SERIES SOLUTION METHOD

The series solution method can be classified into two categories:
(i) Power Series method
(ii) General Series solution method (Frobenius method)

• POWER SERIES

(i) An infinite series of the form: $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots\cdots$, where $a_0, a_1, a_2, \cdots$ are real constants.

(ii) An infinite series of the form: $\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$, where $a_0, a_1, a_2, \cdots$ are real constants, is called the general form of a power series about the point $x = x_0$, where x_0 is a fixed real number.

• Every power series of the form (i) always converges at $x=0$.
• If a power series (i) converges for all $x \in \mathbb{R}$, then the series is called everywhere convergent.
• If a power series (i) is convergent only for $x=0$, then it is called nowhere convergent.
• If R be the radius of convergence of the power series (i), then
the series is absolutely convergent if $|x| < R$,
and " divergent if $|x| > R$.
The interval of convergence $= (-R, R)$.
• A power series represents a continuous function.
• A power series can be differentiated term by term within its interval of convergence.

- ANALYTIC FUNCTION

A function $f(x)$ defined on an interval containing the point x_0 is called analytic at x_0 if its Taylor series about x_0, i.e.,

$$\sum_{n=0}^{\infty} \frac{f^{n}(x_0)}{n!}(x-x_0)^n$$ exists and converges to $f(x)$

for all x in some nbd. of x_0.

- All polynomial functions e^x, $\sin x$, $\cos x$, $\sinh x$, etc. are analytic everywhere.
- A rational function $\frac{P(x)}{Q(x)}$ is analytic except at those values of x for which $Q(x)=0$

- ORDINARY POINTS and SINGULAR POINTS.

Let us consider the 2nd order homogeneous linear D.E. with variable coefficients as given by

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0, \quad a_0(x) \neq 0 \longrightarrow (1)$$

In the equivalent normalized form (canonical form) it becomes as

$$\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \longrightarrow (2), \text{ where } P_1(x)=\frac{a_1(x)}{a_0(x)}, P_2(x)=\frac{a_2(x)}{a_0(x)}.$$

A point $x=x_0$ is called an ordinary point/regular point of the D.E. (1), if both the functions $P_1(x)$ and $P_2(x)$ are analytic at $x=x_0$.

If either or both $P_1(x)$ and $P_2(x)$ is not analytic at $x=x_0$, then the point x_0 is called a singular point.

Two types of singular points:

(i) regular singular points; if both $(x-x_0)P_1(x)$ and $(x-x_0)^2P_2(x)$ are analytic at $x=x_0$.

(ii) irregular singular point; if it is not regular singular point.

POWER SERIES SOLUTIONS

Theorem about the Existence of a power series solution of the D.E. 1 about an ordinary point $x = x_0$. This theorem gives a sufficient condition.

<u>Statement</u>: If $x = x_0$ be an ordinary point of the D.E. 1, then it has two non-trivial l.i.d. power series solutions of the form

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n, \quad c_n\text{'s are constants, and}$$

these power series converge in some interval $|x - x_0| < R$ $(R > 0)$ about x_0, R being the radius of convergence of the power series.

POWER SERIES SOLUTION METHOD OF O.D.E.

- Power Series → An infinite series of the form $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$, where $a_0, a_1, a_2, \cdots$ are real numbers, is called a power series in x.

  The power series about the point x_0 is of the form $\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$, where x_0 is a fixed real number, a_i $(i=0,1,2,\ldots,n)$ are real nos.

- Everywhere Convergent → if a P.S. $\sum_{n=0}^{\infty} a_n x^n$ converges for all $x \in \mathbb{R}$.
- Nowhere convergent → " " " " " only for $x=0$. [Note: Every P.S. $\sum_{n=0}^{\infty} a_n x^n$ always converges at $x=0$].
- Absolutely convergent → when $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ exists, $=R$, where R be the radius of convergence of the P.S. Then the P.S. is absolutely convergent if $|x|<R$, and " " " divergent if $|x|>R$.

  The interval of convergence is $(-R,R)$.
- A p.s. represents a continuous function on $(-R,R)$
- A p.s. can be differentiated terwise within $(-R,R)$

- Analytic function →

  A function $f(x)$ is said to be analytic at the point x_0 ($\in$ its domain of definition), if its Taylor series about x_0, $\sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!}(x-x_0)^n$ exists and converges to $f(x)$, $\forall\, x \in (x_0-\delta, x_0+\delta)$ for some $\delta>0$.

- Analytic functions → e^x, $\sin x$, $\cos x$, $\sinh x$, $\cosh x$, all polynomial functions are analytic everywhere.
- Rational function $\frac{P(x)}{Q(x)}$ is analytic except at the roots of $Q(x)=0$.
- A single-valued function $f(x)$ will be analytic in the domain D if it is defined and differentiable at each point of the domain D.
- A function $f(x)$ will be analytic at a point x_0 if $f'(x)$ exists for all $x \in (x_0-\delta, x_0+\delta)$ for some $\delta>0$.

