Sequence and Series of Functions

You are already familiar with sequence and series of numbers. So, all that we do here is that replace the numbers by functions.

Sequence is a function whose domain is the set of all natural numbers and codomain can be anything. When the codomain is a real number we call it sequequence of real numbers.

If it is a complex it will be sequence of complex numbers. So, similarly you can talk of ~~complex nu~~ sequence of anything, sequence of matrices, sequence of vectors and also sequence of functions.

Each functions defined on different sets, but with that kind of thing we will not be able to do anything further.

So, we shall just fix one set.

Suppose E is a non-empty set, $E \neq \phi$ and we shall assume that all functions are defined on this set E.

In some cases E will be subset of some metric space.

As we are studying in real line, in most of the application it will be some interval in the real line, closed and bounded interval,

$I = [a,b]$, or $[a,b)$, $(a,b]$, (a,b).

So, for each $n \in \mathbb{N}$ we have real valued function $f_n : E \to \mathbb{R}$, $\forall n \in \mathbb{N}$.
Then $\{f_n\}$ is a sequence of functions.

Once you have a sequence of fun^{n}s $\{f_n\}$ then if you take any $x \in E$, then $f_n(x)$ is a real number.
Then $\{f_n(x)\}$ is a sequence of real numbers.
Now suppose for each $x \in E$. $\{f_n(x)\}$ converges and suppose it converges to some real number. Then that will define a new function from E to $\mathbb{R}$, because for each x you have a new number which is limit of $\{f_n(x)\}$.

<u>Def^n</u>:- Let $f_n : E \to \mathbb{R}$ be a function for each $n \in \mathbb{N}$. Then sequence of functions $\{f_n\}$ is said to be converges pointwise if for each $x \in E$, $\{f_n(x)\}$ converges.
We can define a fun^n $f : E \to \mathbb{R}$ s.t.

$$f(x) = \lim_{n \to \infty} f_n(x), \quad x \in E.$$

When this happen we say $\{f_n\}$ converges to the function f pointwise.

We say $f_n \to f$ pointwise ~~means~~
$\equiv$ for each x, $\{f_n(x)\}$ of real numbers converges to the real number $f(x)$
$\equiv f_n(x) \to f(x) \quad \forall x \in E$

Ex1:- $E=[0,1]$, $f_n(x)=x^n$, $0\le x\le 1$.

suppose we take,

$$f(x)=\begin{cases}0, & 0\le x<1\\ 1, & x=1\end{cases}$$

then $f_n\to f$ pointwise.

If we take $E=[0,2]$, then for $x>1$, sequence does not converges.

That means it is not necessary that given any sequence it should converge to some function f.

Whether sequence converges pointwise depends on what is the set E.

2. $E=[-1,1]$, $f_n(x)=x^n$

For $-1<x<1$, $\lim_{n\to\infty} f_n(x)=0$

$x=1$, $\Rightarrow \{1,1,\cdots\}\to$ converges to 1

But for $x=-1$ the sequence becomes $\{-1,1,-1,1,\cdots\}$, which is a divergent sequence.

What are the kinds of ~~that~~ problems that we are interested. The problems are as follow:
What we want to know is that given a sequence, suppose each f_n has same property, then does the limit also have that property?
Usual properties, for example, continuity, differentiability and integribility.
Suppose our question is each f_n is continuous does it follow that f is continuous.
Similarly about the differentiability -
suppose each f_n is differentiable does it follow that f is differentiable
and suppose each f_n is integrable on E, can we say that f is integrable on E, etc.

Now when we start analysing these questions it turns out that this pointwise convergence is not a very satisfied thing.

Let $f_n(x) = x^n$, $x \in [0,1]$

$$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$$

Here each $f_n(x)$ is continuous in $\boxed{[0,1]}$ but $f(x)$ is not continuous at $x=1$

If each f_n is continuous on E and $f_n \to f$ pointwise. Does it follow f is continuous?

f is continuous at $x \in E$

$\equiv \lim_{t \to x} f(t) = f(x)$

[as f is a pointwise limit of $f_n(t)$]

$\equiv \lim_{t \to x} \lim_{n \to \infty} f_n(t) = f(x) = \lim_{n \to \infty} f_n(x)$ — ⊛

But we have assumed that each $f_n(x)$ is continuous. So, we can replace $f_n(x)$ by $\lim_{t \to x} f_n(t)$.

should be same

So, ⊛ $\equiv \lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t)$

So, to say that f is continuous at x where f is pt.wise limit of f_n, it's same as saying whether you can interchange these two limits.

So, asking whether f is continuous at x is basically asking whether these two limiting process can be interchanged.

Each of these questions (differentiability, Integrability) basically can be transformed into question like this, whether certain limiting process can be interchanged.

Unless you put some additional restrictions ~~on them~~ we can't in general interchange the way in which we take the limit.

Suppose we take the example

$$\lim_{m\to\infty} \frac{m}{m+n} = \lim_{m\to\infty} \frac{1}{1+\frac{n}{m}} = 1$$

$$\lim_{n\to\infty} \lim_{m\to\infty} \frac{m}{m+n} = 1$$

$$\lim_{n\to\infty} \frac{m}{n+n} = 0$$, independent of m

$$\lim_{m\to\infty} \lim_{n\to\infty} \frac{m}{m+n} = 0$$

So, for the same funⁿ, if you take the limits in two different orders you will get / you may get different limits

of course, in some cases the limits may be the same, but you need conditions for that.

So, what are the extra conditions?

We will talk of some different kind of convergence which is supposed to be stronger than the pointwise convergence and under which these properties are preserved. That is if each f_n is continuous, then f is continuous etc.

