Sequence – continued   Lecture 3

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Sandwich Theorem:-

Suppose ∃ a sequence $\{z_n\}$ such that
$x_n \le y_n \le z_n \quad \forall n$.
Suppose $\{x_n\}$ & $\{z_n\}$ converges to the same limit, say $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = l$.
Then $\lim_{n\to\infty} y_n = l$.

[The idea is that if the sequence $\{y_n\}$ is ~~sa~~ sandwiched between two convergent sequency ~~then~~ with same limit then that should also converge to the same limit.]

Proof:- ~~Since $\lim_{n\to\infty} x_n = l$. & $\lim_{n\to\infty} y_n = l$~~

Since $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = l$
$\forall \varepsilon > 0$ ∃ natural numbers n_1, n_2 s.t.

$|x_n - l| < \varepsilon \quad \forall n \geq n_1$
$|z_n - l| < \varepsilon \quad \forall n \geq n_2$

Now, take $n_0 = \max\{n_1, n_2\}$
Then $\forall n \geq n_0$, $|x_n - l| < \varepsilon$ & $|z_n - l| < \varepsilon$

$\Rightarrow \forall n \geq n_0, \quad l-\varepsilon < x_n < l+\varepsilon$ & $l-\varepsilon < z_n < l+\varepsilon$

$\Rightarrow \forall n \geq n_0, \quad l-\varepsilon < x_n < z_n < l+\varepsilon$

$\Rightarrow \forall n \geq n_0, \quad l-\varepsilon < x_n < y_n < z_n < l+\varepsilon$

$\Rightarrow \forall n \geq n_0, \quad |y_n - l| < \varepsilon$

$\Rightarrow \lim_{n\to\infty} y_n = l$.

Another form of Sandwich Theorem

Suppose $\exists$ a sequence $\{z_n\}$ such that $x_n \le y_n \le z_n \ \forall n \ge m$, for some natural number m.

Suppose $\{x_n\}$ & $\{z_n\}$ converges to the same limit, say $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = l$.

Then $\lim_{n\to\infty} y_n = l$.

Proof is left for you.

Note:-

Ex1:- Show that

$$\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{1}{(n+1)^2}+\cdots+\frac{1}{(2n)^2}\right)=0$$

$\Rightarrow$ We know

$$\frac{(n+1)}{n^2}<\frac{1}{n^2}+\frac{1}{(n+1)^2}+\cdots+\frac{1}{(2n)^2}<\frac{n+1}{(2n)^2}$$

$(n+1)^2>n^2$

$\Rightarrow \frac{1}{(n+1)^2}<\frac{1}{n^2}$

$\frac{1}{(2n)^2}<\frac{1}{n^2}$

$n+\cancel{n}<n+n$

$\Rightarrow \frac{1}{n^2}<\frac{1}{(2n)^2}$

$\frac{1}{(n+1)^2}<\frac{1}{2n^2}$

now $x_n=\frac{n+1}{n^2}$

$\lim_{n\to\infty}x_n=0$

$z_n=\frac{n+1}{(2n)^2}$

$\lim_{n\to\infty}z_n=0$

By sandwich thm.

$\Rightarrow \lim_{n\to\infty}y_n=0$

Using sandwich theorem ~~prove the~~ find the following limits

① $\lim_{n\to\infty}\frac{\sin n}{n}=0$, $\lim_{n\to\infty}\frac{\cos n}{n}=0$

② $\lim_{n\to\infty}x_n=1$

where $x_n=\frac{n^2}{n^3+n+1}+\frac{n^2}{n^3+n+2}+\cdots+\frac{n^2}{n^3+2n}$

③ $\lim_{n\to\infty}x^n=0$, $0<x<1$.