2.

- **ORDINARY POINT:**

Let us consider the 2nd order homogeneous linear D.E. with variable coefficients as given by

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \longrightarrow (1)$$

The equivalent normalized form of (1) is given by

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \longrightarrow (2), \text{ where } P(x) = \frac{P_1(x)}{P_0(x)}$$

$$\& \quad Q(x) = \frac{P_2(x)}{P_0(x)}.$$

A point $x = x_0$ is called an ordinary point of the differential equation (1) if both $P(x)$ and $Q(x)$ are analytic at x_0. It is also called a regular point of the D.E.

- **SINGULAR POINT:-**

A point $x = x_0$ is called a singular point of the D.E. (1) if either or both $P(x)$ and $Q(x)$ is not analytic at x_0.

- regular singular point → A singular point $x = x_0$ is called regular if both $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ are analytic at $x = x_0$
- irregular singular point → A singular point which is not regular singular point.

Note: Any P.S. is the Taylor's Series of its sum-function f: $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $\forall x \in |x| < R$; $a_n = \frac{f^n(0)}{n!}$.

Exercise.

2. Show that $x=0$ is a regular singular point and $x=1$ is an irregular singular point of
$$x(x-1)^3y'' + 2(x-1)^3y' + 3y = 0. \longrightarrow (1)$$

The normalized form of (1) is $y'' + \frac{2}{x}y' + \frac{3}{x(x-1)^3}y = 0$

Comparing this equ with $y'' + P(x)y' + Q(x)y = 0$, we have
$$P(x) = \frac{2}{x}, \quad Q(x) = \frac{3}{x(x-1)^3}.$$
$P(x)$ is not analytic at $x=0$, $Q(x)$ is also not analytic at $x=0$.
Also, $Q(x)$ is not analytic at $x=1$.

$\therefore$ $x=0$ and $x=1$ are singular points.

Now, $(x-0)P(x) = x\cdot\frac{2}{x} = 2$, which is analytic at $x=0$
& $(x-0)^2Q(x) = x^2\cdot\frac{3}{x(x-1)^3} = \frac{3x}{(x-1)^3} = 0$, " " at $x=0$

$\therefore$ $x=0$ is a regular singular point.

$(x-1)P(x) = \frac{(x-1)\cdot 2}{x} = 2 - \frac{2}{x}$, which is not analytic at $x=1$.
$(x-1)^2Q(x) = (x-1)^2\cdot\frac{3}{x(x-1)^3} = \frac{3}{x(x-1)}$, " " " " at $x=1$.

$\therefore$ $x=1$ is a irregular singular point.

3. Locate and classify the singular points:

(a) $(2x-1)xy'' - (x+1)y' + y = 0$

or, $y'' - \frac{x+1}{(2x-1)x}y' + \frac{1}{x(2x-1)}y = 0$; $P(x) = -\frac{(x+1)}{(2x-1)x}$; $Q(x) = \frac{1}{x(2x-1)}$

$P(x)$ & $Q(x)$ are not analytic at $x=0$, $x=\frac{1}{2}$.

$\therefore$ $x=0$ and $x=\frac{1}{2}$ are singular points.

$\left.\begin{aligned} &(x-0)P(x) = -\frac{(x+1)}{(2x-1)}, \text{ is analytic at } x=0 \\ &(x-0)^2Q(x) = \frac{x}{2x-1}, \text{ is " at } x=0 \end{aligned}\right\}$

$\therefore$ $x=0$ is a regular singular point.

$\left.\begin{aligned} &(x-\tfrac{1}{2})P(x) = -\frac{(x+1)}{2x}, \text{ is analytic at } x=\tfrac{1}{2} \\ &(x-\tfrac{1}{2})^2Q(x) = \frac{2x-1}{4x}, \text{ " " at } x=\tfrac{1}{2} \end{aligned}\right\}$

$x=\frac{1}{2}$ is a regular singular point.

(b) $x^3(x^2-1)y'' + 2x^4y' + 4y = 0$

or, $y'' + \frac{2x}{x^2-1}y' + \frac{4}{x^3(x^2-1)}y = 0$, $P(x) = \frac{2x}{x^2-1}$, $Q(x) = \frac{4}{x^3(x^2-1)}$

$P(x), Q(x)$ are not analytic at $x=\pm 1$, $Q(x)$ is not analytic also at $x=0$

$\left.\begin{aligned} &(x-1)P(x) = \frac{2x}{x+1}, \text{ is analytic at } x=1 \\ &(x-1)^2Q(x) = \frac{4(x-1)}{x^3(x+1)}, \text{ " " " } x=1 \end{aligned}\right\}$ $x=1$, regular S.P.

