

Theorem:- If a sequence converges, then its limit is unique.

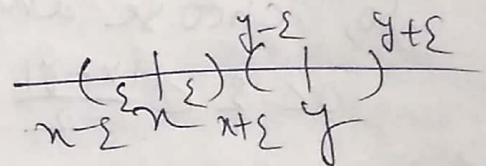
A sequence can't converge to two different limits.

or
Theorem:- Limit of a convergent sequence is unique.

Proof:- To show this you just consider that a the sequence converges to two different limits and then show it is contradiction.

So, suppose $\{x_n\}$ is a convergent sequence and it converges to two limits x & y , where $x \neq y$.

Let $x < y$



So, for every $\epsilon > 0$ we can find some n_1 such that for $n > n_1$, $|x_n - x| < \epsilon$

$$\Rightarrow x_n \in (x - \epsilon, x + \epsilon) \quad \forall n > n_1$$

Similarly we can find some n_2 such that for $n > n_2$, $|x_n - y| < \epsilon$

$$\Rightarrow x_n \in (y - \epsilon, y + \epsilon) \quad \forall n > n_2$$

Now we can always find a n s.t.

$n_0 = \max(n_1, n_2)$ and for which

$$x_n \in (x - \epsilon, x + \epsilon) \quad \& \quad x_n \in (y - \epsilon, y + \epsilon) \quad \forall n > n_0$$

That means for $n > n_0$, x_n lies on

both the interval $(x - \epsilon, x + \epsilon)$ & $(y - \epsilon, y + \epsilon)$

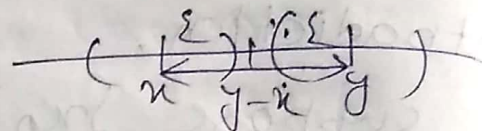
Now for $n \geq$,
 Now if we can show that these two
 intervals $(x-\epsilon, x+\epsilon)$ & $(y-\epsilon, y+\epsilon)$ are
 disjoint then $\forall n \geq n_0$ both
 x_n can't lie on both the intervals.
 Which will contradict our assumption.

Now, can you find an ϵ s.t.

~~$(x-\epsilon, x+\epsilon)$~~ & $(x-\epsilon, x+\epsilon)$ and $(y-\epsilon, y+\epsilon)$ are
 disjoint. Total length of each of the intervals
 is 2ϵ .

$$\text{If } 2\epsilon < \frac{y-x}{2}$$

$$\text{If } \epsilon < \frac{y-x}{2}$$



then they will be
 disjoint.

So, choose any positive ϵ with s.t.

$$0 < \epsilon < \frac{|x-y|}{2} \rightarrow \text{mod is taken because that will work for both } x < y \text{ \& } y < x$$

So, If we choose

$0 < \epsilon < \frac{|x-y|}{2}$ then x_n can't be on both the
 intervals will be disjoint and so
 x_n will not be able to lie on the
 both the intervals $(x-\epsilon, x+\epsilon)$ &
 $(y-\epsilon, y+\epsilon)$ for $n \geq n_0$.

Which is a contradiction.

(As we have considered $\forall \epsilon > 0$

x_n should lie in $(x-\epsilon, x+\epsilon)$ & $(y-\epsilon, y+\epsilon)$
 $\forall \epsilon > 0$)

Hence the proof.

Limit pt. of a sequence :-

A number l is said to be a limit pt. of a sequence $\{x_n\}$ if ~~every neighbourhood~~ for any $\epsilon > 0$, ~~$x_n \in (l - \epsilon, l + \epsilon)$~~

$x_n \in (l - \epsilon, l + \epsilon)$, ~~for infinitely~~
for infinitely many values of n .

Note:- Limit of a sequence is unique.
But limit point of a sequence may
can be more than 1.

Examp

Note:- The sequence $\{(-1)^n\}$ has two limit
1. ~~points~~ $\frac{1}{3}$ the elements of the sequence
are either 1 or -1. So there are infinitely
many 1 & ~~not~~ -1.

So, $1 \in (1 - \epsilon, 1 + \epsilon) \forall \epsilon > 0$, for infinitely
many n

Similarly $-1 \in (-1 - \epsilon, -1 + \epsilon) \forall \epsilon > 0$, $\frac{1}{3}$
for infinitely
many n

But $\because$ it has two limit
points it is not a convergent sequence.
We $\because$ we know limit of a convergent sequence
is unique.

2. If a sequence has unique limit point
then that does not imply the sequence
is convergent.

Suppose $\left\{ \frac{n}{2} (1 + (-1)^n) \right\}$
 $= \{0, 1, 0, 2, 0, 3, 0, 4, 0, 5, \dots\}$

The sequence has unique limit point 0. But it is not a convergent sequence as it is not bounded above.

(3) Consider the sequence $\{x_n\}$

$$x_n = \begin{cases} 1 & \text{for } n=1 \\ n - \frac{k(k+1)}{2} & \text{for } \frac{k(k+1)}{2} < n \leq \frac{(k+1)(k+2)}{2} \end{cases}$$

$$\{x_n\} = \{1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \dots\}$$

We see $\{x_n\}$ is a sequence of natural numbers.

The limit points of the sequence are the set $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ of natural numbers.

$$(3) \quad x_n = \begin{cases} \frac{1}{k+2} & \text{for } n=k \\ \frac{n}{k(k+2)} & \text{for } \frac{k(k+1)}{2} < n \leq \frac{(k+1)(k+2)}{2} \end{cases}$$

$$\text{Suppose } \{x_n\} = \left\{ \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{3}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \dots \right\}$$

The limit points of the sequence are the rational numbers in $(0, 1)$.

Find the limit points of the following if exists:

(1) $\{n^2\}$, (2) $\{1, 2, 4, 4, 1, 6, 1, 8, \dots\}$

(3) $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\}$

(4) $\{1, 3, 5, \dots\}$ (5) $\{2 + (-1)^n\}$

(6) $\{1, 2, 3, 1, 2, 3, 1, 2, 3, \dots\}$ (7) $\{2 - \frac{1}{2^{n-1}}\}$

(8) $\{x_n\}$, $x_n = \begin{cases} 2 & \text{when } n \text{ is even} \\ \frac{1}{n} & \text{when } n \text{ is odd} \end{cases}$

(9) $\{x_n\}$, $x_n = \begin{cases} 2 & \text{when } n \text{ is even} \\ \frac{1}{n} & \text{when } n \text{ is odd} \end{cases}$ (10) $\{m + \frac{1}{n}\}$ (11) $\{ \frac{1}{m} + \frac{1}{n} \}$ (12) $\{ \frac{1}{m} + \frac{1}{n} \}$

Some Algebraic Properties of Limit of a Sequence:—

Theorem:- Let $\{x_n\}$ and $\{y_n\}$ be two convergent sequences such that $\{x_n\}$ converges to x and $\{y_n\}$ converges to y .

Then (i) $\{x_n + y_n\}$ converges to $x + y$.

(ii) $\{x_n - y_n\}$ converges to $x - y$.

$$(iii) \lim_{n \rightarrow \infty} (x_n y_n) = x \cdot y = \lim_{n \rightarrow \infty} x_n \cdot \lim_{n \rightarrow \infty} y_n$$

(iv) If $y_n \neq 0$ & $y \neq 0$

$$\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \frac{x}{y} = \frac{\lim_{n \rightarrow \infty} x_n}{\lim_{n \rightarrow \infty} y_n}$$

(v) $k \in \mathbb{R}$, $\{k x_n\}$ converges to kx .

(vi) If $x_n \leq y_n$, then $x \leq y$.

Proof: (i) $\lim_{n \rightarrow \infty} x_n = x$ & $\lim_{n \rightarrow \infty} y_n = y$

Therefore for any $\varepsilon > 0$, $\exists N_1, N_2$ in $\mathbb{N}$ such that

$$\forall n \geq N_1, |x_n - x| < \varepsilon/2,$$

$$\forall n \geq N_2, |y_n - y| < \varepsilon/2.$$

Let $N = \max\{N_1, N_2\}$.

Then $\forall n \geq N$, $|x_n - x| < \varepsilon/2$ & $|y_n - y| < \varepsilon/2$

$$\text{Now, } |(x_n + y_n) - (x + y)| \leq |x_n - x| + |y_n - y| < \varepsilon \quad \forall n \geq N$$

$$\Rightarrow \lim_{n \rightarrow \infty} (x_n + y_n) = (x + y) = \lim_{n \rightarrow \infty} x_n + \lim_{n \rightarrow \infty} y_n$$

The proof of (2) is trivial and it is ~~is given as a task~~.

Try yourself.

(iii) We use the identity

$$(x_n y_n - xy) = (x_n - x)(y_n - y) + x(y_n - y) + y(x_n - x)$$

Given any $\varepsilon > 0$, there are integers N_1, N_2 such that

$$\forall n \geq N_1, |x_n - x| < \sqrt{\varepsilon}, \quad \text{--- (i)}$$

$$\forall n \geq N_2, |y_n - y| < \sqrt{\varepsilon} \quad \text{--- (ii)}$$

Let $N = \max\{N_1, N_2\} \Rightarrow N \geq N_1 \text{ \& } N \geq N_2$

Then, $|x_n - x| < \sqrt{\varepsilon} \text{ \& } |y_n - y| < \sqrt{\varepsilon}$

$$\forall n \geq N$$

[As (i) & (ii) holds $\forall n \geq N_1 \text{ \& } n \geq N_2$ respectively, so (i) & (ii) holds for $\forall n \geq N \Rightarrow N \geq N_1 \text{ \& } N \geq N_2$]

Now, so, $\forall n \geq N, |(x_n - x)(y_n - y)| < \varepsilon$ & x_n

$$\Rightarrow \lim_{n \rightarrow \infty} (x_n - x)(y_n - y) = 0$$

From (1)

$$\lim_{n \rightarrow \infty} (x_n y_n - xy) = \lim_{n \rightarrow \infty} (x_n - x)(y_n - y) + \lim_{n \rightarrow \infty} x(y_n - y) + \lim_{n \rightarrow \infty} y(x_n - x)$$