$0<x<1$

$\Rightarrow \frac{1}{x}>1$, $\frac{1}{x}=1+a$, $a>0$

$x=\frac{1}{(1+a)}$

$(1+a)^n=1+na+\cdots$
$>na$

$\frac{1}{(1+a)^n}<\frac{1}{na}$

$0<x^n=\frac{1}{(1+a)^n}<\frac{1}{na}\to 0$ as $n\to\infty$

$\Rightarrow \lim_{n\to\infty}x^n=0$

(4) $\lim_{n\to\infty} x^{1/n} = 1,\ x>0,\ x\in\mathbb{R}.$

(18) $\cancel{\lim_{n\to\infty} n^{1/n} = 1}$

$\Rightarrow \quad x>0$

Let $\cancel{0<x<1}$ $\quad x>1$

$\Rightarrow x^{1/n} > 1,\ x^{1/n} = 1+a_n;\ a_n>0$

$$\Rightarrow x = (1+a_n)^n$$
$$= 1+na_n+\frac{n(n-1)}{2!}a_n^2+\cdots$$
$$> 1+na_n > na_n$$
$$\Rightarrow \quad 0<a_n<\frac{x}{n}$$
$$\Rightarrow \lim_{n\to\infty} a_n = 0$$
$$\Rightarrow \lim_{n\to\infty} x^{1/n} = 1$$

Now, let $0<x<1$

$$\Rightarrow \frac{1}{x} > 1$$
$$\Rightarrow \lim_{n\to\infty}\left(\frac{1}{x}\right)^{1/n} = 1$$
$$\Rightarrow \lim_{n\to\infty} x^{1/n} = 1.$$

(5) $\lim_{n\to\infty} n^{1/n} = 1.$

(6) $\lim_{n\to\infty}\left(\frac{1}{1+n^2}+\frac{2}{2+n^2}+\cdots+\frac{n}{n+n^2}\right) = 1/2$

(7) $\lim_{n\to\infty}\left[(\sqrt{2}-2^{1/3})(\sqrt{2}-2^{1/5})\cdots(\sqrt{2}-2^{\frac{1}{2n+1}})\right] = 0$

(8) $\lim_{n\to\infty}\left[n^{\alpha}-(n+1)^{\alpha}\right] = 0$, for some $\alpha\in(0,1)$

⑨ $\lim_{n\to\infty} \frac{2^n}{n!} = 0$

① $\lim_{n\to\infty} \frac{\sin n}{n} = 0$

we know

$-1 \le \sin n \le 1 \quad \forall n$

$\Rightarrow 0 \leftarrow -\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n} \to 0 \quad \forall n.$

$\Rightarrow \lim_{n\to\infty} \frac{\sin n}{n} = 0$

Similarly for $\lim_{n\to\infty} \frac{\cos n}{n} = 0$

② $\lim_{n\to\infty} \left(\frac{n^2}{n^3+n+1} + \frac{n^2}{n^3+n+2} + \cdots + \frac{n^2}{n^3+2n} \right)$

Sol^n:- $\cancel{n^3}$ $n^3+n+1 \not> n^3+n+n$

$\Rightarrow \frac{n^2}{n^3+n+1} \not> \frac{n^2}{n^3+2n}$.

$\vdots$

$\frac{n^2}{n^3+2n} \not> \frac{n^2}{n^3+2n}$

$\Rightarrow \cancel{\frac{n^2}{n^3+n+1} + \cdots + \frac{n^2}{n^3+2n}} > \frac{n \cdot n^2}{n^3+2n}$

$\Rightarrow \cancel{\frac{n^2}{}}$

Again $n^3+n+1 \ge n^3+n+1, \; n^3+n+2 > n^3+n+1$

$\Rightarrow \frac{n^2}{n^3+n+1} \le \frac{n^2}{n^3+n+1}$

$\frac{n^2}{n^3+n+2} \le \frac{n^2}{n^3+n+1}$

$\frac{n^2}{n^3+n+n} \le \frac{n^2}{n^3+n+1}$

Therefore,

$$\frac{n \cdot n^2}{n^3+2n} \le \frac{n^2}{n^3+n+1} + \frac{n^2}{n^3+n+2} + \cdots + \frac{n^2}{n^3+n+n} \le \frac{n \cdot n^2}{n^3+n^2+1} \quad \forall n.$$

$$\Rightarrow \frac{1}{1+2/n^2} \le x_n \le \frac{1}{1+1/n^2+1/n^3}$$

$$\Rightarrow \lim_{n\to\infty} x_n = 1$$

(5) $\lim_{n\to\infty} n^{1/n} = 1$

$n \ge 1$ When $n = 1$ limit is 1

$\Rightarrow n^{1/n} >$

When $n > 1$

$\Rightarrow n^{1/n} > 1$

$\Rightarrow n^{1/n} = 1 + a_n, \; a_n > 0$ when $n > 1$.

