\phi, \quad \tau_y = \frac{\partial U}{\partial \theta_y}\bigg|_\phi \quad \text{and} \quad \tau_z = \frac{\partial U}{\partial \theta_z}\bigg|_\phi. \tag{4.9-10}$$

An example of the above energy method

Suppose we are to find the force on the plates of a parallel plate or capacitor. If Q be the charge on the positive plate, V be the potential difference between the two plates and C be the capacitance then electrostatic energy stored in the capacitor is given by (See, Section 4.13)

$$U = \frac{Q^2}{2C} \tag{4.9-11}$$

$$\text{or} \quad U = \tfrac{1}{2}CV^2, \tag{4.9-12}$$

where the capacitance C is given by

$$C = \frac{\epsilon_0 A}{x}$$

in which A is the area of each plate, x is the plate separation and ϵ_0 is the permittivity of the space (assumed to be air) between the plates.

Now if the charge Q is maintained constant then force on the plate will be

$$F_x = -\left.\frac{\partial U}{\partial x}\right|_Q = -\frac{\partial}{\partial x}\left(\frac{Q^2}{2C}\right)_Q = -\frac{Q^2}{2}\frac{\partial}{\partial x}\left(\frac{x}{\epsilon_0 A}\right) = -\frac{Q^2}{2\epsilon_0 A}.$$

If the potential V is kept constant then the force on the plate will be

$$F_x = \left.\frac{\partial U}{\partial x}\right|_V = \frac{\partial}{\partial x}\left(\frac{1}{2}CV^2\right)_V = \frac{1}{2}V^2\frac{\partial C}{\partial x} = \frac{1}{2}V^2\frac{\partial}{\partial x}\left(\frac{\epsilon_0 A}{x}\right) = -\frac{\epsilon_0 AV^2}{x^2} = -\frac{C^2V^2}{2\epsilon_0 A}.$$

4.10 Capacitors

A capacitor is a system of conductors and dielectric, which can store electric charge. Usually it consists of a pair of conductors containing equal and opposite charges ($\pm Q$) with a potential difference (V) between them, which is independent of the presence of other conductors.

Let us consider a capacitor formed by two such conductors 1 and 2 carrying charges $Q_1 = +Q$ and $Q_2 = -Q$, respectively. Total charge of the system is zero but by convention the charge on the positive conductor is called the *charge on the capacitor.* Now the potentials of the conductors in terms of the coefficients p_{ij} of potential can be written as

$$\begin{aligned}\phi_1 &= p_{11}Q_1 + p_{12}Q_2 = Q\left(p_{11} - p_{12}\right)\\ \phi_2 &= p_{21}Q_1 + p_{22}Q_2 = Q\left(p_{21} - p_{22}\right).\end{aligned}$$

Therefore, potential difference

$$V = \phi_1 - \phi_2 = Q\left(p_{11} - p_{12} - p_{21} + p_{22}\right).$$

Thus, the potential difference between the conductors is proportional to the charge on the capacitor. The above relation is conventionally written as

$$Q = CV,$$

where $C = (p_{11} - p_{12} - p_{21} + p_{22})^{-1}$ is called the *capacitance of the capacitor.* Occasionally the word *condenser* is also used for capacitor. It is a purely geometrical quantity determined by the shape, size and separation of the two conductors. The capacitance C of a capacitor is numerically equal to the charge required to be placed on the capacitor to raise its potential by unity. The SI unit of capacitance is the coulomb per volt, called the *farad* (F). *One farad is the capacitance of a capacitor, which requires 1C charge to establish a pd of one volt across it.* For practical purposes it is a too large unit. For this units like micro farad (μ F), nano farad (nF) or pico farad (pF) are used: 1 μF = 10^{-6}F, 1nF= 10^{-9}F, 1pF= 10^{-12}F.

Sometimes we speak of the capacitance of an isolated single conductor. Such a conductor is assumed to be the part of a capacitor whose other conductor is at infinity.