[As we know $\lim_{n \rightarrow \infty} (x_n + y_n) = \lim_{n \rightarrow \infty} x_n + \lim_{n \rightarrow \infty} y_n$ as we know $\lim_{n \rightarrow \infty} (x_n y_n) = \lim_{n \rightarrow \infty} x_n \cdot \lim_{n \rightarrow \infty} y_n$]

$$\lim_{n \rightarrow \infty} x_n y_n = \lim_{n \rightarrow \infty} x_n \cdot \lim_{n \rightarrow \infty} y_n$$

$$\lim_{n \rightarrow \infty} x_n y_n - xy = 0 + 0 + 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} x_n y_n = xy$$

$$\because \lim_{n \rightarrow \infty} k x_n =$$

$$k \lim_{n \rightarrow \infty} x_n$$

we do this
proof
later

(IV) Since

$$\text{Since } \lim_{n \rightarrow \infty} y_n = y$$

for any $\epsilon > 0 \exists N \in \mathbb{N}$ s.t.

$$|y_n - y| < \epsilon/2 \quad \forall n > N_1 \quad \text{--- (a)}$$

let us take $\epsilon = |y|$ as (a) is true for all $\epsilon > 0$

$$\text{so, } \forall n > N_1, |y_n - y| < \frac{|y|}{2}$$

Hence $\forall n > N_1$

$$\frac{|y|}{2} > |y_n - y| \Rightarrow |y - y_n| > |y| - |y_n|$$

$$\Rightarrow |y_n| > |y| - \frac{1}{2}|y| = \frac{1}{2}|y|, \quad \forall n > N_1 \quad \text{--- (b)}$$

We have for every value of n

$$\left| \frac{x_n}{y_n} - \frac{x}{y} \right| = \left| \frac{y x_n - x y_n}{y_n y} \right|$$

$$= \left| \frac{y(x_n - x) - x(y_n - y)}{y_n y} \right|$$

$$\leq \frac{|y| |x_n - x| + |x| |y_n - y|}{|y_n| |y|}$$

now for $\forall n > N_1$

$$\left| \frac{x_n}{y_n} - \frac{x}{y} \right| \leq \frac{|y| |x_n - x| + |x| |y_n - y|}{|y|} \cdot \frac{2}{|y|}$$

$$\begin{aligned} |y_n| &> \frac{1}{2}|y| \\ \Rightarrow \frac{1}{|y_n|} &< \frac{2}{|y|} \end{aligned}$$

$$= \frac{2}{|y|} |x_n - x| + \frac{2|x|}{|y|^2} |y_n - y| \quad \leftarrow (c) \quad \forall n > N_1$$

Now since $\lim_{n \rightarrow \infty} x_n = x$ & $\lim_{n \rightarrow \infty} y_n = y$

for every $\epsilon > 0$ $\exists N_2, N_3$ in $\mathbb{N}$ such that

$$\begin{aligned} |x_n - x| &< \frac{\epsilon |y|}{4} \quad \leftarrow (d) \quad \forall n > N_2 \\ |y_n - y| &< \frac{|y|^2}{4(|x|+1)} \epsilon \quad \leftarrow (e) \quad \forall n > N_3 \end{aligned} \quad \left\{ \begin{array}{l} \text{we are} \\ \text{taking} \\ \text{this type} \\ \text{of } (d) \text{ \& } (e) \\ \text{we want} \\ \text{to make} \\ (c) \text{ as } \epsilon \\ \text{only} \end{array} \right.$$

$$\text{let } N = \max \{N_1, N_2, N_3\}$$

Then $\forall n > N_0$, (c), (d), (e) holds

$$\text{so, } \left| \frac{x_n}{y_n} \right|$$

so, $\forall n > N_0$

$$\left| \frac{x_n}{y_n} - \frac{x}{y} \right| < \frac{2}{|y|} \frac{|y|}{4} \epsilon + \frac{2|x|}{|y|^2} \frac{|y|^2}{4(|x|+1)} \epsilon$$

$$= \frac{\epsilon}{2} + \frac{\epsilon}{2} \frac{|x|}{|x|+1}$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad \text{as } \frac{|x|}{|x|+1} < 1$$

$$\leq \epsilon$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \frac{x}{y}$$

(v) is left for you.

(1) we have to show that
if $x_n \leq y_n \forall n$ then $x \leq y$.
we will show by contradiction.

↳ suppose $x \neq y$.

Let $x > y$.

as $\lim_{n \rightarrow \infty} x_n = x$ & $\lim_{n \rightarrow \infty} y_n = y$.

then for every $\epsilon > 0 \exists N_1, N_2 \in \mathbb{N}$
such that

$$x_n \in (x - \epsilon, x + \epsilon) \quad \forall n > N_1$$

$$y_n \in (y - \epsilon, y + \epsilon) \quad \forall n > N_2$$

let $N = \max \{N_1, N_2\}$

$$\text{Then } x_n \in (x - \epsilon, x + \epsilon) \quad \forall n > N$$

$$y_n \in (y - \epsilon, y + \epsilon)$$

$$\Rightarrow x_n > y_n \quad \forall n > N$$

But this is false as we are given

$$x_n \leq y_n \forall n.$$

Which is a contradiction.

Hence the proof.

Note:- Even if $x_n < y_n \forall n$ we can't say
that $x < y$. [$x_n < y_n$ does not imply $x < y$]
as an example,

$$\{x_n\} \quad \{y_n\}$$

$$\{1\} \quad \{1\}$$

$$\{0\} \quad \{y_n\}$$

$$x_n < y_n \text{ but}$$

$$x \neq y$$

$$\{x_n\} = \{1/n^2\}$$

$$\{y_n\} = 1/n$$

$$x_n < y_n$$

$$\text{but } x = 0 = y$$

$$\Rightarrow x \neq y.$$

Examples

These are the theorems that can be used very often.

We have seen that the limit of $\{x_n\}$ & $\{c\}$ ~~const~~ converge to 0 & c.

constant
sequence

But using these theorems we can decide convergence or divergence of various sequences and also decide what their limits are.

$$\text{Let } x_n = \frac{3n+5}{7n-1} = \frac{3+\frac{5}{n}}{7-\frac{1}{n}} \rightarrow \frac{3+0}{7-0} \rightarrow 3/7 \text{ as } n \rightarrow \infty$$

For the following ~~exercises~~
using the above theorems at first find the limit and then apply the defⁿ to ~~check~~ verify

$$(1) \lim_{n \rightarrow \infty} \frac{3n+5}{7n-1} = ?$$

$$\begin{aligned} \text{Soln: } &= \lim_{n \rightarrow \infty} \frac{3+\frac{5}{n}}{7-\frac{1}{n}} = \frac{\lim_{n \rightarrow \infty} (3+\frac{5}{n})}{\lim_{n \rightarrow \infty} (7-\frac{1}{n})} \left[\because \frac{\lim_{n \rightarrow \infty} x_n}{\lim_{n \rightarrow \infty} y_n} = \lim_{n \rightarrow \infty} \frac{x_n}{y_n} \right] \\ &= \frac{3+0}{7-0} \left[\because \lim_{n \rightarrow \infty} (x_n \pm y_n) = \lim_{n \rightarrow \infty} x_n \pm \lim_{n \rightarrow \infty} y_n \right] \\ &= 3/7 \end{aligned}$$

$$\text{So, to show } \lim_{n \rightarrow \infty} \frac{3n+5}{7n-1} = 3/7$$

we have

for any $\epsilon > 0$, we have to find

a natural number n_0 s.t.

$$\forall n > n_0, \left| \frac{3n+5}{7n-1} - \frac{3}{7} \right| < \epsilon$$

$$\forall n > n_0, \left| \frac{3n+5}{7n-1} - \frac{3}{7} \right| < \epsilon$$

$$\Leftrightarrow \left| \frac{(3n+5)/7 - 3(7n-1)}{7(7n-1)} \right| < \epsilon$$

$$\Leftrightarrow \left| \frac{38}{7(7n-1)} \right| < \epsilon$$

$$\Leftrightarrow \left| \frac{38}{49(n-\frac{1}{7})} \right| < \epsilon \quad \text{as } n > 1, \quad n - \frac{1}{7} > 0$$

$$\Leftrightarrow (n - \frac{1}{7}) > \frac{38}{49\epsilon}$$

$$\Leftrightarrow n > \frac{38}{49\epsilon} + \frac{1}{7}$$

So, take $n_0 = \left\lceil \frac{38}{49\epsilon} + \frac{1}{7} \right\rceil + 1$

Therefore $\forall n \geq n_0, \left| \frac{3n+5}{7n-1} - \frac{3}{7} \right| < \epsilon$.

Find the limits using the theorem and then verify it by using the ϵ - N definition.

② $\lim_{n \rightarrow \infty}$

② $\lim_{n \rightarrow \infty} \frac{2n+1}{n+1}$

(3) $\lim_{n \rightarrow \infty} \frac{n+1}{2n}$

④ $\lim_{n \rightarrow \infty} \frac{3n-1}{n+2}$

⑤ $\lim_{n \rightarrow \infty} \frac{2n+5}{6n-11}$

⑥ $\lim_{n \rightarrow \infty} \frac{3n}{n+5\sqrt{n}}$

⑦ $\lim_{n \rightarrow \infty} \frac{n^2+n+1}{3n^2+1}$

⑧ $\lim_{n \rightarrow \infty} \frac{n^2-1}{n^2+n+1}$

⑨ $\lim_{n \rightarrow \infty} \frac{2n^2+1}{n^2+3n}$

⑩ $\lim_{n \rightarrow \infty} \frac{n^2+n+1}{n^2-5n-3}$

⑪ $\lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n})$

$$(12) \lim_{n \rightarrow \infty} \frac{n^2 + 3n}{2n^2 + n - 1}$$

$$(13) \lim_{n \rightarrow \infty} \frac{5n^2 + 2}{6n^2 + 4n + 7}$$

$$(13) \lim_{n \rightarrow \infty}$$

$$\begin{aligned} (8) \lim_{n \rightarrow \infty} \frac{n^2 - 1}{n^2 + n + 1} \\ = \lim_{n \rightarrow \infty} \frac{1 - 1/n^2}{1 + 1/n + 1/n^2} \\ = \frac{1 - 0}{1 + 0 + 0} = 1 \end{aligned}$$