$\Rightarrow n = (1+a_n)^n > 1 \;\; \forall n > 1$

By ~~Binom~~ Binomial theorem,

if $n > 1$, $n = (1+a_n)^n = 1 + n a_n + \frac{n(n-1)}{2!} a_n^2 + \cdots$

$$\ge 1 + \frac{n(n-1)}{2!} a_n^2$$

$$\Rightarrow n - 1 \ge \frac{1}{2} n(n-1) a_n^2$$

$$\Rightarrow a_n^2 \le \frac{2}{n}$$

$\Rightarrow$ So, $0 < a_n < \sqrt{2/n}$

$$\Rightarrow \lim_{n\to\infty} a_n = 0$$

$$\Rightarrow \lim_{n\to\infty} n^{1/n} = 1$$

(6) is left for you

(7) $\lim_{n\to\infty} x_n$

where $x_n = (\sqrt{2}-2^{1/3})(\sqrt{2}-2^{1/5})\cdots(\sqrt{2}-2^{\frac{1}{2n+1}})$

$2^{\frac{1}{2n+1}} > 1 \quad \forall n \geq 1$

$\Rightarrow \quad -2^{\frac{1}{2n+1}} < -1 \quad \forall n.$

So, $(\sqrt{2}-2^{1/3}) < (\sqrt{2}-1)$

$(\sqrt{2}-2^{1/5}) < (\sqrt{2}-1)$

$\vdots$

$(\sqrt{2}-2^{\frac{1}{2n+1}}) < (\sqrt{2}-1)$

So, $\quad 0 < x_n < (\sqrt{2}-1)^n$

as $\quad \sqrt{2}-1 < 1$

$\Rightarrow \lim_{n\to\infty} (\sqrt{2}-1)^n = 0$

So, by sandwich theorem

$\lim_{n\to\infty} x_n = 0$

(8) $\lim_{n\to\infty} [n^\alpha - (n+1)^\alpha] \quad , \alpha \in (0,1)$

$x_n = n^\alpha - (n+1)^\alpha$

$= n^\alpha[1-(1+1/n)^\alpha]$ $<$ ~~$n^\alpha[1+\cdot 1$~~

$(1+\frac{1}{n})^\alpha < (1+1/n) \quad , \text{ for } \alpha \in (0,1)$

$\Rightarrow \quad -(1+1/n)^\alpha > -(1+1/n)$

So, $n^\alpha[1-(1+1/n)^\alpha]$

$x_n > n^\alpha[1-(1+1/n)]$

~~$n^\alpha[1+1 < x_n$~~

$(1+n) > n$

$(1+n)^\alpha > n^\alpha$

$\Rightarrow n^\alpha - (1+n)^\alpha < 0$

$0 < \alpha < 1$

$\Rightarrow 1-\alpha > 0$

$\Rightarrow \frac{n^\alpha}{n} < x_n < 0$

$\Rightarrow \frac{1}{n^{1-\alpha}} < x_n < 0$

$\frac{1}{n^{1-\alpha}} \to 0$ as $n \to \infty$

$\Rightarrow \lim_{n\to\infty} x_n = 0.$

(9) $\lim_{n\to\infty} \frac{2^n}{n!} = 0$, by Sandwich Thm.

$x_n = \frac{2^n}{n!}$

$x_1 = \frac{2}{1!}$

$x_2 = \frac{2}{2!} = \frac{2}{1} \cdot \frac{2}{2}$

$x_3 = \frac{2}{3!} = \frac{2}{3} \cdot \frac{2}{2} \cdot \frac{2}{1}$

$x_4 = \frac{2}{4} \cdot \frac{2}{3} \cdot \cancel{\frac{2}{2}} \cdot \frac{2}{1}$

$< \frac{2}{3} \cdot \frac{2}{3} \cdot 2$ $\left[\because 4 > 3,\ \frac{1}{4} < \frac{1}{3},\ \frac{2}{4} < \frac{2}{3}\right]$

$= \left(\frac{2}{3}\right)^2 \cdot 2$

$x_5 = \frac{2}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} \cdot \frac{2}{2} \cdot \frac{2}{1}$

$= \frac{2}{5} \cdot x_4$

$< \left(\frac{2}{3}\right)^3 \cdot 2$ $\left[\because 5 > 3,\ \frac{2}{5} < \frac{2}{3}\right]$