$\left.\begin{aligned} &(x+1)P(x) = \frac{2x}{x-1} \text{ " " " at } x=-1 \\ &(x+1)^2Q(x) = \frac{4(x+1)}{x^3(x-1)} \text{ " " " at } x=-1 \end{aligned}\right\}$ $x=-1$, " " "
$x=0$ irregular S.P.

(c) $(x^2-3x+2)y'' + xy' - 3y = 0$

or, $y'' + \frac{x}{(x-2)(x-1)} y' - \frac{3}{(x-2)(x-1)} y = 0.$

$P(x) = \frac{x}{(x-1)(x-2)}, \quad Q(x) = \frac{-3}{(x-1)(x-2)}.$

$P(x)$ & $Q(x)$ are not analytic at $x=1$ and $x=2$.

$\therefore$ $x=1$ and $x=2$ are singular points.

$(x-1)P(x) = \frac{x}{x-2}$, is analytic at $x=1$ $\Big\}$ $x=1$ is a regular S.P.

$(x-1)^2 Q(x) = -\frac{3(x-1)}{x-2}$ " " at $x=1$

$(x-2)P(x) = \frac{x}{x-1}$ " " at $x=2$ $\Big\}$ $x=2$ is a regular S.P.

$(x-2)^2 Q(x) = -\frac{3(x-2)}{x-1}$ " " at $x=2$

(d) Show that the point at infinity is a regular singular point of the D.E. $x^2y'' + (3x-1)y' + y = 0$.

put $u = \frac{1}{x}$, $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = -\frac{1}{x^2}\frac{dy}{du} = -u^2\frac{dy}{du}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{1}{x^2}\frac{dy}{du}\right) = \frac{2}{x^3}\frac{dy}{du} - \frac{1}{x^2}\frac{d^2y}{du^2}\cdot\frac{du}{dx}$

$= \frac{2}{x^3}\frac{dy}{du} + \frac{1}{x^4}\frac{d^2y}{du^2} = 2u^3\frac{dy}{du} + u^4\frac{d^2y}{du^2}.$

$\frac{1}{u^2}\cdot\left(2u^3\frac{dy}{du} + u^4\frac{d^2y}{du^2}\right) + \left(\frac{3}{u} - 1\right)\cdot\left(-u^2\frac{dy}{du}\right) + y = 0$

or, $u^2\frac{d^2y}{du^2} + (2u - 3u + u^2)\frac{dy}{du} + y = 0$

or, $\frac{d^2y}{du^2} + \frac{u^2-u}{u^2}\cdot\frac{dy}{du} + \frac{1}{u^2}y = 0.$

$P(u) = \frac{u^2-u}{u^2}$, $Q(u) = \frac{1}{u^2}$. Both of which are analytic except $u=0$.

$\therefore$ $u=0$ is a S.P.

Now, $(u-0)P(u) = \frac{u^2-u}{u} = u-1$, $(u-0)^2Q(u) = 1$, both are analytic at $u=0$ or, $x=\infty$.

$\therefore$ $x=\infty$ is a regular singular point.

Note: A P.S. represents a function on its interval of convergence. This function possesses derivatives of all orders. All the derivatives can be computed by termwise differentiation. The coefficients of the P.S. are given by

$$a_K = \frac{f^K(0)}{K!} \quad (K=0,1,2,3,\cdots)$$

A P.S. $\sum_{n=0}^{\infty} a_n x^n = f(x)$ with a non-zero radius of convergence is in fact a Taylor's series for its sum function $f(x)$.

POWER SERIES SOLUTIONS.

Theorem 1.

Existence Theorem for power series solution:

If $x = x_0$ be an ordinary point of the D.E. $y'' + P(x)y' + Q(x)y = 0$, then it has two non-trivial l.i.d. power series solutions of the form $\sum_{n=0}^{\infty} a_n (x-x_0)^n$, which converge in some interval $(x_0 - R, x_0 + R)$ about x_0, R being the radius of convergence of the P.S. $|x - x_0| < R$, $(R > 0)$.

This is a sufficient condition.

Thus the general solution of the D.E. is a linear combination of these two l.i.d. power series.

Theorem 2. The P.S. representation is given by $y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ of the given D.E. about an ordinary point x_0, always converges in $|x - x_0| < R$.

Ex. (1) Find the P.S. solution of the equation $(x^2+1)y'' + xy' - xy = 0$ about $x = 0$.

The normalized form of the given D.E. is

$$y'' + \frac{x}{x^2+1}y' - \frac{x}{x^2+1}y = 0 \longrightarrow (1)$$

Comparing (1) with $y'' + P(x)y' + Q(x)y = 0$, we have

$P(x) = \frac{x}{x^2+1}$, $Q(x) = -\frac{x}{x^2+1}$, which are analytic at $x = 0$.

It follows that $x = 0$ is an ordinary point of (1).