Step now,

$$\left| \frac{n^2 - 1}{n^2 + n + 1} - 1 \right| < \epsilon$$

$$\Leftrightarrow \left| \frac{n^2 - 1 - n^2 - n - 1}{n^2 + n + 1} \right| < \epsilon$$

$$\Rightarrow \left| \frac{-(n+2)}{n^2 + n + 1} \right| < \epsilon$$

$$\Rightarrow \left\{ \frac{n+2}{n^2 + n + 1} \right\} < \epsilon$$

Now, for $n \geq 2$ $n+2 < n+2n \leq 2n$
 $n^2 + n + 1 > n^2 \quad \forall n$

Le So, $\frac{n+2}{n^2 + n + 1} < \frac{2n}{n^2} = 2/n < \epsilon$
 $\Rightarrow n > 2/\epsilon$

So, choose $n_0 = \max \{2, [2/\epsilon]\}$

So, $\forall$ Therefore $\forall n \geq n_0$
 $\left| \frac{n^2 - 1}{n^2 + n + 1} - 1 \right| < \epsilon$

$$(9) \lim_{n \rightarrow \infty} \frac{2n^2+1}{n^2+3n}$$

$$= \lim_{n \rightarrow \infty} \frac{2 + 1/n}{1 + 3/n} = 2$$

$$\left| \frac{2n^2+1}{n^2+3n} - 2 \right| = \left| \frac{1-6n}{n^2+3n} \right| = \left| \frac{6(\frac{1}{6}-n)}{n^2+3n} \right| \quad (*)$$

$$| \frac{1}{6} - n | \leq | \frac{1}{6} | + | n | = n + \frac{1}{6}$$

$$\forall n \geq 1$$

$$| \frac{1}{6} - n | \leq \frac{1}{6} + n \leq n + n = 2n$$

$$n^2+3n > n^2$$

$$\Rightarrow \frac{1}{n^2+3n} < \frac{1}{n^2}$$

$$\text{So, } \left| \frac{2n^2+1}{n^2+3n} - 2 \right| \leq \frac{2n}{n^2} = \frac{2}{n} < \epsilon$$

$$\Rightarrow n > \frac{2}{\epsilon}$$

$$\text{Let us take } n_0 = \left\lceil \frac{2}{\epsilon} \right\rceil + 1$$

$\exists n_0 \text{ s.t.}$

$$\text{So, } \forall n > n_0, \left| \frac{2n^2+1}{n^2+3n} - 2 \right| < \epsilon$$

$$(10) \lim_{n \rightarrow \infty} \frac{n^2+n+1}{n^2-5n-3}$$

$$= \lim_{n \rightarrow \infty} \frac{1 + 1/n + 1/n^2}{1 - 5/n - 3/n^2} = 1$$

$$\left| \frac{n^2+n+1}{n^2-5n-3} - 1 \right| = \left| \frac{6n+4}{n^2-5n-3} \right| = \left| \frac{6(n+\frac{2}{3})}{n^2-5n-3} \right|$$

$$\forall n \geq 1, \quad n + \frac{2}{3} = n + \frac{2}{3} < n + n = 2n \quad (1)$$

$$\begin{aligned} |n^2-5n-3| &\geq n^2 - (5n+3) & \because |a-b| \geq |a|-|b| \\ &\geq n^2 - \frac{1}{2}n^2 & \text{if } 5n+3 < \frac{n^2}{2} \\ &= \frac{1}{2}n^2 & \text{if } 5n+3 < \frac{n^2}{2} \end{aligned}$$

$$1+3 < \frac{n^2}{2}$$

$$n^2 - 6n - 6 > 0$$

$$\Rightarrow n^2 - 2 \cdot 3 \cdot n + 9 - 6 - 9 > 0$$

$$\Rightarrow (n-3)^2 - 15 > 0$$

$$\Rightarrow (n-3)^2 > 15$$

$$2n^2 > 5n+3 < \frac{n^2}{2}$$

$$\Rightarrow 10n+6 < n^2$$

$$\Rightarrow n^2 - 10n - 6 > 0$$

$$\Rightarrow (n^2 - 2 \cdot 5 \cdot n + 25) - 6 - 25 > 0$$

$$\Rightarrow (n-5)^2 - 31 > 0$$

$$\Rightarrow (n-5) > \sqrt{31}$$

$$\Rightarrow n > 5 + \sqrt{31}$$

$$\Rightarrow n > 5 + \sqrt{31} \quad [\because 5 - \sqrt{31} < 0 \text{ \& \textit{one } n \geq 1}]$$

$$\text{So, we take } n_0 = [5 + \sqrt{31}] + 1$$

$$\text{So, for } n > n_0$$

$$|n^2 - 5n - 3| \geq \frac{n^2}{2}$$

$$\Rightarrow \frac{1}{|n^2 - 5n - 3|} \leq \frac{2}{n^2} \quad (3)$$

So from (1) applying (2) & (3) we get

$$\left| \frac{n^2 + n + 1}{n^2 - 5n - 3} - 1 \right| \leq \frac{2n \times 2}{n^2} = \frac{4}{n} < \frac{4}{n} \quad \text{wherever } n > n_0$$

$$\Rightarrow n > \frac{4}{\epsilon}$$

So, we take

$$N = \max \{ [5 + \sqrt{31}] + 1, [4/\epsilon] + 1 \}$$

so, Therefore $\forall n > N$

$$\left| \frac{n^2 + n + 1}{n^2 - 5n - 3} - 1 \right| < \epsilon$$

$$(11) \lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n})$$

$$= \lim_{n \rightarrow \infty} \left(\frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} \right)$$

$$= \lim_{n \rightarrow \infty} \left[\frac{n+1 - n}{\sqrt{n+1} + \sqrt{n}} \right] = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}(1 + \sqrt{1 + \frac{1}{n}})}$$

$$\text{now, } |\sqrt{n+1} - \sqrt{n} - 0| = 0$$

$$= \left| \frac{1}{\sqrt{n} + \sqrt{n+1}} \right| < \frac{1}{2\sqrt{n}} \left[\because \sqrt{n+1} > \sqrt{n} \right]$$

$$< \epsilon$$

$$\Rightarrow n > \frac{1}{4\epsilon^2}$$

$$\text{So, we take } n_0 = \left[\frac{1}{4\epsilon^2} \right] + 1$$

$$(14) \lim_{n \rightarrow \infty} \frac{6n^3 + 2n + 1}{n^3 + n^2}$$

$$(15) \lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n}) \sqrt{n + \frac{1}{2}}$$

$$(16) \lim_{n \rightarrow \infty} (\sqrt[3]{n+1} - \sqrt[3]{n})$$

$$(17) \lim_{n \rightarrow \infty} (\sqrt{n^2 + n} - n)$$

$$(18) \lim_{n \rightarrow \infty} \frac{3 + 2\sqrt{n+1}}{\sqrt{n+1}}$$

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Theorem:- If a sequence converges, then its limit is unique.

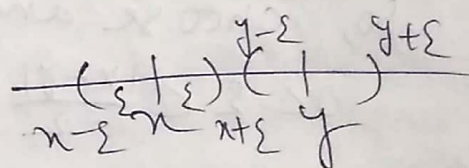
A sequence can't converge to two different limits.

or
Theorem:- Limit of a convergent sequence is unique.

Proof:- To show this you just consider that a sequence converges to two different limits and then show it is contradiction.

So, suppose $\{x_n\}$ is a convergent sequence and it converges to two limits x & y , where $x \neq y$.

Let $x < y$



So, for every $\epsilon > 0$

we can find some n_1 such that for $n > n_1$, $|x_n - x| < \epsilon$

$$\Rightarrow x_n \in (x - \epsilon, x + \epsilon) \quad \forall n > n_1$$

Similarly we can find some n_2 such that for $n > n_2$, $|x_n - y| < \epsilon$

$$\Rightarrow x_n \in (y - \epsilon, y + \epsilon) \quad \forall n > n_2$$

Now we can always find a n s.t.

$n_0 = \max(n_1, n_2)$ and for which

$$x_n \in (x - \epsilon, x + \epsilon) \quad \& \quad x_n \in (y - \epsilon, y + \epsilon) \quad \forall n > n_0$$

That means for $n > n_0$, x_n lies on

both the interval $(x - \epsilon, x + \epsilon)$ & $(y - \epsilon, y + \epsilon)$

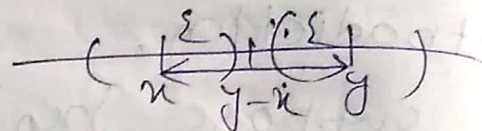
Now for $n \geq$,
 Now if we can show that these two
 intervals $(x-\epsilon, x+\epsilon)$ & $(y-\epsilon, y+\epsilon)$ are
 disjoint then $\forall n \geq n_0$ both
 x_n can't lie on both the intervals.
 Which will contradict our assumption.

Now, can you find an ϵ s.t.

~~$(x-\epsilon, x+\epsilon)$~~ & $(x-\epsilon, x+\epsilon)$ and $(y-\epsilon, y+\epsilon)$ are
 disjoint. Total length of each of the intervals
 is 2ϵ .

$$\text{If } 2\epsilon < \frac{y-x}{2}$$

$$\text{If } \epsilon < \frac{y-x}{2}$$



then they will be
 disjoint.

So, choose any positive ϵ with s.t.

$$0 < \epsilon < \frac{|x-y|}{2} \rightarrow \text{mod is taken because that will work for both } x < y \text{ \& } y < x$$

So, If we choose

$0 < \epsilon < \frac{|x-y|}{2}$ then x_n can't be on both the
 intervals will be disjoint and so
 x_n will not be able to lie on the
 both the intervals $(x-\epsilon, x+\epsilon)$ &
 $(y-\epsilon, y+\epsilon)$ for $n \geq n_0$.

Which is a contradiction.

(As we have considered $\forall \epsilon > 0$

x_n should lie in $(x-\epsilon, x+\epsilon)$ & $(y-\epsilon, y+\epsilon)$
 $\forall \epsilon > 0$)

Hence the proof.