$x_6 = \frac{2}{6} \cdot x_5$

$< \left(\frac{2}{3}\right)^4 \cdot 2$ ... ~~$x_n \le$~~

$x_n = \frac{2^n}{n!} = \frac{2}{n} \cdot x_{n-1}$

$< \frac{2}{3} \cdot \left(\frac{2}{3}\right)^{n-3} \cdot 2$

$< \left(\frac{2}{3}\right)^{n-2} \cdot 2$

So, $0 < x_n < \left(\frac{2}{3}\right)^{n-2} \cdot 2, \quad n \geq 4$

$\downarrow \qquad\qquad \downarrow$

$0 \qquad\qquad 0 \quad [\because 2/3 < 1]$

$\Rightarrow \lim_{n \to \infty} x_n = 0$.

(10) $\lim_{n \to \infty} n^{1/n^2}$

$n^2 \geq n \quad \forall n \geq 1$

$\Rightarrow \frac{1}{n^2} \leq \frac{1}{n}$

$n \geq 1$

$\Rightarrow n^{1/n^2} \geq 1$, $\Rightarrow n^{1/n^2} \leq n^{1/n}$ ~~$\forall n \geq 1$~~ $\forall n \in \mathbb{N}$

~~$\Rightarrow$~~ So, $1 \leq n^{1/n^2} \leq n^{1/n}$

~~So~~ So, As we know $\lim_{n \to \infty} n^{1/n} = 1$

So, by Sandwich thm.

$\lim_{n \to \infty} n^{1/n^2} = 1$

(11) $\lim_{n \to \infty} (n!)^{1/n^2} \rightarrow$ Do in your own.

(12) Show, $\lim_{n \to \infty} (\sqrt{n+1} - \sqrt{n}) = 0$ ~~by~~

$x_n = \sqrt{n+1} - \sqrt{n}$

$= \frac{1}{\sqrt{n+1} + \sqrt{n}}$

$n < n+1$

$\sqrt{n} < \sqrt{n+1}$

$\sqrt{n} + \sqrt{n+1} < \sqrt{n+1} + \sqrt{n+1}$

$\Rightarrow \sqrt{n} + \sqrt{n} < \sqrt{n} + \sqrt{n+1} < 2\sqrt{n+1}$

$\Rightarrow 2\sqrt{n} < \sqrt{n} + \sqrt{n+1} < 2\sqrt{n+1}$

$\Rightarrow \dfrac{1}{2\sqrt{n+1}} < \dfrac{1}{\sqrt{n} + \sqrt{n+1}} < \dfrac{1}{2\sqrt{n}}$

$\cancel{z_n < y_n < x_n}$

$\cancel{x_n < y_n < z_n}$

$\Rightarrow y_n < x_n < z_n$

$\lim\limits_{n\to\infty} y_n = 0, \quad \lim\limits_{n\to\infty} z_n = 0.$

So, $\lim\limits_{n\to\infty} x_n = 0.$

(13) $\lim\limits_{n\to\infty} (2^n + 3^n)^{1/n}$

$\cancel{2 < 0 < 3}$ $\quad 0 < 2 < 3$

$\cancel{2^n <}$ $\quad 2^n < 3^n$

$3^n < 2^n + 3^n < 2 \cdot 3^n$

$3 < (2^n + 3^n)^{1/n} < 2^{1/n} \cdot 3$

$x_n < y_n < z_n$

$\lim\limits_{n\to\infty} x_n = 3, \quad \lim\limits_{n\to\infty} z_n = 3$

So, $\lim\limits_{n\to\infty} (2^n + 3^n) = 3.$

(14) $\lim_{n\to\infty} \frac{1\cdot 3\cdot 5\cdot \cdots (2n-1)}{2\cdot 4\cdot 6\cdot \cdots 2n} = 0$

Let $y_n = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\cdots\frac{2n-1}{2n}$

Now, $(n+1)^2 = n^2+2n+1 > n^2+2n$

$= n(n+2)$

$\therefore \frac{n}{n+1} < \frac{n+1}{n+2} \quad \forall\, n \geq 1$

Putting $n = 1, 3, 5, \ldots, 2n-1$

we have, $\frac{1}{2} < \frac{2}{3}$

$\frac{3}{4} < \frac{4}{5}$

$\frac{5}{6} < \frac{6}{7}$

$\vdots$

$\frac{2n-1}{2n} < \frac{2n}{2n+1}$

so, $y_n < \frac{2\cdot 4\cdot 6\cdots 2n}{3\cdot 5\cdot 7\cdots (2n+1)}$

$y_n\cdot y_n < \frac{1\cdot 3\cdot 5\cdots (2n-1)}{2\cdot 4\cdot 6\cdots 2n} \times \frac{2\cdot 4\cdot 6\cdots 2n}{3\cdot 5\cdot 7\cdots (2n+1)}$