4.11 Calculation of Capacitance for Different Geometries

1. Parallel plate capacitor

A parallel plate capacitor consists of two parallel metal plates separated by a dielectric medium. Suppose the plates are of area A each and separated by a distance d. We assume that d is very small compared with the linear dimensions of the plates so that the end effects may be neglected and the electric field between the plates may be taken to be uniform. Suppose that equal and opposite charges $\pm Q$ are put on the plates. The charges get distributed on the plates in such a way that the field inside the thickness of the plates becomes zero. Thus, charges spread only over the inner surfaces of the plates and give uniform charge densities $+\sigma = Q/A$ and $-\sigma = -Q/A$ as shown in Fig 4.11-1. The electric field will be uniform, normal to the plates and of magnitude σ/ϵ between the plates and zero elsewhere, where ϵ is the permittivity of the medium.

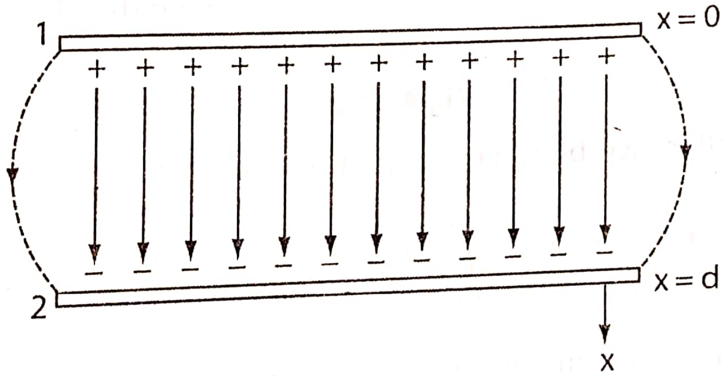

Fig 4.11-1

The potential difference between the plates is given by

$$V = \phi_1 - \phi_2 = -\int_2^1 \vec{E} \cdot d\vec{r} = -\int_2^1 \frac{\sigma}{\epsilon}\hat{\imath} \cdot d\vec{r} = -\int_2^1 \frac{\sigma}{\epsilon} dx$$

$$= -\left[\frac{\sigma}{\epsilon} x\right]_d^0 = \frac{\sigma}{\epsilon} \cdot d = \frac{Qd}{\epsilon A}.$$

Now by definition the capacitance of this capacitor is

$$C = \frac{Q}{V} = \frac{\epsilon A}{d} \tag{4.11-1}$$

Special cases

(i) Air capacitor

If the space between the plates is air then its capacitance would be

$$C = \frac{\epsilon_0 A}{d}. \tag{4.11-2}$$

(ii) Capacitor with a composite dielectric medium

Let the medium between the plates consists of two linear dielectrics of thickness d_1 and d_2 and permittivities ϵ_1 and ϵ_2, respectively (Fig 4.11-2). The field intensities within the first and second media are given by

$$E_1 = \frac{\sigma}{\epsilon_1} \quad \text{and} \quad E_2 = \frac{\sigma}{\epsilon_2}.$$

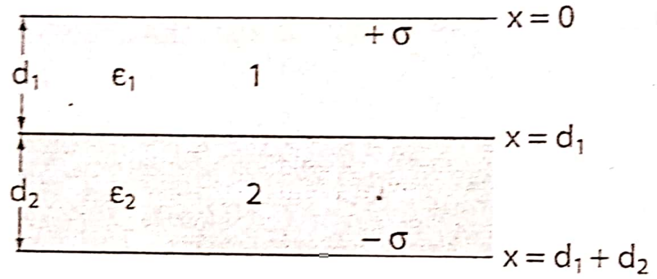

Fig 4.11-2

So the potential difference between the plates would be

$$V = \phi_1 - \phi_2 = -\int_{d_1+d_2}^{d_1} E_2 dx - \int_{d_1}^{0} E_1 dx = \frac{\sigma}{\epsilon_1} \cdot d_1 + \frac{\sigma}{\epsilon_2} \cdot d_2 = \frac{Q}{A}\left(\frac{d_1}{\epsilon_1} + \frac{d_2}{\epsilon_2}\right).$$

