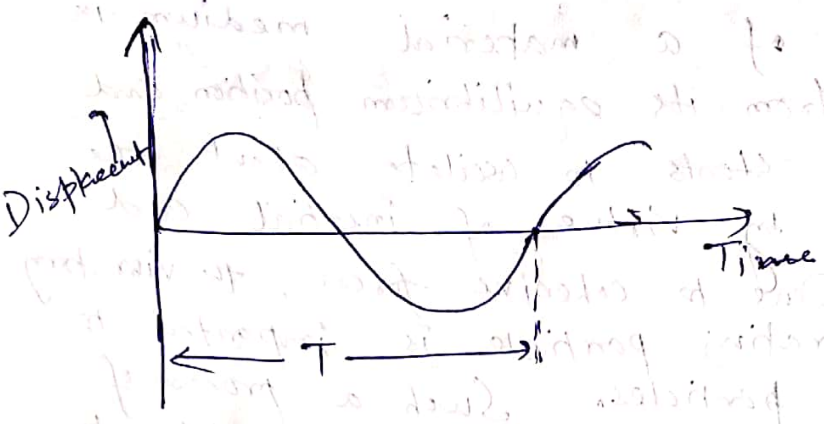
Wave motions :

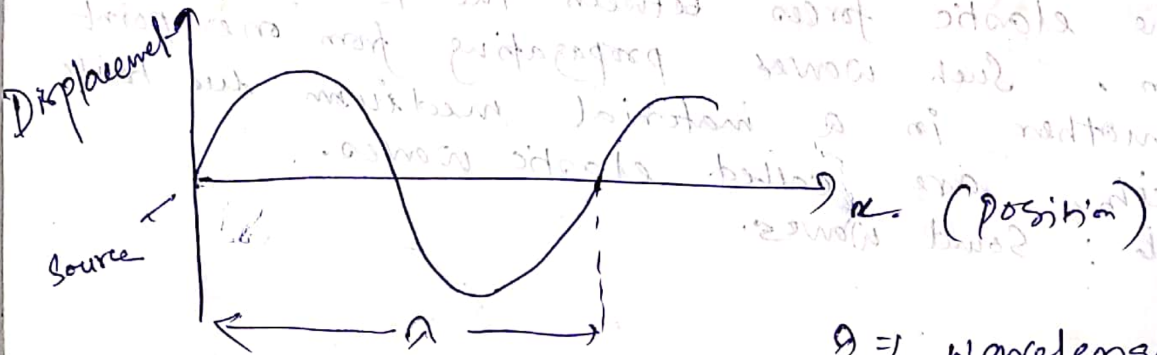
If a particle of a material medium is slightly displaced from its equilibrium position and then released, it starts to oscilate about the equilibrium position by virtue of inertial and elastic properties. Due to cohesive forces, the vibratory motion of the vibrating particle is imported to the neighbouring particles. Such a process of transference of the vibration from particle to particle is known as a Wave motion.

The wave motion is generated by the tensile or the elastic forces between the particles of the medium. Such waves propagating from one point to another in a material medium due to its elasticity, are called elastic waves.
Example : Sound waves.



T => Time period

Time displacement curve at a fixed position i.e for one particle



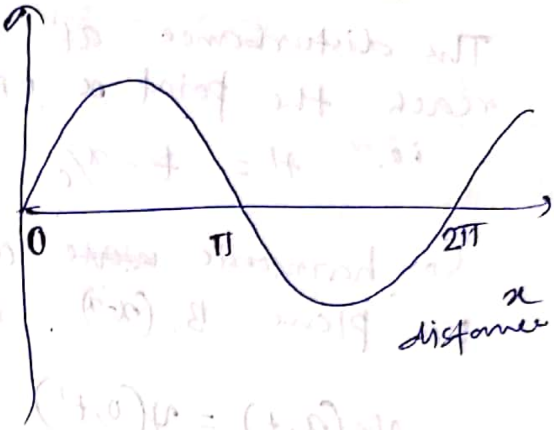
λ => wavelength.

Space-displacement curve at a particular instant of time.

For a wave of constant type, all the particles on the path of propagation will have identical time-displacement curves. But a particle further from the source will attain at a latter time the displacement of a particle nearer the source. This phase lag in a homogeneous medium is proportional to the distance between the particles.

Space-displacement curve gives the shape or the form of the wave, So it is also called the wave form or the wave profile

Wavelength(λ): The shortest distance between any two particles on the path of wave having the same phase i.e. in the same state of motion at any instant of time is termed as wavelength.



for path difference λ phase diff. is 2π

" " " 1 " " $\frac{2\pi}{\lambda}$

" " " x " $= \frac{2\pi}{\lambda}.x$.

Phase difference between two oscilating particles a unit distance appart is $2\pi/\lambda$.

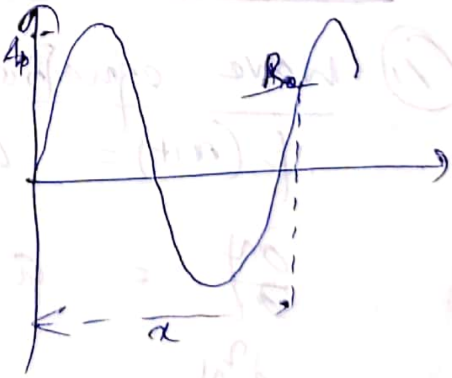
phase diff $= \frac{2\pi}{\lambda}$ path difference

Progressive wave:

A wave propagating from one point to another in a medium without being subjected to any boundary condition, is called progressive wave.

Mathematical description of wave motion

Let us consider one dimensional plane progressive harmonic waves travelling in a medium with a phase velocity C.



Consider two planes A & B seperated by x.