$y_n^2 < \frac{1}{2n+1} \quad \forall\, n \geq 1$

$\therefore 0 < y_n < \frac{1}{\sqrt{2n+1}} \quad \forall\, n.$

$\Rightarrow \lim_{n\to\infty} y_n = 0$

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(15) $\lim_{n\to\infty} (\sqrt{n^2+1} - n) = 0$, show by using sandwich theorem.

(16) $\lim_{n\to\infty} a^{1/n} = 1$ for $a > 0$

suppose $a \geq 1$.

Then for $n \geq a$, we have

$$1 \leq a^{1/n} \leq n^{1/n}$$

since $\lim_{n\to\infty} n^{1/n} = 1$

so, $\lim_{n\to\infty} a^{1/n} = 1$.

~~Then~~ $\cancel{\frac{1}{a}}$ suppose that $0 < a < 1$

then $\frac{1}{a} > 1$

so, $\lim_{n\to\infty} \left(\frac{1}{a}\right)^{1/n} = 1$

$\Rightarrow \lim_{n\to\infty} \frac{1^{1/n}}{a^{1/n}} = 1$

$\Rightarrow \frac{1}{\lim_{n\to\infty} a^{1/n}} = 1$

$\Rightarrow \lim_{n\to\infty} a^{1/n} = 1$

(17) Show that $\lim_{n\to\infty} x_n = 1$, where

$$x_n = \frac{1}{1^2+1} + \frac{1}{2^2+2} + \cdots + \frac{1}{n^2+n}.$$

(18) Show If $a > 0$, then $\lim \left(\frac{1}{1+na}\right) = 0$

Note:- By Mathematical Induction we can show that

(i) if $A = \{a_n\}$, $B = \{b_n\}, \ldots, Z = \{z_n\}$ are convergent sequences of real numbers then their sum

$$A + B + \cdots + Z = \{a_n + b_n + \cdots + z_n\}$$ is a

convergent sequence and

(i) $\lim_{n\to\infty} (a_n \pm b_n \pm \cdots \pm z_n) = \lim_{n\to\infty} a_n \pm \lim b_n \pm \cdots \pm \lim z_n$

Also their product $A \cdot B \cdots Z = (a_n b_n \cdots z_n)$ is a convergent sequence and

(ii) $\lim (a_n b_n \cdots z_n) = (\lim a_n)(\lim b_n) \cdots (\lim z_n)$

Hence, if $k \in \mathbb{N}$ and if $A = \{a_n\}$ is a convergent sequence, then

$$\lim (a_n^k) = (\lim (a_n))^k .$$

— ~~Prove~~ Try to prove.

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① Let $X = \{x_n\}$ be a sequence of real numbers that converges to $x \in \mathbb{R}$, let p be a polynomial; for enamble, let

$$p(t) = a_k t^k + a_{k-1} t^{k-1} + \cdots + a_1 t + a_0,$$

where $k \in \mathbb{N}$ and $a_j \in \mathbb{R}$ for $j = 0, 1, \ldots, k$.

Then prove that $\{p(x_n)\}$ converges to $p(x)$.

→ As we know if k is a natural number then

$$\lim_{n\to\infty} (a_n^k) = (\lim(a_n))^k .$$

therefore, $\{(x_n)^k\}$ converges to x^k,

$\{(x_n)^{k-1}\}$ converges to x^{k-1}

⋮ and so on.

$\{x_n\}$ $\cancel{\{x}$

Also, $\{a_k(x_n)^k\} \to a_k x^k$ [∵ $\cancel{x^k}$

$\{a_{k-1}(x_n)^{k-1}\} \to a_{k-1}x^{k-1}$

⋮

$\{a_1 (x_n)\} \to a_1 x$

$\{a\} \to a \Rightarrow$ constant sequence.

