Velocity of plane longitudinal sound waves in a fluid in a pipe.

A longitudinal wave (sound) propagates through an elastic fluid medium as an alternation in pressure, particle displacement or particle velocity.

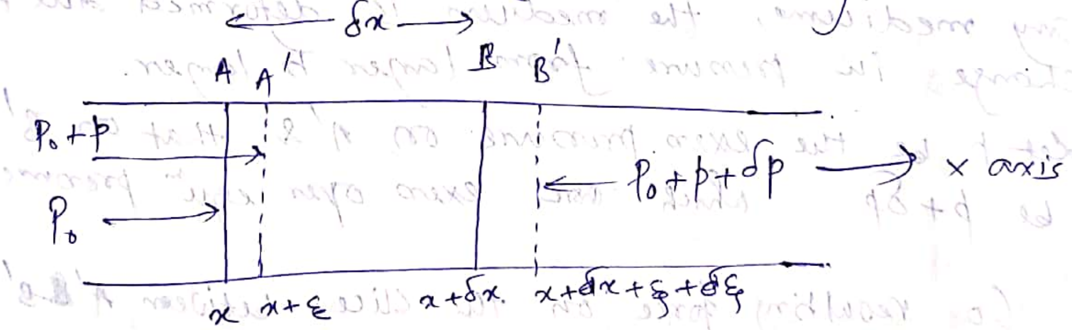
Let us assume –

(i) The medium is homogeneous, isotropic and has no dissipative force.

(ii) The medium is continuous i.e. the wavelength is much greater than the mean free path.

(iii) The equilibrium pressure P_0 and density ρ_0 are the same everywhere i.e. gravity is neglected.

(iv) The wave is of small amplitude such that the strain developed in the medium is so small that the Hooke's Law is obeyed.



Let us consider a tube of fluid of unit crosssection with its axis in the direction of propagation of the wave. Two planes A & B are at x & $x+\delta x$ positi initially. Such that $\delta x << \lambda$ and

$\delta x >>$ mean molecular seperation.

Under influence of sound wave let all the particles in the layer A are displaced parallel to the axis of the tube by a distance ξ to A'. and those at B are displaced by $\xi + \delta\xi$ to B'.

$$\xi << \delta x, \qquad \xi = \xi(x,t)$$

This disturbance causes change in volume of the fluid between two planes and also a variation of pressure from point to point along the x axis.

The final volume of the slice $A'B'$ is

$(x + \delta x + \xi + \delta\xi).1 - (x+\xi).1$ AS cross section is unity.

$= \delta x + \delta\xi$

$= \delta x + \frac{\partial \xi}{\partial x} \cdot \delta x.$

The fractional change in volume i.e ~~diatation~~ dilatation

$$\Delta = \frac{\text{Change in volume}}{\text{Original volume}}$$

$$= \frac{\left(\delta x + \frac{\partial \xi}{\partial x} \cdot \delta x\right) - \delta x}{\delta x}$$

$$= \frac{\partial \xi}{\partial x}$$

When an accoustic wave passes through any mediume, the medium is deformed due to change in pressure from layer to layer.

Let p be the exess pressure on A' & that on B' be $p + \delta p$ which are exess oper equ^n pressure P_0.

So resulting force on the slice between $A'B'$ along x axis is

$p.1 - (p+\delta p).1$ $\quad f = p \times area = p \times 1 = p$

$= p - \left(p + \frac{\partial p}{\partial x} \cdot \delta x\right)$

$= -\frac{\partial p}{\partial x} \cdot \delta x.$

From Newton's Second Law of motion -

$$\rho_0\, \delta x \cdot \frac{\partial^2 \xi}{\partial t^2} = -\frac{\partial p}{\partial x}\, \delta x.$$

$(\rho_0 \delta x) \Rightarrow$ mass of slice AB

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = -\frac{\partial p}{\partial x}$$

$\frac{\partial^2 \xi}{\partial t^2} \Rightarrow$ accelation.