Limit pt. of a sequence :-

A number l is said to be a limit pt. of a sequence $\{x_n\}$ if ~~every neighbourhood~~ for any $\epsilon > 0$, ~~$x_n \in (l-\epsilon, l+\epsilon)$~~

$x_n \in (l-\epsilon, l+\epsilon)$, ~~for infinitely~~
for infinitely many values of n .

Note:- Limit of a sequence is unique.
But limit point of a sequence may
can be more than 1.

Examp

Note:- The sequence $\{(-1)^n\}$ has two limit
1. ~~points~~ ^{pts} ~~the~~ elements of the sequence
are either 1 or -1. So there are infinitely
many 1 & ~~inf~~ -1.

So, $1 \in (1-\epsilon, 1+\epsilon) \forall \epsilon > 0$, for infinitely
many n

Similarly $-1 \in (-1-\epsilon, -1+\epsilon) \forall \epsilon > 0$, $\frac{(-1)^n}{1-\epsilon \quad 1 \quad 1+\epsilon}$

for infinitely
many n

~~But~~ As it has two limit
points it is not a convergent sequence.
We know limit of a convergent sequence
is unique.

2. If a sequence has unique limit point
then that does not imply the sequence
is convergent.

Suppose $\left\{ \frac{n}{2} (1 + (-1)^n) \right\}$
 $= \{0, 1, 0, 2, 0, 3, 0, 4, 0, 5, \dots\}$

The sequence has unique limit point 0. But it is not a convergent sequence as it is not bounded above.

(3) Consider the sequence $\{x_n\}$

$$x_n = \begin{cases} 1 & \text{for } n=1 \\ n - \frac{k(k+1)}{2} & \text{for } \frac{k(k+1)}{2} < n \leq \frac{(k+1)(k+2)}{2} \end{cases}$$

$$\{x_n\} = \{1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \dots\}$$

We see $\{x_n\}$ is a sequence of natural numbers.

The limit points of the sequence are the set $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ of natural numbers.

$$(3) \quad x_n = \begin{cases} \frac{1}{k+2} & \text{for } n=k \\ \frac{n}{k(k+2)} & \text{for } \frac{k(k+1)}{2} < n \leq \frac{(k+1)(k+2)}{2} \end{cases}$$

$$\text{Suppose } \{x_n\} = \left\{ \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{3}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \dots \right\}$$

The limit points of the sequence are the rational numbers in $(0, 1)$.

Find the limit points of the following if exists:

(1) $\{n^2\}$, (2) $\{1, 2, 4, 4, 1, 6, 1, 8, \dots\}$

(3) $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\}$

(4) $\{1, 3, 5, \dots\}$ (5) $\{2 + (-1)^n\}$

(6) $\{1, 2, 3, 1, 2, 3, 1, 2, 3, \dots\}$ (7) $\{2 - \frac{1}{2^{n-1}}\}$

(8) $\{x_n\}$, $x_n = \begin{cases} 2 & \text{when } n \text{ is even} \\ \frac{1}{n} & \text{when } n \text{ is odd} \end{cases}$

(9) $\{x_n\}$, $x_n = \begin{cases} 2 & \text{when } n \text{ is even} \\ \frac{1}{n} & \text{when } n \text{ is odd} \end{cases}$ (10) $\{m + \frac{1}{n}\}$ (11) $\{ \frac{1}{m} + \frac{1}{n} \}$ (12) $\{ \frac{1}{m} + \frac{1}{n} \}$

Some Algebraic Properties of Limit of a Sequence:—

Theorem:- Let $\{x_n\}$ and $\{y_n\}$ be two convergent sequences such that $\{x_n\}$ converges to x and $\{y_n\}$ converges to y .

Then (i) $\{x_n + y_n\}$ converges to $x + y$.

(ii) $\{x_n - y_n\}$ converges to $x - y$.

$$(iii) \lim_{n \rightarrow \infty} (x_n y_n) = x \cdot y = \lim_{n \rightarrow \infty} x_n \cdot \lim_{n \rightarrow \infty} y_n$$

(iv) If $y_n \neq 0$ & $y \neq 0$

$$\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \frac{x}{y} = \frac{\lim_{n \rightarrow \infty} x_n}{\lim_{n \rightarrow \infty} y_n}$$

(v) $k \in \mathbb{R}$, $\{k x_n\}$ converges to kx .

(vi) If $x_n \leq y_n$, then $x \leq y$.

Proof: (i) $\lim_{n \rightarrow \infty} x_n = x$ & $\lim_{n \rightarrow \infty} y_n = y$

Therefore for any $\varepsilon > 0$, $\exists N_1, N_2$ in $\mathbb{N}$ such that

$$\forall n \geq N_1, |x_n - x| < \varepsilon/2,$$

$$\forall n \geq N_2, |y_n - y| < \varepsilon/2.$$

Let $N = \max\{N_1, N_2\}$.

Then $\forall n \geq N$, $|x_n - x| < \varepsilon/2$ & $|y_n - y| < \varepsilon/2$

$$\text{Now, } |(x_n + y_n) - (x + y)| \leq |x_n - x| + |y_n - y| < \varepsilon \quad \forall n \geq N$$

$$\Rightarrow \lim_{n \rightarrow \infty} (x_n + y_n) = (x + y) = \lim_{n \rightarrow \infty} x_n + \lim_{n \rightarrow \infty} y_n$$

The proof of (2) is trivial and it is ~~is given as a task~~.

Try yourself.

(iii) We use the identity

$$(x_n y_n - xy) = (x_n - x)(y_n - y) + x(y_n - y) + y(x_n - x)$$

Given any $\varepsilon > 0$, there are integers N_1, N_2 such that

$$\forall n \geq N_1, |x_n - x| < \sqrt{\varepsilon}, \quad \text{--- (i)}$$

$$\forall n \geq N_2, |y_n - y| < \sqrt{\varepsilon} \quad \text{--- (ii)}$$

Let $N = \max\{N_1, N_2\} \Rightarrow N \geq N_1 \text{ \& } N \geq N_2$

Then, $|x_n - x| < \sqrt{\varepsilon} \text{ \& } |y_n - y| < \sqrt{\varepsilon}$

$$\forall n \geq N$$

[As (i) & (ii) holds $\forall n \geq N_1 \text{ \& } n \geq N_2$ respectively, so (i) & (ii) holds for $\forall n \geq N \Rightarrow N \geq N_1 \text{ \& } N \geq N_2$]

Now, so, $\forall n \geq N, |(x_n - x)(y_n - y)| < \varepsilon$ & x_n

$$\Rightarrow \lim_{n \rightarrow \infty} (x_n - x)(y_n - y) = 0$$

From (1)

$$\lim_{n \rightarrow \infty} (x_n y_n - xy) = \lim_{n \rightarrow \infty} (x_n - x)(y_n - y) + \lim_{n \rightarrow \infty} x(y_n - y) + \lim_{n \rightarrow \infty} y(x_n - x)$$

$$\left[\text{As we know } \lim_{n \rightarrow \infty} (x_n + y_n) = \lim_{n \rightarrow \infty} x_n + \lim_{n \rightarrow \infty} y_n \right] \quad \left[\text{As we know } \lim_{n \rightarrow \infty} (x_n y_n) = \lim_{n \rightarrow \infty} x_n \lim_{n \rightarrow \infty} y_n \right]$$

$$\lim_{n \rightarrow \infty} x_n y_n = \lim_{n \rightarrow \infty} x_n \cdot \lim_{n \rightarrow \infty} y_n$$

$$\lim_{n \rightarrow \infty} x_n y_n - xy = 0 + 0 + 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} x_n y_n = xy$$

$$\because \lim_{n \rightarrow \infty} k x_n =$$

$$k \lim_{n \rightarrow \infty} x_n$$

we do this
proof
later

(IV) Since

$$\text{Since } \lim_{n \rightarrow \infty} y_n = y$$

for any $\epsilon > 0 \exists N \in \mathbb{N}$ s.t.

$$|y_n - y| < \epsilon/2 \quad \forall n > N_1 \quad \text{--- (a)}$$

let us take $\epsilon = |y|$ as (a) is true for all $\epsilon > 0$

$$\text{so, } \forall n > N_1, |y_n - y| < \frac{|y|}{2}$$

Hence $\forall n > N_1$

$$\frac{|y|}{2} > |y_n - y| \Rightarrow |y - y_n| > |y| - |y_n|$$

$$\Rightarrow |y_n| > |y| - \frac{1}{2}|y| = \frac{1}{2}|y|, \quad \forall n > N_1 \quad \text{--- (b)}$$

We have for every value of n

$$\left| \frac{x_n}{y_n} - \frac{x}{y} \right| = \left| \frac{y x_n - x y_n}{y_n y} \right|$$

$$= \left| \frac{y(x_n - x) - x(y_n - y)}{y_n y} \right|$$

$$\leq \frac{|y| |x_n - x| + |x| |y_n - y|}{|y_n| |y|}$$

now for $\forall n > N_1$

$$\left| \frac{x_n}{y_n} - \frac{x}{y} \right| \leq \frac{|y| |x_n - x| + |x| |y_n - y|}{|y|} \cdot \frac{2}{|y|}$$

$$\begin{aligned} |y_n| &> \frac{1}{2}|y| \\ \Rightarrow \frac{1}{|y_n|} &< \frac{2}{|y|} \end{aligned}$$

$$= \frac{2}{|y|} |x_n - x| + \frac{2|x|}{|y|^2} |y_n - y| \quad \leftarrow (c) \quad \forall n > N_1$$

Now since $\lim_{n \rightarrow \infty} x_n = x$ & $\lim_{n \rightarrow \infty} y_n = y$

for every $\epsilon > 0$ $\exists N_2, N_3$ in $\mathbb{N}$ such that

$$\begin{aligned} |x_n - x| &< \frac{\epsilon}{4} \quad \forall n > N_2 \\ |y_n - y| &< \frac{|y|^2}{4(|x|+1)} \epsilon \quad \forall n > N_3 \end{aligned} \quad \left\{ \begin{array}{l} \text{we are} \\ \text{taking} \\ \text{this type} \\ \text{of } (\epsilon) \text{ as} \\ \text{we want} \\ \text{to make} \\ (c) \text{ as } \epsilon \\ \text{only} \end{array} \right.$$

$$\text{let } N = \max \{N_1, N_2, N_3\}$$

Then $\forall n > N_0$, (a), (b), (c) holds

$$\text{so, } \left| \frac{x_n}{y_n} \right| < \frac{x}{y}$$

So, $\forall n > N_0$

$$\left| \frac{x_n}{y_n} - \frac{x}{y} \right| < \frac{2}{|y|} \frac{\epsilon}{4} + \frac{2|x|}{|y|^2} \frac{|y|^2 \epsilon}{4(|x|+1)}$$

$$= \frac{\epsilon}{2} + \frac{\epsilon}{2} \frac{|x|}{|x|+1}$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad \text{as } \frac{|x|}{|x|+1} < 1$$

$$\leq \epsilon$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \frac{x}{y}$$

(v) is left for you.