We assume a P.S. solution of (1) be of the form:

$$y = \sum_{n=0}^{\infty} a_n x^n \longrightarrow (2), \quad a_0, a_1, a_2, \ldots \text{ are constants}$$

Now differentiating (2) twice in succession w.r.t. x,

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \longrightarrow (3)$$

Substituting (2) & (3) in (1), we get

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} n a_n x^{n-1} - x\sum_{n=0}^{\infty} a_n x^n = 0$$

or, $$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

or, $$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = 0$$

$$\therefore \sum_{n=2}^{\infty}\{n(n-1)a_n+(n+1)(n+2)a_{n+2}\}x^n + 2a_2 + 6a_3x + \sum_{n=2}^{\infty} n a_n x^n + a_1$$
$$- \sum_{n=2}^{\infty} a_{n-1}x^n - a_0 = 0$$

$$\therefore \sum_{n=2}^{\infty}\{n^2a_n - \cancel{na_n} + (n+1)(n+2)a_{n+2} + \cancel{na_n} - a_{n-1}\}x^n + x(a_1+6a_3) + 2a_2 - a_0 = 0$$

$$\therefore \sum_{n=2}^{\infty}\{n^2a_n - a_{n-1} + (n+1)(n+2)a_{n+2}\}x^n + x(a_1+6a_3-a_0) + 2a_2 = 0 \longrightarrow (4)$$

This is an identity, equating the coefficients of like powers of x from both sides of (4), we have

$$2a_2 = 0 \Rightarrow \cancel{a_0 = 2a_2 \Rightarrow}\ a_2 = 0 \longrightarrow (5)$$

$$-a_0 + a_1 + 6a_3 = 0 \Rightarrow a_3 = \frac{1}{6}(a_0 - a_1) \longrightarrow (6)$$

& $$n^2a_n - a_{n-1} + (n+1)(n+2)a_{n+2} = 0.$$

$$\therefore\ a_{n+2} = \frac{a_{n-1} - n^2a_n}{(n+1)(n+2)},\ \forall n \geq 2 \longrightarrow (7)$$

This (7) is ~~an~~ a recurrence relation.

There are no relation between a_0 & a_1,
So they are arbitrary constants, and l.i.d.
We express each coefficients in terms of a_0 & a_1.

from (7), $n=2$, $$a_4 = \frac{a_1 - 4a_2}{3\cdot4} = \frac{a_1 - 4\cdot 0}{12} = \frac{a_1}{12}.$$

$n=3$, $$a_5 = \frac{a_2 - 9a_3}{4\cdot5} = \frac{-9\cdot\frac{1}{6}(a_0-a_1)}{20} = -\frac{3}{40}(a_0-a_1)$$

$n=4$, $$a_6 = \frac{a_3 - 16a_4}{5\cdot6} = \frac{1}{30}\cdot\frac{1}{6}(a_0-a_1) - \frac{8}{15}\cdot\frac{a_1}{12}$$
$$= \frac{1}{180}a_0 - \frac{9}{180}a_1 = \frac{1}{180}a_0 - \frac{1}{20}a_1.$$

Substituting all the values of $a_2, a_3, a_4, a_5, \ldots$ in (2),

$$y = a_0 + a_1x + \frac{1}{6}(a_0-a_1)x^3 + \frac{1}{12}a_1x^4 - \frac{3}{40}(a_0-a_1)x^5 + \frac{1}{180}(a_0-9a_1)x^6 + \cdots$$

$$= a_0\left(1 + \frac{1}{6}x^3 - \frac{3}{40}x^5 + \frac{1}{180}x^6 - \cdots\right) + a_1\left(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{3}{40}x^5 - \frac{1}{20}x^6 + \cdots\right)$$

g.s. $\Rightarrow$ $y = a_0y_1(x) + a_1y_2(x)$; a_0, a_1 are arbitrary constants $\longrightarrow (8)$

where $y_1(x) = 1 + \frac{1}{6}x^3 - \frac{3}{40}x^5 + \frac{1}{180}x^6 - \cdots$

& $y_2(x) = x - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{3}{40}x^5 - \frac{1}{20}x^6 + \cdots$

$\therefore$ Eqn (8) gives the g.s. of D.E. (1) in powers of x, where $y_1(x)$ & $y_2(x)$ are the P.S. expansions of two l.i.d. solns. of (1).

② Obtain the series solution of $y''+xy'+x^2y=0$ in powers of x → ①

Comparing the giving eqn with $y''+P(x)y'+Q(x)y=0$, we have, $P(x)=x$, $Q(x)=x^2$, which are both analytic at $x=0$. Therefore, $x=0$ is an ordinary point of ①

Let $y=\sum_{n=0}^{\infty} a_n x^n$ → ② be a P.S. solution in powers of x.