Now we know Bulk modulus,

$$K = \frac{\text{Volume stress}}{\text{Volume strain}}$$

$$= \frac{p}{-\Delta}$$

$$p = -K\frac{\partial \xi}{\partial x}$$

$$\therefore \rho_0 \frac{\partial^2 \xi}{\partial t^2} = -\frac{\partial}{\partial x}\left(-K\frac{\partial \xi}{\partial x}\right)$$

$$= K\frac{\partial^2 \xi}{\partial x^2}$$

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{K}{\rho_0}\frac{\partial^2 \xi}{\partial x^2}$$

$$\boxed{\frac{\partial^2 \xi}{\partial t^2} = v^2 \frac{\partial^2 \xi}{\partial x^2}}$$

$v \Rightarrow$ Velocity of wave $= \sqrt{\frac{K}{\rho_0}}$

↳ Differential wave equation for longitudinal wave (Sound wave) in elastic fluid.

<u>Condensation</u> : The ratio of the increase in density $\delta\rho$ of a layer to the initial density ρ_0 is known as condensation (s).

$$s = \frac{\delta\rho}{\rho_0}$$

The final density $\rho = \rho_0 + \delta\rho$
$$= \rho_0(1+s)$$

If V_0 and V be the initial and final volumes of the fluid slab respectively, then

$\rho V = \rho_0 V_0$ as mass is same

Again $V = V_0 + \delta V$ $\qquad \frac{\delta V}{V_0} = \Delta$
$$= V_0(1+\Delta)$$

So. $\rho_0(1+s)\cdot V_0(1+\Delta) = \rho_0 V_0$

$\Rightarrow (1+s)(1+\Delta) = 1$

$\Rightarrow \quad 1+s+\Delta = 1$ $\quad$ $s\Delta$ is negligible

$s = -\Delta.$

Again Bulk modulus $K = -\frac{p}{\Delta} = \frac{p}{s}.$

$p = Ks.$

$= v_0^2 \rho_0 s.$

Differential wave equation in 1-D

A plane progressive wave propagating in +ve x direction is represented by

$$\psi = f(ct - x) \qquad \psi \Rightarrow \text{wave field parameter.}$$

Putting $z = ct - x$.

$$\frac{\partial z}{\partial t} = c, \qquad \frac{\partial z}{\partial x} = -1$$

Hence, $\dfrac{\partial \psi}{\partial x} = \dfrac{d\psi}{dz} \cdot \dfrac{\partial z}{\partial x} = -\dfrac{d\psi}{dz}$

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{\partial}{\partial x}\left(\frac{d\psi}{dz}\right)$$

$$= -\frac{d}{dz}\left(\frac{d\psi}{dz}\right) \cdot \frac{\partial z}{\partial x}$$

$$= \frac{d^2\psi}{dz^2} \longleftarrow (1)$$

And. $\dfrac{\partial \psi}{\partial t} = \dfrac{d\psi}{dz} \cdot \dfrac{\partial z}{\partial t} = c\,\dfrac{d\psi}{dz}$

$$\frac{\partial^2 \psi}{\partial t^2} = c\,\frac{\partial}{\partial t}\left(\frac{d\psi}{dz}\right)$$

$$= c\,\frac{d}{dz}\left(\frac{d\psi}{dz}\right) \cdot \frac{\partial z}{\partial t}$$

$$= c^2\,\frac{d^2\psi}{dz^2} \longleftarrow (2)$$

From equation (1) & (2) we get

$$\frac{\partial^2 \psi}{\partial t^2} = c^2\,\frac{\partial^2 \psi}{\partial x^2}$$

Similarly if the wave propagates along −ve X direction then $\psi = f(ct + x)$, and we get same equation.

Properties of the differential wave equation

We have seen that particle displacement ξ obeys the differential equation for plane wave.

$$\frac{\partial^2 \xi}{\partial t^2} = v^2 \frac{\partial^2 \xi}{\partial x^2} \quad \text{--- (1)} \qquad \xi = \xi(x,t)$$

Now we can show that, the particle velocity, dilatation, condensation, accoustic pressure etc. satisfy the same differential equation.

(i) The particle velocity
$$u = \frac{\partial \xi}{\partial t}$$

Differentiating equation (1) w.r.t. t

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 \xi}{\partial t^2}\right) = v^2 \frac{\partial}{\partial t}\left(\frac{\partial^2 \xi}{\partial x^2}\right)$$

Since x, t are independent variables, the order of differentiation w.r.t. x & t can be interchanged.

$$\text{So. } \frac{\partial^2}{\partial t^2}\left(\frac{\partial \xi}{\partial t}\right) = v^2 \frac{\partial^2}{\partial x^2}\left(\frac{\partial \xi}{\partial t}\right)$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$$

So. particle velocity u obeys the wave equation.