(1) we have to show that
if $x_n \leq y_n \forall n$ then $x \leq y$.
we will show by contradiction.

Let suppose $x \neq y$.

Let $x > y$.

as $\lim_{n \rightarrow \infty} x_n = x$ & $\lim_{n \rightarrow \infty} y_n = y$.

then for every $\epsilon > 0 \exists N_1, N_2 \in \mathbb{N}$
such that

$$x_n \in (x - \epsilon, x + \epsilon) \quad \forall n > N_1$$

$$y_n \in (y - \epsilon, y + \epsilon) \quad \forall n > N_2$$

let $N = \max \{N_1, N_2\}$

$$\text{Then } x_n \in (x - \epsilon, x + \epsilon) \quad \forall n > N$$

$$y_n \in (y - \epsilon, y + \epsilon)$$

$$\Rightarrow x_n > y_n \quad \forall n > N$$

But this is false as we are given

$$x_n \leq y_n \forall n.$$

Which is a contradiction.

Hence the proof.

Note: Even if $x_n < y_n \forall n$ we can't say
that $x < y$. [$x_n < y_n$ does not imply $x < y$]
as an example,

$$\{x_n\} \quad \{y_n\}$$

$$\{1\}$$

$$\{0\}$$

$$\{y_n\}$$

$$x_n < y_n \text{ but } x \neq y$$

$$\{x_n\} = \{1/n^2\}$$

$$\{y_n\} = 1/n$$

$$x_n < y_n$$

$$\text{but } x = 0 = y$$

$$\Rightarrow x \neq y$$

Examples

These are the theorems that can be used very often.

We have seen that the limit of $\{x_n\}$ & $\{c\}$ ~~const~~ converge to 0 & c.

constant
sequence

But using these theorems we can decide convergence or divergence of various sequences and also decide what their limits are.

$$\text{Let } x_n = \frac{3n+5}{7n-1} = \frac{3+5/n}{7-1/n} \rightarrow \frac{3+0}{7-0} \rightarrow 3/7 \text{ as } n \rightarrow \infty$$

For the following ~~exercises~~
using the above theorems at first find the limit and then apply the defⁿ to ~~check~~ verify

$$(1) \lim_{n \rightarrow \infty} \frac{3n+5}{7n-1} = ?$$

$$\begin{aligned} \text{Soln: } &= \lim_{n \rightarrow \infty} \frac{3+5/n}{7-1/n} = \frac{\lim_{n \rightarrow \infty} (3+5/n)}{\lim_{n \rightarrow \infty} (7-1/n)} \left[\because \lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \frac{\lim_{n \rightarrow \infty} x_n}{\lim_{n \rightarrow \infty} y_n} \right] \\ &= \frac{3+0}{7-0} \left[\because \lim_{n \rightarrow \infty} (x_n \pm y_n) = \lim_{n \rightarrow \infty} x_n \pm \lim_{n \rightarrow \infty} y_n \right] \\ &= 3/7 \end{aligned}$$

$$\text{So, to show } \lim_{n \rightarrow \infty} \frac{3n+5}{7n-1} = 3/7$$

we have

for any $\epsilon > 0$, we have to find

a natural number n_0 s.t.

$$\forall n > n_0, \left| \frac{3n+5}{7n-1} - \frac{3}{7} \right| < \epsilon$$

$$\forall n > n_0, \left| \frac{3n+5}{7n-1} - \frac{3}{7} \right| < \epsilon$$

$$\Leftrightarrow \left| \frac{(3n+5)/7 - 3(7n-1)}{7(7n-1)} \right| < \epsilon$$

$$\Leftrightarrow \left| \frac{38}{7(7n-1)} \right| < \epsilon$$

$$\Leftrightarrow \left| \frac{38}{49(n-\frac{1}{7})} \right| < \epsilon \quad \text{as } n > 1, \quad n - \frac{1}{7} > 0$$

$$\Leftrightarrow (n - \frac{1}{7}) > \frac{38}{49\epsilon}$$

$$\Leftrightarrow n > \frac{38}{49\epsilon} + \frac{1}{7}$$

So, take $n_0 = \left\lceil \frac{38}{49\epsilon} + \frac{1}{7} \right\rceil + 1$

Therefore $\forall n \geq n_0, \left| \frac{3n+5}{7n-1} - \frac{3}{7} \right| < \epsilon$.

Find the limits using the theorem and then verify it by using the ϵ - N definition.

(2) $\lim_{n \rightarrow \infty}$

(2) $\lim_{n \rightarrow \infty} \frac{2n+1}{n+1}$

(3) $\lim_{n \rightarrow \infty} \frac{n+1}{2n}$

(4) $\lim_{n \rightarrow \infty} \frac{3n-1}{n+2}$

(5) $\lim_{n \rightarrow \infty} \frac{2n+5}{6n-1}$

(6) $\lim_{n \rightarrow \infty} \frac{3n}{n+5\sqrt{n}}$

(7) $\lim_{n \rightarrow \infty} \frac{n^2+n+1}{3n^2+1}$

(8) $\lim_{n \rightarrow \infty} \frac{n^2-1}{n^2+n+1}$

(9) $\lim_{n \rightarrow \infty} \frac{2n^2+1}{n^2+3n}$

(10) $\lim_{n \rightarrow \infty} \frac{n^2+n+1}{n^2-5n-3}$

(11) $\lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n})$

$$(12) \lim_{n \rightarrow \infty} \frac{n^2 + 3n}{2n^2 + n - 1}$$

$$(13) \lim_{n \rightarrow \infty} \frac{5n^2 + 2}{6n^2 + 4n + 7}$$

$$(13) \lim_{n \rightarrow \infty}$$

$$\begin{aligned} (8) \lim_{n \rightarrow \infty} \frac{n^2 - 1}{n^2 + n + 1} \\ = \lim_{n \rightarrow \infty} \frac{1 - 1/n^2}{1 + 1/n + 1/n^2} \\ = \frac{1 - 0}{1 + 0 + 0} = 1 \end{aligned}$$

Step now,

$$\left| \frac{n^2 - 1}{n^2 + n + 1} - 1 \right| < \epsilon$$

$$\Leftrightarrow \left| \frac{n^2 - 1 - n^2 - n - 1}{n^2 + n + 1} \right| < \epsilon$$

$$\Rightarrow \left| \frac{-(n+2)}{n^2 + n + 1} \right| < \epsilon$$

$$\Rightarrow \left\{ \frac{n+2}{n^2 + n + 1} \right\} < \epsilon$$

Now, for $n \geq 2$ $n+2 < n+2n \leq 2n$
 $n^2 + n + 1 > n^2 \quad \forall n$

Le So, $\frac{n+2}{n^2 + n + 1} < \frac{2n}{n^2} = 2/n < \epsilon$
 $\Rightarrow n > 2/\epsilon$

So, choose $n_0 = \max \{2, [2/\epsilon]\}$

So, $\forall$ Therefore $\forall n \geq n_0$
 $\left| \frac{n^2 - 1}{n^2 + n + 1} - 1 \right| < \epsilon$

$$(9) \lim_{n \rightarrow \infty} \frac{2n^2+1}{n^2+3n}$$

$$= \lim_{n \rightarrow \infty} \frac{2 + 1/n}{1 + 3/n} = 2$$

$$\left| \frac{2n^2+1}{n^2+3n} - 2 \right| = \left| \frac{1-6n}{n^2+3n} \right| = \left| \frac{6(\frac{1}{6}-n)}{n^2+3n} \right| \quad (*)$$

$$| \frac{1}{6} - n | \leq | \frac{1}{6} | + | n | = n + \frac{1}{6}$$

$$\forall n \geq 1$$

$$| \frac{1}{6} - n | \leq \frac{1}{6} + n \leq n + n = 2n$$

$$n^2+3n > n^2$$

$$\Rightarrow \frac{1}{n^2+3n} < \frac{1}{n^2}$$

$$\text{So, } \left| \frac{2n^2+1}{n^2+3n} - 2 \right| \leq \frac{2n}{n^2} = \frac{2}{n} < \epsilon$$

$$\Rightarrow n > \frac{2}{\epsilon}$$

$$\text{Let us take } n_0 = \left\lceil \frac{2}{\epsilon} \right\rceil + 1$$

$\exists n_0 \text{ s.t.}$

$$\text{So, } \forall n > n_0, \left| \frac{2n^2+1}{n^2+3n} - 2 \right| < \epsilon$$

$$(10) \lim_{n \rightarrow \infty} \frac{n^2+n+1}{n^2-5n-3}$$

$$= \lim_{n \rightarrow \infty} \frac{1 + 1/n + 1/n^2}{1 - 5/n - 3/n^2} = 1$$

$$\left| \frac{n^2+n+1}{n^2-5n-3} - 1 \right| = \left| \frac{6n+4}{n^2-5n-3} \right| = \left| \frac{6(n+\frac{2}{3})}{n^2-5n-3} \right|$$

$$\forall n \geq 1, \quad n + \frac{2}{3} = n + \frac{2}{3} < n + n = 2n \quad (1)$$

$$\begin{aligned} |n^2-5n-3| &\geq n^2 - (5n+3) & \because |a-b| \geq |a|-|b| \\ &\geq n^2 - \frac{1}{2}n^2 & \text{if } 5n+3 < \frac{n^2}{2} \\ &= \frac{1}{2}n^2 & \text{if } 5n+3 < \frac{n^2}{2} \end{aligned}$$