Therefore, the capacitance in this case, would be

$$C = \frac{Q}{V} = \frac{A}{\frac{d_1}{\epsilon_1} + \frac{d_2}{\epsilon_2}} = \frac{\epsilon_0 A}{\frac{d_1}{K_1} + \frac{d_2}{K_2}}, \qquad (4.11\text{-}3)$$

where K_1 and K_2 are the dielectric constants of the two media. For a capacitor with several dielectric media,

$$C = \frac{\epsilon_0 A}{\sum_i \frac{d_i}{K_i}}. \qquad (4.11\text{-}4)$$

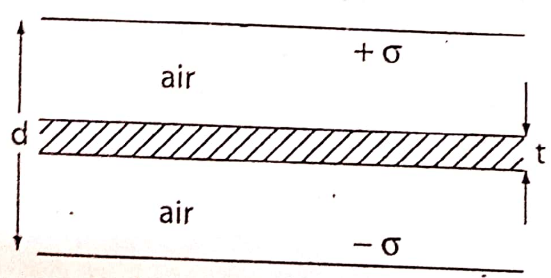

Fig 4.11-3

As a special case let us consider a parallel plate air capacitor with a dielectric slab of thickness $t (t < d)$ and permittivity ϵ introduced between the plates (Fig 4.11-3). In this case, pd between the plates would be

$$V = \frac{\sigma}{\epsilon} \cdot t + \frac{\sigma}{\epsilon_0}(d - t) = \frac{Q}{A}\left[\frac{t}{\epsilon} + \frac{d-t}{\epsilon_0}\right]$$

and capacitance

$$C = \frac{Q}{V} = \frac{\epsilon_0 A}{d - t + \frac{t}{K}} = \frac{\epsilon_0 A}{d - t\left(1 - \frac{1}{K}\right)}. \qquad (4.11\text{-}5)$$

As the dielectric constant $K > 1$, comparison with Eq. (4.11-2) shows that the capacitance increases with the introduction of the dielectric slab.

(*iii*) Capacitor with a nonuniform dielectric

Suppose the space between the plates is filled with a nonuniform dielectric whose dielectric constant varies linearly from one plate to the other. Let the dielectric constant at P at a distance x from the plate 1 be K and $\frac{dK}{dx} = \alpha$ (constant). Therefore, $K = \alpha x + \beta$ (constant).

Now at $x = 0$, let $K = K_1$ and at $x = d$, $K = K_2$. So $\beta = K_1$ and $\alpha = \frac{K_2 - K_1}{d}$. Thus,

$$K = \frac{K_2 - K_1}{d}x + K_1.$$

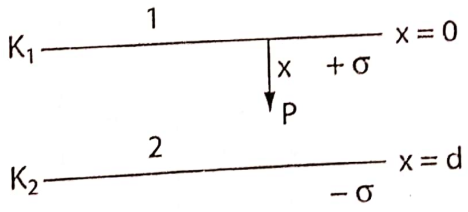

Fig 4.11-4

Now field at P is

$$E = \frac{\sigma}{\epsilon} = \frac{\sigma}{\epsilon_0 K} = \frac{\sigma}{\epsilon_0\left[\frac{K_2 - K_1}{d}x + K_1\right]}$$

Therefore, pd between the plates is

$$\begin{aligned} V = -\int_2^1 E\,dx &= -\frac{\sigma}{\epsilon_0}\int_d^0 \frac{dx}{\frac{K_2 - K_1}{d}x + K_1} \\ &= -\frac{\sigma}{\epsilon_0} \cdot \frac{d}{K_2 - K_1} \cdot \ln\left(\frac{K_2 - K_1}{d}x + K_1\right)\Bigg|_d^0 \\ &= \frac{Qd}{\epsilon_0 A(K_2 - K_1)} \ln\frac{K_2}{K_1}. \end{aligned}$$

Now by definition the capacitance is

$$C = \frac{Q}{V} = \frac{\epsilon_0 A(K_2 - K_1)}{d \ln(K_2/K_1)}. \tag{4.11-6}$$

2. Cylindrical capacitor

A cylindrical capacitor consists of a pair of long concentric metal cylinders, the space between them being filled with a dielectric. The coaxial cable used in communication system is a common example of such a capacitor. Let a and b be the radii of the inner and outer cylinders respectively and ϵ be the permittivity of the dielectric medium (Fig 4.11-5). Suppose equal and opposite charges $+Q$ and $-Q$ are put on the inner and outer cylinders respectively. By symmetry the electric field will be radial from the axis if we are far away from the ends. If λ be the charge per unit length on the inner cylinder then electric field is given by

$$\vec{E} = \frac{\lambda}{2\pi\epsilon r}\hat{r} \quad \text{for } a < r < b$$

$$\text{and} \quad \vec{E} = 0 \quad \text{for } r < a \quad \text{and} \quad r > b.$$