Differentiating ② twice successively w.r.t. x,

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y''=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \rightarrow ③$$

Using ② & ③ in ①, we have

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}+\sum_{n=1}^{\infty} n a_n x^n+\sum_{n=0}^{\infty} \cancel{n(n-1)} a_n x^{n+2}=0$$

$$\text{or, } \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty} n a_n x^n+\sum_{n=2}^{\infty} a_{n-2}x^n=0$$

$$\text{or, } 2a_2+6a_3x+a_1x+\sum_{n=2}^{\infty} n a_n x^n+\sum_{n=2}^{\infty} a_{n-2}x^n+\sum_{n=2}^{\infty}(n+1)(n+2)a_{n+2}x^n=0$$

$$\text{or, } 2a_2+x(a_1+6a_3)+\sum_{n=2}^{\infty}\left[n a_n+a_{n-2}+(n+1)(n+2)a_{n+2}\right]x^n=0$$

It is an identity, comparing the coeffs. of like powers of x from bothsides, we get

$$2a_2=0, \quad a_1+6a_3=0, \quad n a_n+a_{n-2}+(n+1)(n+2)a_{n+2}=0$$

$$\text{or, } a_2=0, \quad a_1=-6a_3, \Rightarrow a_3=-\tfrac{1}{6}a_1, \quad a_{n+2}=\frac{-n a_n-a_{n-2}}{(n+1)(n+2)} \ (n\geq 2)$$

(It is a recurrence relation)

Put $n=2$, $a_4=\dfrac{-2a_2-a_0}{3\cdot 4}=\dfrac{-a_0}{12}$

$n=3$, $a_5=\dfrac{-3a_3-a_1}{4\cdot 5}=\dfrac{\frac{1}{2}a_1-a_1}{20}=-\dfrac{a_1}{40}$

$n=4$, $a_6=\dfrac{-4a_4-a_2}{5\cdot 6}=\dfrac{\frac{a_0}{3}-0}{30}=\dfrac{a_0}{90}$.

Putting the values of $a_2, a_3, a_4, a_5, a_6, \ldots$ in ②,

$$y=a_0+a_1x+(-\tfrac{1}{6}a_1)x^3-\frac{a_0}{12}x^4-\frac{a_1}{40}x^5+\frac{a_0}{90}x^6-\cdots$$

$$=a_0\left(1-\frac{1}{12}x^4+\frac{1}{90}x^6-\cdots\right)+a_1\left(x-\frac{1}{6}x^3-\frac{1}{40}x^5-\cdots\right)$$

where a_0 & a_1 are arbitrary constants.

$y=a_0y_1(x)+a_1y_2(x)$ [$y_1(x)$ & $y_2(x)$ are two l.i.d. Power Series ~~expansion~~ solutions of ①

③ Solve the D.E. $y'' + (x-1)^2 y' - 4(x-1)y = 0$ in P.S. about the point $x = 1$. $\longrightarrow$ ①

Comparing ① with $y'' + P(x)y' + Q(x)y = 0$, we get $P(x) = (x-1)^2$, $Q(x) = -4(x-1)$. Which are both analytic at $x = 1$. So, $x = 1$ is an ordinary point.

Let $y = \sum_{n=0}^{\infty} a_n (x-1)^n \longrightarrow$ ② be a P.S. solution of ①.

$\therefore y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$, $\quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} \longrightarrow$ ③

Using ② & ③ in ①, we get

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} + (x-1)^2 \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} - 4(x-1) \cdot \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

or, $$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n + \sum_{n=1}^{\infty} n a_n (x-1)^{n+1} - 4\sum_{n=0}^{\infty} a_n (x-1)^{n+1} = 0$$

or, $$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} (x-1)^n + \sum_{n=2}^{\infty} (n-1) a_{n-1} (x-1)^n - 4\sum_{n=1}^{\infty} a_{n-1} (x-1)^n = 0.$$

or, $$2a_2 + 6a_3(x-1) - 4a_0(x-1) + \sum_{n=2}^{\infty} \left[(n+1)(n+2)a_{n+2} + (n-1)a_{n-1} - 4a_{n-1}\right](x-1)^n = 0$$

This is an identity. Equating the coefficients of like powers of $(x-1)$ from both sides, we get

$$2a_2 = 0, \quad 6a_3 - 4a_0 = 0, \quad a_{n+2} = \frac{4a_{n-1} - (n-1)a_{n-1}}{(n+1)(n+2)} \ (n \geq 2)$$

or, $a_2 = 0$, $\quad a_3 = \frac{2}{3} a_0$

$$a_{n+2} = \frac{(5-n)a_{n-1}}{(n+1)(n+2)}$$

$n=2$, $a_4 = \frac{3a_1}{3.4} = \frac{a_1}{4}$.