Let $\psi(x,t)$ be the field parameter, $\psi(x,t)$ may be displacement, velocity, pressure or any other physical parameter that varies with space and time (i.e. periodically)

The harmonic oscillations of the particles at the plane A ($x=0$) at any time instant t is given by

$$\psi(0,t) = a \sin \omega t$$

$a \Rightarrow$ amplitude
$\omega \Rightarrow$ angular freq^n.

The disturbance at $x=0$ takes a time x/c to reach the point x on the plane B.

i.e. $t' = t - x/c$

So harmonic ~~wave~~ oscillations of the particles at the plane B. ($x=x$) will be

$$\psi(x,t) = \psi(0,t')$$
$$= a \sin \omega t'$$
$$= a \sin \omega (t - x/c) \qquad k = \frac{\omega}{c}$$
$$= a \sin(\omega t - kx) \qquad = \frac{2\pi}{T \cdot \nu\lambda}$$
$$= a \sin \frac{2\pi}{\lambda}(ct - x) \qquad = \frac{2\pi}{\lambda}$$

$k \Rightarrow$ Angular wave number.

If wave moves along $-x$ direction.

$$\psi(x,t) = a \sin(\omega t + kx)$$

In other form, $\psi = a e^{i(\omega t - kx)}$

$\boxed{* \ \frac{1}{\lambda} = \bar{\nu} = \text{wave no.} = \frac{\nu}{c}.}$

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(1) Wave equation

$$\psi(x,t) = a \sin(\omega t + kx)$$

$$\frac{\partial \psi}{\partial t} = a \cos(\omega t - kx) \cdot \omega$$

$$\frac{\partial^2 \psi}{\partial t^2} = -\omega^2 a \sin(\omega t - kx)$$

$$\frac{\partial \psi}{\partial x} = a \cos(\omega t - kx) \cdot (-k)$$

$$\frac{\partial^2 \psi}{\partial x^2} = -k^2 a \sin(\omega t - kx)$$

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{\omega^2}{k^2} \frac{\partial^2 \psi}{\partial x^2}$$

$$\boxed{\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{\partial^2 \psi}{\partial x^2}} \quad \text{--- (1)} \qquad c = \frac{\omega}{k}$$

One dimensional differential wave equation.

This equation is applicable only to plane wave of constant type. It can not describe waves such as large amplitude waves, flexural waves, attenuated waves etc.

Plane wave solution:

$$\psi(x,t) = F_1(x)\, F_2(t)$$

Putting this in equation (1)

$$F_1 \frac{d^2 F_2}{dt^2} = c^2 F_2 \frac{d^2 F_1}{dx^2}$$

Deviding both side by $F_1 F_2$.

$$\frac{1}{F_2} \frac{d^2 F_2}{dt^2} = \frac{c^2}{F_1} \frac{d^2 F_1}{dx^2} = -\omega^2 \text{ (say)}$$

-ve sign is chosed in order to get periodic solⁿ.

$$\frac{d^2 F_1}{dx^2} + \frac{\omega^2}{c^2} F_1 = 0$$

$$\Rightarrow \quad \frac{d^2 F_1}{dx^2} + k^2 F_1 = 0 \longrightarrow (2) \qquad k = \frac{\omega}{c}$$

and

$$\frac{d^2 F_2}{dt^2} + \omega^2 F_2 = 0 \longrightarrow (3)$$

$$\therefore F_1(x) = A_1 e^{ikx} + A_2 e^{-ikx}$$

and

$$F_2(t) = B_1 e^{i\omega t} + B_2 e^{-i\omega t}$$

$$\psi(x,t) = F_1(x) F_2(t)$$

$$= A_1 B_1 e^{i(\omega t + kx)} + A_2 B_2 e^{-i(\omega t + kx)} + A_1 B_2 e^{-i(\omega t - kx)} + A_2 B_1 e^{i(\omega t - kx)}$$

$$= A_1 B_1 e^{ik(ct+x)} + A_2 B_2 e^{-ik(ct+x)} + A_1 B_2 e^{-ik(ct-x)} + A_2 B_1 e^{ik(ct-x)}$$

$$\psi(x,t) = f_1(ct-x) + f_2(ct+x)$$

Spherical wave solution:

Three dimensional wave equation

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \nabla^2 \psi \qquad \psi = \psi(r, \theta, \phi, t)$$

If the waves have spherical symmetry then ψ will be independent of θ & ϕ

So, $$\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right)$$

Putting $\psi = \frac{\psi'}{r}$

$$\frac{\partial^2}{\partial t^2}\left(\frac{\psi'}{r}\right) = c^2 \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r}\left(\frac{\psi'}{r}\right)\right]$$

or, $$\frac{1}{r}\frac{\partial^2 \psi'}{\partial t^2} = \frac{c^2}{r^2}\frac{\partial}{\partial r}\left[r\frac{\partial \psi'}{\partial r} - \psi'\right]$$

$$= \frac{c^2}{r^2}\left[r\frac{\partial^2 \psi'}{\partial r^2} + \frac{\partial \psi'}{\partial r} - \frac{\partial \psi'}{\partial r}\right]$$

$$= \frac{c^2}{r}\frac{\partial^2 \psi'}{\partial r^2}$$

$$\therefore \frac{\partial^2 \psi'}{\partial t^2} = c^2 \frac{\partial^2 \psi'}{\partial r^2}$$

$$\psi' = f_1(ct-r) + f_2(ct+r)$$

$$\psi = \frac{1}{r} f_1(ct-r) + \frac{1}{r} f_2(ct+r)$$

In spherical Polar Co-ordinate

$$\nabla^2 \psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2}$$

In Cylindrical

$$\nabla^2 \psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{}$$