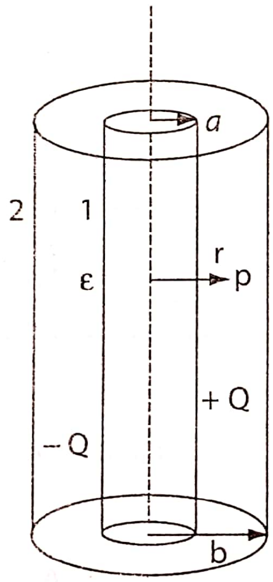

Fig 4.11-5

Now the potential difference between the two cylinders is

$$V = \phi_1 - \phi_2 = -\int_2^1 \vec{E} \cdot d\vec{r} = \int_a^b E\, dr = \frac{\lambda}{2\pi\epsilon}\int_a^b \frac{dr}{r} = \frac{\lambda}{2\pi\epsilon}\ln\frac{b}{a}.$$

By definition the capacitance per unit length is

$$C = \frac{\lambda}{V} = \frac{2\pi\epsilon}{\ln\frac{b}{a}}. \tag{4.11-7}$$

It is interesting to note that if the gap between the cylinders is very small compared with a, i.e., if $b - a \ll a$ then we can write

$$C = \frac{2\pi\epsilon}{\ln\left(1 + \frac{b-a}{a}\right)} \approx \frac{2\pi\epsilon}{\frac{b-a}{a}} = \frac{\epsilon \cdot 2\pi a}{b - a}.$$

Since $2\pi a$ is the area per unit length, the formula can be identified to be similar to that of a parallel plate capacitor.

3. Spherical capacitor

A spherical capacitor consists of a pair concentric metal spheres, the space between them being filled with a dielectric. Let a and b be the radii of the inner and outer spheres respectively and ϵ be the permittivity of the dielectric medium (Fig 4.11-6). Suppose equal and opposite charges $+Q$ and $-Q$ are put on the inner and outer spheres respectively. By symmetry the field lines will be radial. The electric field at a distance r from the common centre O is

$$\vec{E} = \begin{cases} \frac{Q}{4\pi\epsilon r^2}\hat{r}, & \text{for } a < r < b \\ 0, & \text{for } r < a \text{ and } r > b. \end{cases}$$

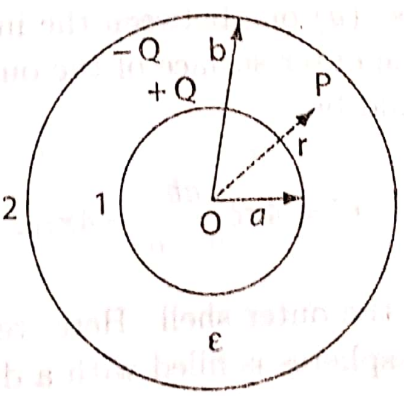

Fig 4.11-6

Now the pd between the spheres is

$$V = \phi_1 - \phi_2 = -\int_2^1 \vec{E} \cdot d\vec{r} = \int_a^b E\,dr = \frac{Q}{4\pi\epsilon}\int_a^b \frac{dr}{r^2} = \frac{Q}{4\pi\epsilon}\left(\frac{1}{a} - \frac{1}{b}\right)$$

By definition the capacitance of the capacitor is

$$C = \frac{Q}{V} = \frac{4\pi\epsilon}{\frac{1}{a} - \frac{1}{b}} = \frac{4\pi\epsilon ab}{b - a} \qquad (4.11\text{-}8)$$

Special cases

(*i*) Isolated sphere

An isolated sphere may be assumed to form a spherical capacitor whose outer spherical shell is at infinity. So putting $b = \infty$ in Eq. (4.11-8) we may get the capacitance of an isolated sphere as

$$C = 4\pi\epsilon a \tag{4.11-9}$$

(*ii*) Outer sphere is earthed

If the outer sphere is grounded its charge does not flow to the ground because it is held by the opposite charge of the inner sphere. If the system is initially uncharged and we put a charge $+Q$ on the inner sphere, this will induce a charge $-Q$ on the outer sphere. So in this case, capacitance will be the same as given by the Eq. (4.11-8).