$$1+3 < \frac{n^2}{2}$$

$$n^2 - 6n - 6 > 0$$

$$\Rightarrow n^2 - 2 \cdot 3 \cdot n + 9 - 6 - 9 > 0$$

$$\Rightarrow (n-3)^2 - 15 > 0$$

$$\Rightarrow (n-3)^2 > 15$$

$$2n^2 > 5n+3 < \frac{n^2}{2}$$

$$\Rightarrow 10n+6 < n^2$$

$$\Rightarrow n^2 - 10n - 6 > 0$$

$$\Rightarrow (n^2 - 2 \cdot 5 \cdot n + 25) - 6 - 25 > 0$$

$$\Rightarrow (n-5)^2 - 31 > 0$$

$$\Rightarrow (n-5) > \sqrt{31}$$

$$\Rightarrow n > 5 + \sqrt{31}$$

$$\Rightarrow n > 5 + \sqrt{31} \quad [\because 5 - \sqrt{31} < 0 \text{ \& \textit{owe } } n \geq 1]$$

$$\text{So, we take } n_0 = [5 + \sqrt{31}] + 1$$

$$\text{So, for } n > n_0$$

$$|n^2 - 5n - 3| \geq \frac{n^2}{2}$$

$$\Rightarrow \frac{1}{|n^2 - 5n - 3|} \leq \frac{2}{n^2} \quad (3)$$

So from (1) applying (2) & (3) we get

$$\left| \frac{n^2 + n + 1}{n^2 - 5n - 3} - 1 \right| \leq \frac{2n \times 2}{n^2} = \frac{4}{n} < \frac{4}{n} \quad \text{wherever } n > n_0$$

$$\Rightarrow n > \frac{4}{\epsilon}$$

So, we take

$$N = \max \{ [5 + \sqrt{31}] + 1, [4/\epsilon] + 1 \}$$

so, Therefore $\forall n > N$

$$\left| \frac{n^2 + n + 1}{n^2 - 5n - 3} - 1 \right| < \epsilon$$

$$(11) \lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n})$$

$$= \lim_{n \rightarrow \infty} \left(\frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} \right)$$

$$= \lim_{n \rightarrow \infty} \left[\frac{n+1 - n}{\sqrt{n+1} + \sqrt{n}} \right] = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}(1 + \sqrt{1 + \frac{1}{n}})}$$

$$\text{now, } |\sqrt{n+1} - \sqrt{n} - 0| = 0$$

$$= \left| \frac{1}{\sqrt{n} + \sqrt{n+1}} \right| < \frac{1}{2\sqrt{n}} \left[\because \sqrt{n+1} > \sqrt{n} \right]$$

$$< \epsilon$$

$$\Rightarrow n > \frac{1}{4\epsilon^2}$$

$$\text{So, we take } n_0 = \left[\frac{1}{4\epsilon^2} \right] + 1$$

$$(14) \lim_{n \rightarrow \infty} \frac{6n^3 + 2n + 1}{n^3 + n^2}$$

$$(15) \lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n}) \sqrt{n + \frac{1}{2}}$$

$$(16) \lim_{n \rightarrow \infty} (\sqrt[3]{n+1} - \sqrt[3]{n})$$

$$(17) \lim_{n \rightarrow \infty} (\sqrt{n^2 + n} - n)$$

$$(18) \lim_{n \rightarrow \infty} \frac{3 + 2\sqrt{n+1}}{\sqrt{n+1}}$$

Anjana Mondal
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Theorem:- If a sequence converges, then its limit is unique.

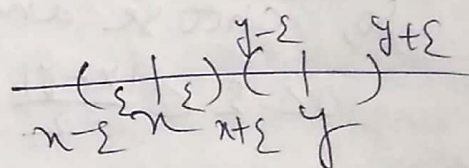
A sequence can't converge to two different limits.

or
Theorem:- Limit of a convergent sequence is unique.

Proof:- To show this you just consider that a the sequence converges to two different limits and then show it is contradiction.

So, suppose $\{x_n\}$ is a convergent sequence and it converges to two limits x & y , where $x \neq y$.

Let $x < y$



So, for every $\epsilon > 0$

we can find some n_1 such that for $n > n_1$, $|x_n - x| < \epsilon$

$$\Rightarrow x_n \in (x - \epsilon, x + \epsilon) \quad \forall n > n_1$$

Similarly we can find some n_2 such that for $n > n_2$, $|x_n - y| < \epsilon$

$$\Rightarrow x_n \in (y - \epsilon, y + \epsilon) \quad \forall n > n_2$$

Now we can always find a n s.t.

$n_0 = \max(n_1, n_2)$ and for which

$$x_n \in (x - \epsilon, x + \epsilon) \quad \& \quad x_n \in (y - \epsilon, y + \epsilon) \quad \forall n > n_0$$

That means for $n > n_0$, x_n lies on

both the interval $(x - \epsilon, x + \epsilon)$ & $(y - \epsilon, y + \epsilon)$

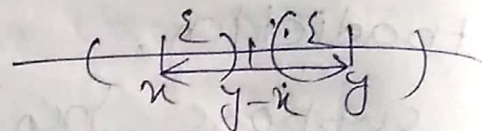
Now for $n \geq$,
 Now if we can show that these two
 intervals $(x-\epsilon, x+\epsilon)$ & $(y-\epsilon, y+\epsilon)$ are
 disjoint then $\forall n \geq n_0$ both
 x_n can't lie on both the intervals.
 Which will contradict our assumption.

Now, can you find an ϵ s.t.

~~$(x-\epsilon, x+\epsilon)$~~ & $(x-\epsilon, x+\epsilon)$ and $(y-\epsilon, y+\epsilon)$ are
 disjoint. Total length of each of the intervals
 is 2ϵ .

$$\text{If } 2\epsilon < \frac{y-x}{2}$$

$$\text{If } \epsilon < \frac{y-x}{2}$$



then they will be
 disjoint.

So, choose any positive ϵ with s.t.

$$0 < \epsilon < \frac{|x-y|}{2} \rightarrow \text{mod is taken because that will work for both } x < y \text{ \& } y < x$$

So, If we choose

$0 < \epsilon < \frac{|x-y|}{2}$ then x_n can't be on both the
 intervals will be disjoint and so
 x_n will not be able to lie on the
 both the intervals $(x-\epsilon, x+\epsilon)$ &
 $(y-\epsilon, y+\epsilon)$ for $n \geq n_0$.

Which is a contradiction.

(As we have considered $\forall \epsilon > 0$

x_n should lie in $(x-\epsilon, x+\epsilon)$ & $(y-\epsilon, y+\epsilon)$
 $\forall \epsilon > 0$)

Hence the proof.

Limit pt. of a sequence :-

A number l is said to be a limit pt. of a sequence $\{x_n\}$ if ~~every neighbourhood~~ for any $\epsilon > 0$, ~~$x_n \in (l-\epsilon, l+\epsilon)$~~

$x_n \in (l-\epsilon, l+\epsilon)$, ~~for infinitely~~
for infinitely many values of n .

Note:- Limit of a sequence is unique.
But limit point of a sequence may
can be more than 1.

Examp

Note:- The sequence $\{(-1)^n\}$ has two limit
1. ~~points~~ ^{pts} $\frac{1}{2}$ As the elements of the sequence
are either 1 or -1. So there are infinitely
many 1 & ~~not~~ -1.

So, $1 \in (1-\epsilon, 1+\epsilon) \forall \epsilon > 0$, for infinitely
many n

Similarly $-1 \in (-1-\epsilon, -1+\epsilon) \forall \epsilon > 0$, $\frac{(-1)}{1-\epsilon} \quad 1 \quad 1+\epsilon$

for infinitely
many n

~~But~~ As it has two limit
points it is not a convergent sequence.
We know limit of a convergent sequence
is unique.

2. If a sequence has unique limit point
then that does not imply the sequence
is convergent.

Suppose $\left\{ \frac{n}{2} (1 + (-1)^n) \right\}$
 $= \{0, 1, 0, 2, 0, 3, 0, 4, 0, 5, \dots\}$

The sequence has unique limit point 0. But it is not a convergent sequence as it is not bounded above.

(3) Consider the sequence $\{x_n\}$

$$x_n = \begin{cases} 1 & \text{for } n=1 \\ n - \frac{k(k+1)}{2} & \text{for } \frac{k(k+1)}{2} < n \leq \frac{(k+1)(k+2)}{2} \end{cases}$$

$$\{x_n\} = \{1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \dots\}$$

We see $\{x_n\}$ is a sequence of natural numbers.

The limit points of the sequence are the set $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ of natural numbers.

$$(3) \quad x_n = \begin{cases} \frac{1}{k+2} & \text{for } n=k \\ \frac{n}{k(k+2)} & \text{for } \frac{k(k+1)}{2} < n \leq \frac{(k+1)(k+2)}{2} \end{cases}$$

$$\text{Suppose } \{x_n\} = \left\{ \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{3}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \dots \right\}$$

The limit points of the sequence are the rational numbers in $(0, 1)$.

Find the limit points of the following if exists:

(1) $\{n^2\}$, (2) $\{1, 2, 4, 4, 1, 6, 1, 8, \dots\}$

(3) $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\}$

(4) $\{1, 3, 5, \dots\}$ (5) $\{2 + (-1)^n\}$

(6) $\{1, 2, 3, 1, 2, 3, 1, 2, 3, \dots\}$ (7) $\{2 - \frac{1}{2^{n-1}}\}$

(8) $\{x_n\}$, $x_n = \begin{cases} 2 & \text{when } n \text{ is even} \\ \frac{1}{n} & \text{when } n \text{ is odd} \end{cases}$

(9) $\{x_n\}$, $x_n = \begin{cases} \frac{1}{n} & \text{when } n \text{ is odd} \\ \frac{1}{n} & \text{when } n \text{ is even} \end{cases}$ (10) $\{m + \frac{1}{n}\}$ (11) $\{m + \frac{1}{n}\}$ (12) $\{m + \frac{1}{n}\}$

Some Algebraic Properties of Limit of a Sequence:—

Theorem:- Let $\{x_n\}$ and $\{y_n\}$ be two convergent sequences such that $\{x_n\}$ converges to x and $\{y_n\}$ converges to y .