$n=3$, $a_5 = \frac{2a_2}{4.5} = 0$

$n=4$, $a_6 = \frac{1.a_3}{5.6} = \frac{2}{3.5.6}a_0 = \frac{1}{45}a_0$

$n=5$, $a_7 = 0$

$n=6$, $a_8 = -\frac{a_5}{7.8} = 0$

$n=7$, $a_9 = \frac{-2a_6}{8.9} = -\frac{2}{8.9.45}a_0$

$= -\frac{1}{1620}a_0$

$n=8$, $a_{10} = \frac{-3a_7}{9.10} = 0$

$n=9$, $a_{11} = \frac{-4a_8}{10.11} = 0$

$n=10$, $a_{12} = \frac{-5a_9}{11.12} = \frac{+a_0}{132.324}$

$$y = a_0 + a_1(x-1) + a_3(x-1)^3 + a_4(x-1)^4 + a_6(x-1)^6 + a_9(x-1)^9 + a_{12}(x-1)^{12} + \cdots$$

$$= a_0\left(1 + \frac{2}{3}(x-1)^3 + \frac{1}{45}(x-1)^6 - \frac{1}{1620}(x-1)^9 + \frac{1}{132 \cdot 324}(x-1)^{12} + \cdots\right) + a_1\left((x-1) + \frac{1}{4}(x-1)^4\right)$$

which is the required P.S. solution about $x = 1$.

④ Find the P.S. solution of the IVP.

$y'' + xy' + 2y = 0 \rightarrow ①$, $y(0) = 1$, $y'(0) = 0$.

Comparing ① with $y'' + P(x)y' + Q(x)y = 0$, we get $P(x) = x$, $Q(x) = 2$, which are analytic at $x = 0$.

So, $x = 0$ is an ordinary point.

Let $y = \sum_{n=0}^{\infty} a_n x^n \longrightarrow ②$ be a P.S. solution of ①.

$$\therefore y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \longrightarrow ③.$$

Using ② & ③ in ①, we get:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + 2 \cdot \sum_{n=0}^{\infty} a_n x^n = 0$$

$+2\sum a_n x^n = 0.$

$$\text{or,} \quad \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\text{or,} \quad 2a_2 + \sum_{n=1}^{\infty} (n+1)(n+2) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + 2a_0 + 2\sum_{n=1}^{\infty} a_n x^n = 0.$$

$$\text{or,} \quad 2a_2 + 2a_0 + \sum_{n=1}^{\infty} \left[(n+1)(n+2) a_{n+2} + (n+2) a_n\right] x^n = 0$$

$$\Rightarrow 2a_2 + 2a_0 = 0; \quad (n+1)(n+2) a_{n+2} = -(n+2) a_n \quad (n \geq 1)$$

$$\text{or,} \quad a_2 = -a_0; \qquad \text{or,} \quad a_{n+2} = \frac{-a_n}{n+1} \quad (n \geq 1)$$

$n = 1$, $a_3 = -\frac{a_1}{2}$

$n = 2$, $a_4 = \frac{-a_2}{3} = \frac{a_0}{3}$,

$n = 3$, $a_5 = -\frac{a_3}{4} = \frac{a_1}{8}$

$n = 4$, $a_6 = -\frac{a_4}{5} = \frac{-a_0}{15}$;

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$$\therefore y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$$

$$= a_0 + a_1 x - a_0 x^2 - \frac{a_1}{2} x^3 + \frac{a_0}{3} x^4 + \frac{a_1}{8} x^5 - \frac{a_0}{15} x^6 + \cdots$$

$$y = a_0\left(1 - x^2 + \frac{x^4}{3} - \frac{x^6}{15}\right) + a_1\left(x - \frac{x^3}{2} + \frac{x^5}{8} - \cdots\right);$$ a_0, a_1 are arbitrary constants

$$\therefore y' = \cancel{2a_0} \; a_0\left(-2x + \frac{4}{3}x^3 - \cdots\right) + a_1\left(1 - \frac{3}{2}x^2 + \frac{5}{8}x^4 - \cdots\right)$$

Putting the initial conditions: $y(0) = 1$ & $y'(0) = 0$, we get

$a_0 = 1$, $a_1 = 0$

$$\therefore y = 1 - x^2 + \frac{x^4}{3} - \frac{x^6}{15} + \cdots$$

FROBENIUS METHOD FOR SERIES SOLUTION about Regular Singular Point

THEOREM: Let $x = x_0$ be a regular singular point of the D.E. $y'' + P(x)y' + Q(x)y = 0$. Then the D.E. has at least one non-trivial solution, which can be expressed as $y = |x-x_0|^K \sum_{n=0}^{\infty} a_n (x-x_0)^n$; K being a constant. $(a_0 \neq 0)$

This solution will be valid for all $x \in N_R'(x_0)$, the deleted nbd. of x_0 of radius $R(>0)$. $|x-x_0| < R$

THEOREM: Let $x = x_0$ be a regular singular point of the given above D.E., and α, β be two roots of the indicial equation.