(*iii*) Inner sphere is earthed and outer sphere is charged

Suppose that the total charge on the outer sphere be $+Q$. Due to the presence of inner grounded sphere a part of $+Q$, say, $+q_1$ will lie on the inner surface of the outer sphere and the rest $Q - q_1$ will lie on its outer surface (Fig 4.11-7). The charge $+q_1$ will induce a charge $-q_1$ on the inner sphere. Thus, the system can be considered as a parallel combination of two capacitors: (a) one between the inner sphere and the inner surface of the outer sphere and (b) the outer surface of the outer sphere and the earth. Hence, capacitance of the system would be

$$C = 4\pi\epsilon\frac{ab}{b-a} + 4\pi\epsilon_0 c, \tag{4.11-10}$$

where c is the outer radius of the outer shell. Here we assume that outside medium is air and the space between the spheres is filled with a dielectric of permittivity ϵ.

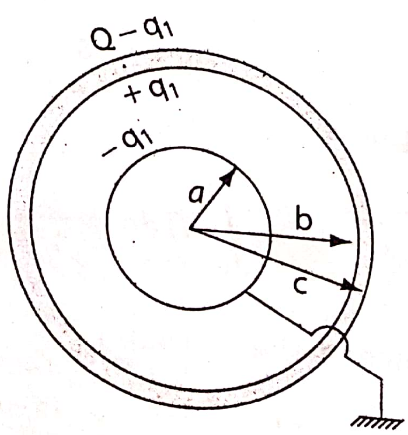

Fig 4.11-7

4. Capacitance between two long thin parallel wires in air

The overhead transmission line is a common example of such a capacitor system. Let a be the radius of each wire, d be the distance between them ($d \gg a$) and $+\lambda$ be the charge per unit length of wire 1 and $-\lambda$ be the charge per unit length of wire 2. The total electric field at a point P at a distance x from the wire 1 is

$$\vec{E} = \left[\frac{\lambda}{2\pi\epsilon_0 x} + \frac{\lambda}{2\pi\epsilon_0 (d-x)}\right]\hat{x}.$$

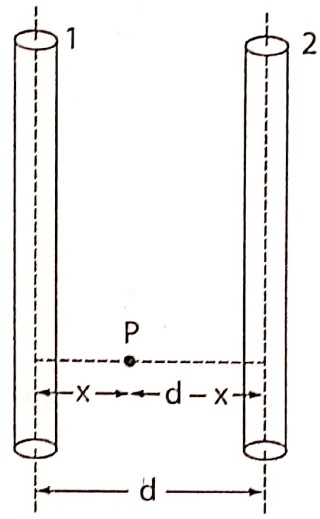

Fig 4.11-8

Hence, the pd between the wires is given by

$$\begin{aligned} V = \phi_1 - \phi_2 = -\int_{d-a}^{a} E\,dx &= \frac{\lambda}{2\pi\epsilon_0}\int_a^{d-a}\left(\frac{1}{x}+\frac{1}{d-x}\right)dx \\ &= \frac{\lambda}{2\pi\epsilon_0}\left[\ln x - \ln(d-x)\right]_a^{d-a} \\ &= \frac{\lambda}{\pi\epsilon_0}\ln\frac{d-a}{a}. \end{aligned}$$

∴ Capacitance per unit length

$$C = \frac{\lambda}{V} = \frac{\pi\epsilon_0}{\ln\frac{d-a}{a}} \qquad (4.11\text{-}11)$$

$$\approx \frac{\pi\epsilon_0}{\ln\frac{d}{a}}. \qquad (4.11\text{-}12)$$

4.12 Combination of Capacitors

Capacitors are often used in combinations. There are two simple combinations—*series* and *parallel*. Fig 4.12-1 shows the parallel combination of two capacitors. Here pd between the plates of each capacitor is the same and charge stored by a capacitor depends on its capacitance. Let $V = V_A - V_B$ be the pd across each capacitor. Charge on C_1 is $q_1 = C_1V$ and that on C_2 is $q_2 = C_2V$. The total charge on the two capacitors is $Q = q_1 + q_2 = C_1V + C_2V = (C_1 + C_2)V$. In terms of a single equivalent capacitor C, which stores a charge Q for a pd V we can write

$$C = \frac{Q}{V} = C_1 + C_2. \tag{4.12-1}$$

In the general case of N capacitors in parallel we have the equivalent capacitor,