Then (i) $\{x_n + y_n\}$ converges to $x + y$.

(ii) $\{x_n - y_n\}$ converges to $x - y$.

$$(iii) \lim_{n \rightarrow \infty} (x_n y_n) = x \cdot y = \lim_{n \rightarrow \infty} x_n \cdot \lim_{n \rightarrow \infty} y_n$$

(iv) If $y_n \neq 0$ & $y \neq 0$

$$\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \frac{x}{y} = \frac{\lim_{n \rightarrow \infty} x_n}{\lim_{n \rightarrow \infty} y_n}$$

(v) $k \in \mathbb{R}$, $\{k x_n\}$ converges to kx .

(vi) If $x_n \leq y_n$, then $x \leq y$.

Proof: (i) $\lim_{n \rightarrow \infty} x_n = x$ & $\lim_{n \rightarrow \infty} y_n = y$

Therefore for any $\varepsilon > 0$, $\exists N_1, N_2$ in $\mathbb{N}$ such that

$$\forall n \geq N_1, |x_n - x| < \varepsilon/2,$$

$$\forall n \geq N_2, |y_n - y| < \varepsilon/2.$$

Let $N = \max\{N_1, N_2\}$.

Then $\forall n \geq N$, $|x_n - x| < \varepsilon/2$ & $|y_n - y| < \varepsilon/2$

$$\text{Now, } |(x_n + y_n) - (x + y)| \leq |x_n - x| + |y_n - y| < \varepsilon \quad \forall n \geq N$$

$$\Rightarrow \lim_{n \rightarrow \infty} (x_n + y_n) = (x + y) = \lim_{n \rightarrow \infty} x_n + \lim_{n \rightarrow \infty} y_n$$

The proof of (2) is trivial and it is ~~is given as a task~~.

Try yourself.

(iii) We use the identity

$$(x_n y_n - xy) = (x_n - x)(y_n - y) + x(y_n - y) + y(x_n - x)$$

Given any $\varepsilon > 0$, there are integers N_1, N_2 such that

$$\forall n \geq N_1, |x_n - x| < \sqrt{\varepsilon}, \quad \text{--- (I)}$$

$$\forall n \geq N_2, |y_n - y| < \sqrt{\varepsilon} \quad \text{--- (II)}$$

Let $N = \max\{N_1, N_2\} \Rightarrow N \geq N_1 \text{ \& } N \geq N_2$

Then, $|x_n - x| < \sqrt{\varepsilon} \text{ \& } |y_n - y| < \sqrt{\varepsilon}$

$$\forall n \geq N$$

[As (I) & (II) holds $\forall n \geq N_1 \text{ \& } n \geq N_2$ respectively, so (I) & (II) holds for $\forall n \geq N \Rightarrow N \geq N_1 \text{ \& } N \geq N_2$]

Now, so, $\forall n \geq N, |(x_n - x)(y_n - y)| < \varepsilon$ & x_n

$$\Rightarrow \lim_{n \rightarrow \infty} (x_n - x)(y_n - y) = 0$$

From (1)

$$\lim_{n \rightarrow \infty} (x_n y_n - xy) = \lim_{n \rightarrow \infty} (x_n - x)(y_n - y) + \lim_{n \rightarrow \infty} x(y_n - y) + \lim_{n \rightarrow \infty} y(x_n - x)$$

[As we know $\lim_{n \rightarrow \infty} (x_n + y_n) = \lim_{n \rightarrow \infty} x_n + \lim_{n \rightarrow \infty} y_n$] [as we know $\lim_{n \rightarrow \infty} (x_n y_n) = \lim_{n \rightarrow \infty} x_n \cdot \lim_{n \rightarrow \infty} y_n$]

$$\lim_{n \rightarrow \infty} x_n y_n = \lim_{n \rightarrow \infty} x_n \cdot \lim_{n \rightarrow \infty} y_n$$

$$\lim_{n \rightarrow \infty} x_n y_n - xy = 0 + 0 + 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} x_n y_n = xy$$

$$\because \lim_{n \rightarrow \infty} k x_n =$$

$$k \lim_{n \rightarrow \infty} x_n$$

we do this
proof
later

(IV) Since

$$\text{Since } \lim_{n \rightarrow \infty} y_n = y$$

for any $\epsilon > 0 \exists N \in \mathbb{N}$ s.t.

$$|y_n - y| < \epsilon/2 \quad \forall n > N_1 \quad \text{--- (a)}$$

let us take $\epsilon = |y|$ as (a) is true for all $\epsilon > 0$

$$\text{so, } \forall n > N_1, |y_n - y| < \frac{|y|}{2}$$

Hence $\forall n > N_1$

$$\frac{|y|}{2} > |y_n - y| \Rightarrow |y - y_n| > |y| - |y_n|$$

$$\Rightarrow |y_n| > |y| - \frac{1}{2}|y| = \frac{1}{2}|y|, \quad \forall n > N_1 \quad \text{--- (b)}$$

We have for every value of n

$$\left| \frac{x_n}{y_n} - \frac{x}{y} \right| = \left| \frac{y x_n - x y_n}{y_n y} \right|$$

$$= \left| \frac{y(x_n - x) - x(y_n - y)}{y_n y} \right|$$

$$\leq \frac{|y| |x_n - x| + |x| |y_n - y|}{|y_n| |y|}$$

now for $\forall n > N_1$

$$\left| \frac{x_n}{y_n} - \frac{x}{y} \right| \leq \frac{|y| |x_n - x| + |x| |y_n - y|}{|y|} \cdot \frac{2}{|y|}$$

$$\begin{aligned} |y_n| &> \frac{1}{2}|y| \\ \Rightarrow \frac{1}{|y_n|} &< \frac{2}{|y|} \end{aligned}$$

$$= \frac{2}{|y|} |x_n - x| + \frac{2|x|}{|y|^2} |y_n - y| \quad \leftarrow (c) \quad \forall n > N_1$$

Now since $\lim_{n \rightarrow \infty} x_n = x$ & $\lim_{n \rightarrow \infty} y_n = y$

for every $\varepsilon > 0$ $\exists N_2, N_3$ in $\mathbb{N}$ such that

$$\begin{aligned} |x_n - x| &< \frac{\varepsilon |y|}{4} \quad \leftarrow (d) \quad \forall n > N_2 \\ |y_n - y| &< \frac{|y|^2}{4(|x|+1)} \varepsilon \quad \leftarrow (e) \quad \forall n > N_3 \end{aligned} \quad \left\{ \begin{array}{l} \text{we are} \\ \text{taking} \\ \text{this type} \\ \text{of } (d) \text{ \& } (e) \\ \text{as we want} \\ \text{to make} \\ (c) \text{ as } \varepsilon \\ \text{only} \end{array} \right.$$

$$\text{let } N = \max \{N_1, N_2, N_3\}$$

Then $\forall n > N_0$, (c), (d), (e) holds

$$\text{so, } \left| \frac{x_n}{y_n} \right| < \frac{x}{y}$$

So, $\forall n > N_0$

$$\left| \frac{x_n}{y_n} - \frac{x}{y} \right| < \frac{2}{|y|} \frac{|y|}{4} \varepsilon + \frac{2|x|}{|y|^2} \frac{|y|^2}{4(|x|+1)} \varepsilon$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \frac{|x|}{|x|+1}$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \text{as } \frac{|x|}{|x|+1} < 1$$

$$\leq \varepsilon$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \frac{x}{y}$$

(v) is left for you.

(1) we have to show that
if $x_n \leq y_n \forall n$ then $x \leq y$.
we will show by contradiction.

↳ suppose $x \neq y$.

Let $x > y$.

as $\lim_{n \rightarrow \infty} x_n = x$ & $\lim_{n \rightarrow \infty} y_n = y$.

then for every $\epsilon > 0 \exists N_1, N_2 \in \mathbb{N}$
such that

$$x_n \in (x - \epsilon, x + \epsilon) \quad \forall n > N_1$$

$$y_n \in (y - \epsilon, y + \epsilon) \quad \forall n > N_2$$

let $N = \max \{N_1, N_2\}$

$$\text{Then } x_n \in (x - \epsilon, x + \epsilon) \quad \forall n > N$$

$$y_n \in (y - \epsilon, y + \epsilon)$$

$$\Rightarrow x_n > y_n \quad \forall n > N$$

But this is false as we are given

$$x_n \leq y_n \forall n.$$

Which is a contradiction.

Hence the proof.

Note:- Even if $x_n < y_n \forall n$ we can't say
that $x < y$. [$x_n < y_n$ does not imply $x < y$]

as an example,

$$\{x_n\} \quad \{y_n\}$$

"

$$\{0\}$$

$$\{y_n\}$$

$$x_n < y_n \text{ but}$$

$$x \neq y$$

$$\{x_n\} = \{1/n^2\}$$

$$\{y_n\} = 1/n$$

$$x_n < y_n$$

$$\text{but } x = 0 = y$$

$$\Rightarrow x \neq y.$$

Examples

These are the theorems that can be used very often.