Therefore,

$$\{p(x_n)\} = \{a_k x_n^k + a_{k-1} x_n^{k-1} + \cdots + a_1 x_n + a\}$$

$$= a_k\{x_n^k\} + a_{k-1}\{x_n^{k-1}\} + \cdots + a_1\{x_n\} + \{a\}$$

$$\to a_k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + a$$

$$= p(x)$$

(2) Let $X = \{x_n\}$ be a sequence of real numbers that converges to $x \in \mathbb{R}$. Let r be a rational ~~...~~ fun^n (that is, $r(t) = p(t)/q(t)$), where p & q are polynomials). Suppose that $q(x_n) \neq 0 \ \forall n \in \mathbb{N}$ and that $q(x) \neq 0$. Then the sequence $\{r(x_n)\}$ converges to $r(x) = p(x)/q(x)$.

$\Rightarrow$ Do in your own.

Thm:- Let the sequence $\{x_n\}$ converges to x. Then the sequence $\{|x_n|\}$ of absolute values of converges to $|x|$.
That is if $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} |x_n| = |x|$.

Proof:- As $\lim_{n\to\infty} x_n = x$, so for any $\varepsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ s.t.

$$|x_n - x| < \varepsilon, \ \forall n \geq n_0 \quad \text{---(1)}$$

Now,

$$\big||x_n| - |x|\big| \leq |x_n - x| < \varepsilon \quad \forall n \geq n_0$$

(by (1))

$$\Rightarrow \lim_{n\to\infty} |x_n| = |x|$$

Note:- converse is not true. $\{\cos n\pi\} = \{(-1)^n\}$ ~~converges~~ does not converge, but $\{|\cos n\pi|\} \to 1$.

Thm:- Let $\{x_n\}$ be a sequence of real numbers that converges to x and suppose that $x_n \geq 0$. Then the sequence $\{\sqrt{x_n}\}$ ~~...~~ of positive integers converges and $\lim(\sqrt{x_n}) = \sqrt{x}$.

Proof:- As $x_n \geq 0$, so

$$\Rightarrow \lim x_n \geq 0$$
$$\Rightarrow x \geq 0$$

So, there are two possibilities -(i) $x = 0$, (ii) $x > 0$

Case (i) :- If $x = 0$. Let $\varepsilon > 0$ be given. Since $x_n \to 0$ there exists a natural number n_0 such that

$$|x_n - 0| < \varepsilon \quad , \forall n \geq n_0$$

$$\Rightarrow 0 \leq x_n < \varepsilon$$

So,

$$|\sqrt{x_n} - 0| < \sqrt{\varepsilon} \quad \forall n \geq n_0$$

$$\Rightarrow \lim_{n \to \infty} \sqrt{x_n} = 0 .$$

Case (ii) :- If $x > 0$, then $\sqrt{x} > 0$, it follows that

$$|\sqrt{x_n} - \sqrt{x}| = \frac{(\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x})}{(\sqrt{x_n} + \sqrt{x})}$$

$$= \frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}$$

Since $\sqrt{x_n} + \sqrt{x} \geq \sqrt{x} > 0$, it follows that

$$|\sqrt{x_n} - \sqrt{x}| \leq \frac{1}{\sqrt{x}} |x_n - x|$$

$$< \varepsilon / \sqrt{x} \quad \forall n \geq n_0$$

[$\because x_n \to x$ so, for given $\varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ $\forall n \geq n_0$]

$$\Rightarrow \lim_{n \to \infty} \sqrt{x_n} = \sqrt{x}$$

Therefore, we get for $x_n \geq 0$

$$\lim_{n \to \infty} \sqrt{x_n} = \sqrt{x} \quad , x \geq 0$$

① If $\{b_n\}$ be a bounded sequence and $\lim_{n\to\infty} a_n = 0$, show that ~~lim~~ $\lim_{n\to\infty} a_n b_n = 0$.

Sol^n :- Since $\{b_n\}$ is bounded, ~~for~~ there exists a real number $M>0$ s.t.

$$|b_n| \leq M \quad \forall n.$$

since $\lim_{n\to\infty} a_n = 0$, therefore for every $\varepsilon>0$ $\exists$ $n_0 \in \mathbb{N}$ s.t.

$$|a_n - 0| < \varepsilon \quad \forall n \geq n_0.$$

Now,
$$|a_n b_n - 0| = |a_n b_n| \leq |b_n| M \quad \forall n$$
$$< M\varepsilon \quad \forall n \geq n_0$$

So, we take $\varepsilon' = M\varepsilon > 0$ s.t.

$$|a_n b_n| < \varepsilon' \quad \forall n \geq n_0$$

$$\Rightarrow \lim_{n\to\infty} a_n b_n = 0.$$

[Note:- Do one correction in the 1st lecture; page no. 4, Section ③ (Bounded sequence).