(i) If α, β are distinct and $\alpha - \beta$ is not an integer, then the D.E. has two non-trivial l.i.d. solutions of the form:

$$u = |x-x_0|^\alpha \sum_{n=0}^{\infty} a_n (x-x_0)^n,\ a_0 \neq 0, \text{ and}$$
$$v = |x-x_0|^\beta \sum_{n=0}^{\infty} a_n' (x-x_0)^n,\ a_0' \neq 0.$$
valid in $x \in N_R'(x_0)$.

(ii) If α, β are distinct and $\alpha - \beta$ is an integer, then the D.E. has two non-trivial solution l.i.d. of the form:

$$u = |x-x_0|^\alpha \sum_{n=0}^{\infty} a_n (x-x_0)^n,\ a_0 \neq 0$$
and
$$v = |x-x_0|^\beta \sum_{n=0}^{\infty} a_n' (x-x_0)^n + a\, u \ln|x-x_0|,\ a_0' \neq 0,$$

(a being a constant), valid for $x \in N_R'(x_0)$.

(iii) If α, β are equal $(\alpha = \beta)$, then the D.E. has two non-trivial l.i.d. solutions of the form:

$$u = |x-x_0|^\alpha \sum_{n=0}^{\infty} a_n (x-x_0)^n,\ a_0 \neq 0$$
and
$$v = |x-x_0|^{\alpha+1} \sum_{n=0}^{\infty} a_n' (x-x_0)^n + u \ln|x-x_0|$$
valid in $x \in N_R'(x_0)$

① Find the series solution (general) of the D.E.

$$2x^2y'' - xy' + (1-x^2)y = 0 \text{ about } x=0.$$

The normalised form of the given D.E. is

$$y'' - \frac{1}{2x}y' + \frac{1-x^2}{2x^2}y = 0 \longrightarrow ①$$

Comparing ① with $y'' + P(x)y' + Q(x)y = 0$, we have $P(x) = -\frac{1}{2x}$, $Q(x) = \frac{1-x^2}{2x^2}$; both of which are not analytic at $x=0$

So, $x=0$ is a singular point.

Now $xP(x) = -\frac{1}{2}$, $x^2Q(x) = \frac{1-x^2}{2}$, which are both analytic at $x=0$.

$\therefore$ $x=0$ is a regular singular point

We apply the method of Frobenius.

Let

$$y = \sum_{n=0}^{\infty} a_n x^{k+n}, \quad a_0 \neq 0 \longrightarrow ② \text{ be a P.S. Soln of ①}$$

Differentiating ② twice in succession w.r.t. x,

$$y' = \sum_{n=0}^{\infty}(k+n)a_n x^{k+n-1} \text{ and } y'' = \sum_{n=0}^{\infty}(k+n)(k+n-1)a_n x^{k+n-2} \longrightarrow ③$$

Using ② & ③ in ①, we obtain

$$2x^2\sum_{n=0}^{\infty}(k+n)(k+n-1)a_n x^{k+n-2} - x\sum_{n=0}^{\infty}(k+n)a_n x^{k+n-1} + (1-x^2)\sum_{n=0}^{\infty} a_n x^{k+n} = 0$$

or, $$\sum_{n=0}^{\infty}2(k+n)(k+n-1)a_n x^{k+n} - \sum_{n=0}^{\infty}(k+n)a_n x^{k+n} + \sum_{n=0}^{\infty}a_n x^{k+n} - \sum_{n=0}^{\infty}a_n x^{k+n+2} = 0$$

or, $$\sum_{n=0}^{\infty}\left[2(k+n)^2 - 3(k+n) + 1\right]a_n x^{k+n} - \sum_{n=0}^{\infty}a_n x^{k+n+2} = 0$$

or, $$\sum_{n=0}^{\infty}(2k+2n-1)(k+n-1)a_n x^{k+n} - \sum_{n=0}^{\infty}a_n x^{k+n+2} = 0 \rightarrow ④$$

Relation ④ is valid in some deleted nbd. $N_R'(0)$.

Thus, equating to zero the coeff. of the lowest degree term of x in ④), we obtain the indicial equation; $a_0(2k-1)(k-1) = 0 \Rightarrow k = \frac{1}{2}, 1$ $(a_0 \neq 0)$

The roots are distinct and their difference is not an integer.