We have seen that the limit of $\{x_n\}$ & $\{c\}$ ~~const~~ converge to 0 & c.

constant
sequence

But using these theorems we can decide convergence or divergence of various sequences and also decide what their limits are.

$$\text{Let } x_n = \frac{3n+5}{7n-1} = \frac{3+5/n}{7-1/n} \rightarrow \frac{3+0}{7-0} \rightarrow 3/7 \text{ as } n \rightarrow \infty$$

For the following ~~exercises~~
using the above theorems at first find the limit and then apply the defⁿ to ~~check~~ verify

$$\begin{aligned} (1) \quad \lim_{n \rightarrow \infty} \frac{3n+5}{7n-1} &= ? \\ \text{Soln:} \quad &= \lim_{n \rightarrow \infty} \frac{3+5/n}{7-1/n} = \frac{\lim_{n \rightarrow \infty} (3+5/n)}{\lim_{n \rightarrow \infty} (7-1/n)} \left[\because \lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \frac{\lim_{n \rightarrow \infty} x_n}{\lim_{n \rightarrow \infty} y_n} \right] \\ &= \frac{3+0}{7-0} \left[\because \lim_{n \rightarrow \infty} (x_n \pm y_n) = \lim_{n \rightarrow \infty} x_n \pm \lim_{n \rightarrow \infty} y_n \right] \\ &= 3/7 \end{aligned}$$

So, to show $\lim_{n \rightarrow \infty} \frac{3n+5}{7n-1} = 3/7$

we have

for any $\epsilon > 0$, we have to find

a natural number n_0 s.t.

$$\forall n > n_0, \left| \frac{3n+5}{7n-1} - \frac{3}{7} \right| < \epsilon$$

$$\forall n > n_0, \left| \frac{3n+5}{7n-1} - \frac{3}{7} \right| < \epsilon$$

$$\Leftrightarrow \left| \frac{(3n+5)/7 - 3(7n-1)}{7(7n-1)} \right| < \epsilon$$

$$\Leftrightarrow \left| \frac{38}{7(7n-1)} \right| < \epsilon$$

$$\Leftrightarrow \left| \frac{38}{49(n-\frac{1}{7})} \right| < \epsilon \quad \text{as } n > 1, \quad n - \frac{1}{7} > 0$$

$$\Leftrightarrow (n - \frac{1}{7}) > \frac{38}{49\epsilon}$$

$$\Leftrightarrow n > \frac{38}{49\epsilon} + \frac{1}{7}$$

So, take $n_0 = \left\lceil \frac{38}{49\epsilon} + \frac{1}{7} \right\rceil + 1$

Therefore $\forall n \geq n_0, \left| \frac{3n+5}{7n-1} - \frac{3}{7} \right| < \epsilon$.

Find the limits using the theorem and then verify it by using the ϵ - N definition.

(2) $\lim_{n \rightarrow \infty}$

(2) $\lim_{n \rightarrow \infty} \frac{2n+1}{n+1}$

(3) $\lim_{n \rightarrow \infty} \frac{n+1}{2n}$

(4) $\lim_{n \rightarrow \infty} \frac{3n-1}{n+2}$

(5) $\lim_{n \rightarrow \infty} \frac{2n+5}{6n-11}$

(6) $\lim_{n \rightarrow \infty} \frac{3n}{n+5\sqrt{n}}$

(7) $\lim_{n \rightarrow \infty} \frac{n^2+n+1}{3n^2+1}$

(8) $\lim_{n \rightarrow \infty} \frac{n^2-1}{n^2+n+1}$

(9) $\lim_{n \rightarrow \infty} \frac{2n^2+1}{n^2+3n}$

(10) $\lim_{n \rightarrow \infty} \frac{n^2+n+1}{n^2-5n-3}$

(11) $\lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n})$

$$(12) \lim_{n \rightarrow \infty} \frac{n^2 + 3n}{2n^2 + n - 1}$$

$$(13) \lim_{n \rightarrow \infty} \frac{5n^2 + 2}{6n^2 + 4n + 7}$$

$$(13) \lim_{n \rightarrow \infty}$$

$$\begin{aligned} (8) \lim_{n \rightarrow \infty} \frac{n^2 - 1}{n^2 + n + 1} \\ = \lim_{n \rightarrow \infty} \frac{1 - 1/n^2}{1 + 1/n + 1/n^2} \\ = \frac{1 - 0}{1 + 0 + 0} = 1 \end{aligned}$$

Step, Now,

$$\left| \frac{n^2 - 1}{n^2 + n + 1} - 1 \right| < \epsilon$$

$$\Leftrightarrow \left| \frac{n^2 - 1 - n^2 - n - 1}{n^2 + n + 1} \right| < \epsilon$$

$$\Rightarrow \left| \frac{-(n+2)}{n^2 + n + 1} \right| < \epsilon$$

$$\Rightarrow \left\{ \frac{n+2}{n^2 + n + 1} \right\} < \epsilon$$

Now, for $n \geq 2$ $n+2 < n+2n \leq 2n$
 $n^2 + n + 1 > n^2 \quad \forall n$

Le So, $\frac{n+2}{n^2 + n + 1} < \frac{2n}{n^2} = 2/n < \epsilon$
 $\Rightarrow n > 2/\epsilon$

So, choose $n_0 = \max \{2, [2/\epsilon]\}$

So, $\forall$ Therefore $\forall n \geq n_0$
 $\left| \frac{n^2 - 1}{n^2 + n + 1} - 1 \right| < \epsilon$

$$(9) \lim_{n \rightarrow \infty} \frac{2n^2+1}{n^2+3n}$$

$$= \lim_{n \rightarrow \infty} \frac{2 + 1/n}{1 + 3/n} = 2$$

$$\left| \frac{2n^2+1}{n^2+3n} - 2 \right| = \left| \frac{1-6n}{n^2+3n} \right| = \left| \frac{6(\frac{1}{6}-n)}{n^2+3n} \right| \quad (*)$$

$$| \frac{1}{6} - n | \leq | \frac{1}{6} | + | n | = n + \frac{1}{6}$$

$$\forall n \geq 1$$

$$| \frac{1}{6} - n | \leq \frac{1}{6} + n \leq n + n = 2n$$

$$n^2+3n > n^2$$

$$\Rightarrow \frac{1}{n^2+3n} < \frac{1}{n^2}$$

$$\text{So, } \left| \frac{2n^2+1}{n^2+3n} - 2 \right| \leq \frac{2n}{n^2} = \frac{2}{n} < \epsilon$$

$$\Rightarrow n > \frac{2}{\epsilon}$$

$$\text{Let us take } n_0 = \left\lceil \frac{2}{\epsilon} \right\rceil + 1$$

$\exists n_0 \text{ s.t.}$

$$\text{So, } \forall n > n_0, \left| \frac{2n^2+1}{n^2+3n} - 2 \right| < \epsilon$$

$$(10) \lim_{n \rightarrow \infty} \frac{n^2+n+1}{n^2-5n-3}$$

$$= \lim_{n \rightarrow \infty} \frac{1 + 1/n + 1/n^2}{1 - 5/n - 3/n^2} = 1$$

$$\left| \frac{n^2+n+1}{n^2-5n-3} - 1 \right| = \left| \frac{6n+4}{n^2-5n-3} \right| = \left| \frac{6(n+\frac{2}{3})}{n^2-5n-3} \right|$$

$$\forall n \geq 1, \quad n + \frac{2}{3} = n + \frac{2}{3} < n + n = 2n \quad (1)$$

$$\begin{aligned} |n^2-5n-3| &\geq n^2 - (5n+3) & \because |a-b| \geq |a|-|b| \\ &\geq n^2 - \frac{1}{2}n^2 & \text{if } 5n+3 < \frac{n^2}{2} \\ &= \frac{1}{2}n^2 & \text{if } 5n+3 < \frac{n^2}{2} \end{aligned}$$

$$1+3 < \frac{n^2}{2}$$

$$n^2 - 6n - 6 > 0$$

$$\Rightarrow n^2 - 2 \cdot 3 \cdot n + 9 - 6 - 9 > 0$$

$$\Rightarrow (n-3)^2 - 15 > 0$$

$$\Rightarrow (n-3)^2 > 15$$

$$2n^2 > 5n+3 < \frac{n^2}{2}$$

$$\Rightarrow 10n+6 < n^2$$

$$\Rightarrow n^2 - 10n - 6 > 0$$

$$\Rightarrow (n^2 - 2 \cdot 5 \cdot n + 25) - 6 - 25 > 0$$

$$\Rightarrow (n-5)^2 - 31 > 0$$

$$\Rightarrow (n-5) > \sqrt{31}$$

$$\Rightarrow n > 5 + \sqrt{31}$$

$$\Rightarrow n > 5 + \sqrt{31} \quad [\because 5 - \sqrt{31} < 0 \text{ \& \textit{one } n \geq 1}]$$

$$\text{So, we take } n_0 = [5 + \sqrt{31}] + 1$$

$$\text{So, for } n > n_0$$

$$|n^2 - 5n - 3| \geq \frac{n^2}{2}$$

$$\Rightarrow \frac{1}{|n^2 - 5n - 3|} \leq \frac{2}{n^2} \quad (3)$$

So from (1) applying (2) & (3) we get

$$\left| \frac{n^2 + n + 1}{n^2 - 5n - 3} - 1 \right| \leq \frac{2n \times 2}{n^2} = \frac{4}{n} < \frac{4}{n} \quad \text{wherever } n > n_0$$

$$\Rightarrow n > \frac{4}{\epsilon}$$

So, we take

$$N = \max \{ [5 + \sqrt{31}] + 1, [4/\epsilon] + 1 \}$$

so, Therefore $\forall n > N$

$$\left| \frac{n^2 + n + 1}{n^2 - 5n - 3} - 1 \right| < \epsilon$$

$$(11) \lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n})$$

$$= \lim_{n \rightarrow \infty} \left(\frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} \right)$$

$$= \lim_{n \rightarrow \infty} \left[\frac{n+1 - n}{\sqrt{n+1} + \sqrt{n}} \right] = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}(1 + \sqrt{1 + \frac{1}{n}})}$$

$$\text{now, } |\sqrt{n+1} - \sqrt{n} - 0| = 0$$

$$= \left| \frac{1}{\sqrt{n} + \sqrt{n+1}} \right| < \frac{1}{2\sqrt{n}} \left[\because \sqrt{n+1} > \sqrt{n} \right]$$

$$< \epsilon$$

$$\Rightarrow n > \frac{1}{4\epsilon^2}$$

$$\text{So, we take } n_0 = \left[\frac{1}{4\epsilon^2} \right] + 1$$

$$(14) \lim_{n \rightarrow \infty} \frac{6n^3 + 2n + 1}{n^3 + n^2}$$

$$(15) \lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n}) \sqrt{n + \frac{1}{2}}$$

$$(16) \lim_{n \rightarrow \infty} (\sqrt[3]{n+1} - \sqrt[3]{n})$$

$$(17) \lim_{n \rightarrow \infty} (\sqrt{n^2 + n} - n)$$

$$(18) \lim_{n \rightarrow \infty} \frac{3 + 2\sqrt{n+1}}{\sqrt{n+1}}$$

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