Similarly we can show acceleration $\frac{\partial u}{\partial t}$ can satisfy wave equation.

(ii) The ~~dialatation~~ dialatation $\Delta = \frac{\partial \xi}{\partial x} = -s$.

Differentiating eqn (1) w.r.t. x

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 \xi}{\partial t^2}\right) = v^2 \frac{\partial}{\partial x}\left(\frac{\partial^2 \xi}{\partial x^2}\right)$$

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial \xi}{\partial x}\right) = v^2 \frac{\partial^2}{\partial x^2}\left(\frac{\partial \xi}{\partial x}\right)$$

$$\Rightarrow \frac{\partial^2 \Delta}{\partial t^2} = v^2 \frac{\partial^2 \Delta}{\partial x^2}$$

So Δ and s obey the wave equation.

(iii) Accoustic pressure $p = Ks$.

K = Bulk Modulas.

Since s satisfy wave equation p must obey the wave equation.

Similarly pressure gradient $\frac{\partial p}{\partial x}$, exess density $\delta\rho = \rho_0 s$ obviously obey wave equation.

Simple harmonic solution of the wave equation.

Differential equation for the wave field parameter ψ associated with a plane progressive wave with the phase velocity is

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial x^2}$$

ψ may be particle displacement ξ, the particle velocity, ~~dil~~ dilatation, accoustic pressure etc.

For small displacements, when Hooke's Law is valid, i.e. restoring force is proportional to the displacement the particle displacement ξ varies simple harmonically.

$$\xi = A \cos(\omega t - kx)$$

particle velocity $u = \frac{\partial \xi}{\partial t} = -\omega A \sin(\omega t - kx)$

$$= \omega A \cos\left(\omega t - kx + \frac{\pi}{2}\right)$$

The dilatation

$$\Delta = \frac{\partial \xi}{\partial x} = KA \sin(\omega t - Kx)$$

$$= KA \cos\left(\omega t - Kx \cancel{\mp} \frac{\pi}{2}\right)$$

The condensation,

$$s = -\Delta = -KA \sin(\omega t - Kx)$$

$$= KA \cos\left(\omega t - Kx + \frac{\pi}{2}\right)$$

The accoustic pressure,

$$p = Ks = -K\,KA \sin(\omega t - Kx)$$

$$= -v^2 \rho_0 KA \sin(\omega t - Kx)$$

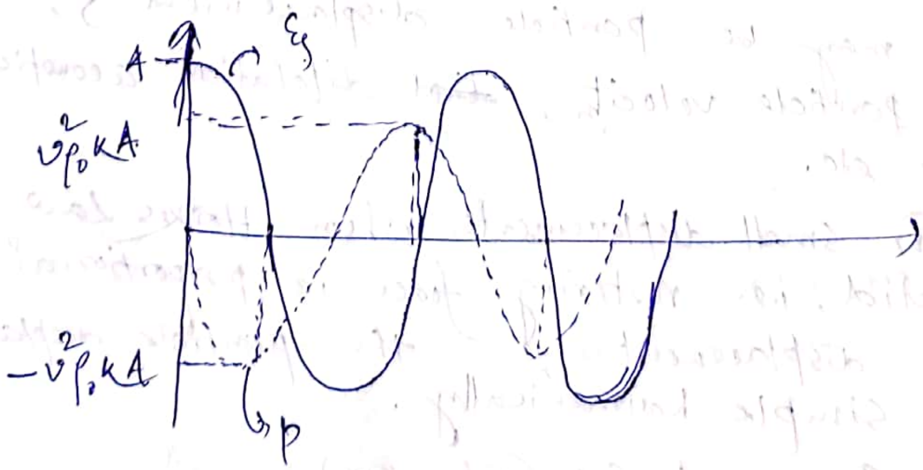
$$= v^2 \rho_0 KA \cos\left(\omega t - Kx + \frac{\pi}{2}\right)$$

The excess density,

$$d = \rho_0 s = -\rho_0 KA \sin(\omega t - Kx)$$

$$= \rho_0 KA \cos\left(\omega t - Kx + \frac{\pi}{2}\right)$$

So. u, s, p and d are in phase and all of them lead ~~lags~~ the ξ by $\pi/2$ in time. Only Δ lags ξ by $\pi/2$.



when ξ is $+A$ or $-A$, $p = 0$

at $\xi = 0$ $p = +v^2 \rho_0 KA$ or $-v^2 \rho_0 KA$