Now equating to zero the coeff. of x^{k+n}, we obtain the recurrence relation

$$(2k+2n-1)(k+n-1)a_n - a_{n-2} = 0$$

i.e., $$a_n = \frac{a_{n-2}}{(2k+2n-1)(k+n-1)}, \text{ for } n \geq 2. \longrightarrow (5)$$

Again equating to zero the coeff of x^{k+1}, we get $\longrightarrow (6)$.

$$(2k+1)k a_1 = 0 \Rightarrow a_1 = 0, \ (\text{for } k = \tfrac{1}{2}, 1)$$

Using (5) & (6), we have,

$$a_1 = a_3 = a_5 = \cdots = 0.$$

Now putting $n=2$ in (5),

$$a_2 = \frac{a_0}{(2k+3)(k+1)}$$

putting $n=4$ in (5), $a_4 = \dfrac{a_2}{(2k+7)(k+3)} = \dfrac{a_0}{(2k+7)(2k+3)(k+3)(k+1)}$

From (2), we obtain:

$$\therefore\ y = a_0 x^k\left[1 + \frac{x^2}{(k+1)(2k+3)} + \frac{x^4}{(2k+7)(2k+3)(k+3)(k+1)} + \cdots\right] = c_1 u \text{ (say)}$$

Put $k=\frac{1}{2}$ & $a_0 = c_1$; $y = c_1\sqrt{x}\left[1 + \dfrac{x^2}{2.3} + \dfrac{x^4}{2.3.4.7} + \cdots\right] = c_2 v$, (say)

Put $k=1$, & $a_0 = c_2$; $y = c_2 x\left[1 + \dfrac{x^2}{2.5} + \dfrac{x^4}{2.4.5.9} + \cdots\right]$

$\therefore$ g.s. of (1) is given by $y = c_1 u + c_2 v$; c_1, c_2 being arbitrary constants.

(2) Use the method of Frobenious to solve the D.E.

$$(2x+x^3)y'' - y' - 6xy = 0 \text{ near } x=0 \longrightarrow (1)$$

We see that $x=0$ is a regular s.p. of (1).

We assume the soln of (1) near $x=0$ as

$$y = \sum_{n=0}^{\infty} a_n x^{n+k} \quad (a_0 \neq 0) \longrightarrow (2)$$

Differentiating (2) twice w.r.t. x, we get

$$y' = \sum_{n=0}^{\infty}(n+k)a_n x^{n+k-1}, \quad y'' = \sum_{n=0}^{\infty}(n+k)(n+k-1)a_n x^{n+k-2} \longrightarrow (3)$$

Using (2) & (3) in (1), we get

$$2\sum_{n=0}^{\infty}(n+k)(n+k-1)a_n x^{n+k-1} + \sum_{n=0}^{\infty}(n+k)(n+k-1)a_n x^{n+k+1} - \sum_{n=0}^{\infty}(n+k)a_n x^{n+k-1} - 6\sum_{n=0}^{\infty} a_n x^{n+k+1} = 0.$$

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or, $\sum_{n=0}^{\infty}(n+k)(2n+2k-3)a_n x^{n+k-1} + \sum_{n=0}^{\infty}[(n+k)(n+k-1)-6]a_n x^{n+k} = 0 \rightarrow (4)$

Relation (4) is an identity.
Equating the coeff. of the term of the lowest power of x to zero, the indicial eqn is obtained as

$$K(2K-3)=0 \quad [\because a_0 \neq 0] \rightarrow (5)$$

The roots of (5) are $K=0$, $K=3/2$. They are real and unequal and their difference is not an integer.
Thus the eqn (1) has two l.i.d. solutions.

Equating the coeff. of the term of the next highest power of x to zero, we get

$$(2K-1)(K+1)a_1 = 0 \quad \text{or,} \quad a_1 = 0.$$

Equating the coeffs. of x^{n+K-1} to zero, we get

$$(n+K)(2n+2K-3)a_n + (n+K)(n+K-5)a_{n-2} = 0$$

$$\text{or,} \quad a_n = -\frac{n+K-5}{2n+2K-3}a_{n-2} \longrightarrow (6) \quad n \geq 2$$

This is the recurrence formula.
Since $a_1 = 0$, $a_{2K+1} = 0$ for $K = 1, 2, 3, \ldots$
Putting $n = 2, 4, 6, \ldots$, in (6), we get

$$a_2 = -\frac{K-3}{2K+1}a_0, \quad a_4 = -\frac{K-1}{2K+5}a_2 = \frac{(K-1)(K-3)}{(2K+1)(2K+5)}a_0,$$

and so on.
Substituting these values in (2), we have

$$y = x^K\left[a_0 - \frac{K-3}{2K+1}a_0x^2 + \frac{(K-1)(K-3)}{(2K+1)(2K+5)}a_0x^4 + \ldots\right] \rightarrow (7)$$

Putting $K = 0, 3/2$, in (7) we get two l.i.d. Solns.
Hence, the g.s. of (1) is

$$y = a\left[1 + 3x^2 + \frac{3}{5}x^4 + \ldots\right] + bx^{3/2}\left[1 + \frac{3}{8}x^2 - \frac{3}{128}x^4 \ldots\right]$$

where a and b are arbitrary constants.