$$C = \sum_{i=1}^{N} C_i. \tag{4.12-2}$$

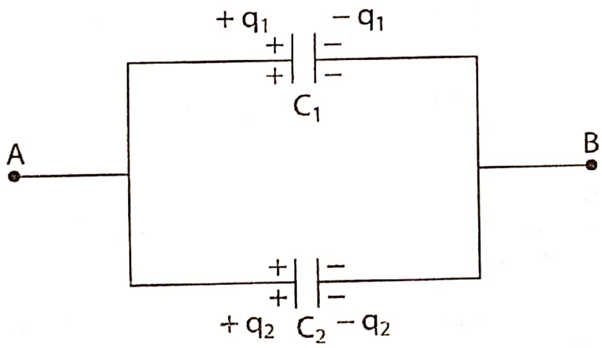

Fig 4.12-1

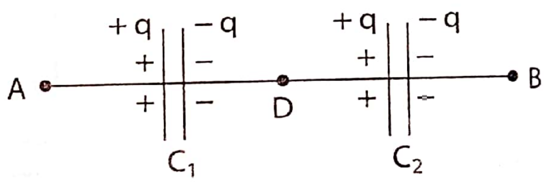

Fig 4.12-2

Fig 4.12-2 shows the series combination of two capacitors. Here, each capacitor acquires the same charge q but pd across each is different. From Fig 4.12-2,

$$\phi_A - \phi_D = \frac{q}{C_1} \quad \text{and} \quad \phi_D - \phi_B = \frac{q}{C_2}.$$

$\therefore$ pd across the combination is

$$V = \phi_A - \phi_B = (\phi_A - \phi_D) + (\phi_D - \phi_B) = \frac{q}{C_1} + \frac{q}{C_2} = q\left(\frac{1}{C_1} + \frac{1}{C_2}\right).$$

In terms of an equivalent single capacitor C, which stores a charge q for a pd V we can write

$$\frac{1}{C} = \frac{V}{q} = \frac{1}{C_1} + \frac{1}{C_2}. \tag{4.12-3}$$

In the general case of N capacitors in series we can write

$$\frac{1}{C} = \sum_{i=1}^{N} \frac{1}{C_i}. \tag{4.12-4}$$

Note that the equivalent capacitance in parallel combination is greater than any single one of them, whereas in series combination the equivalent capacitance is smaller than any single one.

4.13 Energy Stored in a Charged Capacitor

In order to charge up a capacitor work is to be done against the electrostatic forces present. This work is stored in the capacitor as electrostatic potential energy. Let us start with an uncharged capacitor. To charge it up we must remove electrons from one conductor and carry them to the other conductor. Suppose at any intermediate stage during charging the charge on the positive conductor is q and the pd across the capacitor is $\phi = q/C$, where C is the capacitance. Work done to increase the charge by an amount dq is

$$dW = \phi dq = \frac{q}{C} \cdot dq.$$

Therefore, total work done in charing the capacitor from uncharged condition to a state with charge Q would be

$$W = \int_0^Q \frac{1}{C} q dq = \frac{Q^2}{2C}. \qquad (4.13\text{-}1)$$

If V be the final pd of the capacitor then $Q = CV$. So

$$W = \frac{1}{2} CV^2. \qquad (4.13\text{-}2)$$

4.14 Dielectric Strength and Maximum Operating Voltage of a Capacitor

When a dielectric is placed in a sufficiently large electric field it begins to pull electrons completely from the atoms and molecules and the dielectric starts conducting. Then dielectric breakdown is said to have occurred. Dielectric breakdown depends on the nature of material, temperature, humidity etc. *The maximum electric field that a dielectric can withstand or tolerate without breaking down is called the dielectric strength.*

This problem of dielectric breakdown is of considerable importance in electrical technology. Capacitor specifications include the maximum voltage that can be safely applied without having dielectric breakdown. This voltage depends on the nature and thickness of the dielectric. If the capacitor is used beyond the specified voltage the dielectric is likely to breakdown. For air at normal pressure the dielectric strength is about 3×10^6 V/m. By connecting capacitors in series they can be used over higher voltages. By the specification "10μF, 50 V" we mean a capacitor of 10μF and 50 V is the maximum voltage that can be safely applied across it.

SOLVED PROBLEMS

Problem 1. *Find the electrostatic energy stored in the space surrounding a uniformly charged, spherical shell (or a charged conducting sphere) of radius R carrying a total charge Q.* [J.U. 2010]