Correct the def^n of bounded sequence to → ~~A sequence is said~~ Also we can say that a sequence is said to bounded if there exists some positive real number ~~A~~ M→ such that $|x_n| \leq A$ $\forall n$. Take $A = \max\{|K_1|, |K_2|\}$

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instead of $A = \max\{k_1, k_2\}$.

Ex:- ① $\left\{\frac{(-1)^n}{n}\right\}$.

Here $\{a_n\} = \{(-1)^n\}$, $\{b_n\} = \left\{\frac{1}{n}\right\}$

$\{a_n\}$ is a bounded sequence

and $\lim_{n\to\infty} b_n = 0$.

So, $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

② $\left\{\frac{\sin n}{n^2}\right\}$,

$\lim_{n\to\infty} \frac{\sin n}{n^2} = 0$ [$\because \{\sin n\}$ is a bounded sequence as $|\sin n| \leq 1 \ \forall n \in \mathbb{N}$ & $\lim_{n\to\infty} \frac{1}{n^2} = 0$.

③ $\left\{\frac{(-1)^{n+1}}{2^n}\right\} \to 0$.

Note:- But we can't conclude about the convergence of $\{a_n b_n\}$, where $\{a_n\}$ is bounded and $\{b_n\}$ coverges to some non zero limit b.

As an example, $\{(-1)^{n+1}\}$

Here ~~$\{a_n\} = \{(-1)^{n+1}\}$ $\{b_n\} = \{1\}$~~.

we take $a_n = (-1)^{n+1}$, $b_n = 1$

then $\{a_n\}$ is bdd & $\{b_n\}$ converges to 1.

But ~~$\{(-1)^{n+1}\}$ is con~~ $\{a_n b_n\} = \{(-1)^{n+1}\}$ is not convergent.

Some properties of Bounded sequences

1. Let $\{a_n\}$ and $\{b_n\}$ be bounded ~~sequence~~ sequences. prove that $\{a_n \pm b_n\}$ and $\{a_n b_n\}$ are also bounded sequences. ~~What~~ can you say about the sequence $\{\alpha a_n\}$, $\alpha \in \mathbb{R}$.

2. Give examples to show that if $\{a_n\}$ and $\{b_n\}$ are bounded sequences, $b_n \neq 0$ for every n, then $\left\{\frac{a_n}{b_n}\right\}$ need not be bounded sequence.

$\Rightarrow$ We know the sequence $\{\frac{1}{n}\}$ is a bounded ~~But~~ $\{1\}$ is also a bounded sequence.

But $\left\{\frac{1}{1/n}\right\} = \{n\}$ is not a bdd sequence.

Exercise

Give an example of each of the following, or prove that such a thing is impossible —

(a) sequences $\{x_n\}$ and $\{y_n\}$, which both diverge, but whose sum $\{x_n + y_n\}$ converges.

(b) sequences $\{x_n\}$ and $\{y_n\}$, where $\{x_n\}$ converges, $\{y_n\}$ diverges, and $\{x_n + y_n\}$ converges

(c) A convergent sequence $\{b_n\}$ with $b_n \neq 0$ $\forall n \in \mathbb{N}$ such that $\{1/b_n\}$ converges.

(d) An unbounded sequence $\{a_n\}$ and a convergent sequence $\{b_n\}$ with $\{a_n - b_n\}$ bounded.

(e) Two sequences $\{a_n\}$ and $\{b_n\}$, where $\{a_n b_n\}$ and $\{a_n\}$ converge but $\{b_n\}$ does not.

(f) If $\lim_{n\to\infty}(a_n - b_n) = 0$, then $\{a_n\}$ and $\{b_n\}$ converge, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

(g) If $\{b_n\} \to b$ then $\{|b_n|\} \to |b|$

(h) If $\{a_n\} \to a$ and $\{b_n - a_n\} \to 0$, then $\{b_n\} \to a$

(i) If $\{a_n\} \to 0$ and $|b_n - b| \le a_n$ $\forall n$ then $\{b_n\} \to b$