And that is one way of dealing, which is that you strengthen the concept of convergence.

Exercise on Pointwise convergence

① $f_n(x) = nx(1-x^2)^n, \ 0 \leq x \leq 1$

② $f_n(x) = \left(1 - \frac{nx}{n+1}\right)^{n/2}, \ n \geq 1$

$E = (-\infty, 1]$

Uniform Convergence

Let $f_n : E \to \mathbb{R}$ be a sequence of fun's and $f : E \to \mathbb{R}$ is also a funⁿ. We say that $f_n \to f$ uniformly if for every $\varepsilon > 0$ $\exists$ $n_0 \in \mathbb{N}$ s.t.

$$n \geq n_0, \ |f_n(x) - f(x)| < \varepsilon, \ \forall x \in E$$

Since $f_n(x)$ converges to $f(x)$ uniformly $\forall x \in E$ means for every $x \in E$, $f_n(x)$ converges to $f(x)$

$\Rightarrow$ Therefore uniform convergence imply pointwise convergence.

Theorem: If $f_n \to f$ uniformly, then $f_n \to f$ pointwise.

Therefore uniform convergence is stronger that pointwise convergence.

Is the converse true?

If that were true both the notions of convergence coincide, that is not a fact.

So, the converse is false.

As an example $f_n(x) = x^n$, $0 \leq x \leq 1$ is converges pointwise to the funⁿ

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

But $\{f_n\}$ is not ~~not~~ uniformly convergent.

What is the difference?

$f_n \to f$ pointwise $\equiv f_n(x) \to f(x)$ $\forall x \in E$

$\equiv$ $\forall x \in E$, $\forall \varepsilon > 0$ $\exists n_0 \in \mathbb{N}$ s.t.

$\forall n \geq n_0$, $|f_n(x) - f(x)| < \varepsilon$.

We start with a fixed x, and we look at the convergence only at that point. Though it happens at each point, n_0 may depend on x, because given $x \in E$ for this ε we say that some n_0 exist. But this n_0 in general depend on this x. So, we can write it as $n_0(x)$.

~~So, we can~~

But in case of uniform convergence, we are saying that ~~this~~ n_0, that ~~n_0~~ should work for every $x \in E$, that is the main difference.

~~What~~

In case of pointwise convergence, we take different n_0 for different values of x. Suppose it is possible to take max^m of all those n_0's, then that max^m will work for all x.

But this may or may not be possible. If it is possible then that sequence converges uniformly. Otherwise, if it is not possible means there is no maximum. That is if you take $n_0(x)$ for different values of x, it's max^m is infinity. That is what can happen in case of pointwise convergence

let us consider a set

$M_n = \{ |f_n(x) - f(x)| : x \in E \}$ → set of non-negative real numbers

each are $< \varepsilon$,

That means this is a bounded set.

So, we on look at it's supremum or least upper bound.

let $M_n = \sup \{ |f_n(x) - f(x)| : x \in E \}$

so, $M_n < \varepsilon \quad \forall n \geq n_0$

or, for every $\varepsilon > 0 \; \exists \; n_0 \in \mathbb{N}$ s.t. $M_n < \varepsilon$

sequence of real numbers

$\equiv M_n \to 0$ as $n \to \infty$.

So, saying $f_n \to f$ uniformly is equivalent to saying that the sequence of real numbers M_n converges to 0.

This is a very useful criteria for deciding whether a certain sequence converges uniformly or not.

So, $f_n \to f$ uniformly $\equiv M_n \to 0$ as $n \to \infty$

→ this will be useful in many practical problems.

suppose you are given a sequence $\{f_n\}$ and suppose you are asked to check whether that sequence converges uniformly or not. Then the best thing to do is to try to find out M_n. If not M_n, something which

is bounded by M_n, some estimate of M_n and then try to conclude from that whether the sequence converges uniformly or not. In most of the cases you will be able to find out M_n.

<u>Theorem</u>:- Let $E \subseteq \mathbb{R}$ and $\{f_n\}$ be a sequence of function pointwise converges to a func f on E. Suppose $M_n = \sup_{x \in E} |f_n(x) - f(x)|$ then $\{f_n\}$ is uniformly convergent on E iff $\lim_{n\to\infty} M_n = 0$.

<u>Proof</u>:- Suppose $\{f_n\}$ converges uniformly to f on E.

Let $\varepsilon > 0$ be given.

Then $\exists\, n_0 \in \mathbb{N}$ such that

$|f_n(x) - f(x)| < \varepsilon/2 \quad \forall n \geq n_0, \forall x \in E$

So, $\sup |f_n(x) - f(x)| \leq \varepsilon/2 < \varepsilon \quad \forall n \geq n_0$

$\Rightarrow M_n < \varepsilon \quad \forall n \geq n_0$

$\Rightarrow |M_n - 0| < \varepsilon \quad \forall n \geq n_0$

i.e., $\lim_{n\to\infty} M_n = 0$

Conversely, suppose $\lim_{n\to\infty} M_n = 0$.

then for each $\varepsilon > 0$ $\exists\, n_0 \in \mathbb{N}$ such that

$|M_n - 0| < \varepsilon \quad \forall n \geq n_0$

$\Rightarrow M_n < \varepsilon \quad \forall n \geq n_0$

$\Rightarrow |f_n(x) - f(x)| < \varepsilon \quad \forall n \geq n_0 \;\forall x \in E$

$\Rightarrow |f_n(x) - f(x)| < \varepsilon \quad \forall n \geq n_0 \quad \forall x \in E$

Hence $\{f_n\}$ converges uniformly to f on E

① $f_n(x) = x^n, \quad 0 \le x \le 1$

$$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$$

$M_n = \sup\{|f_n(x) - f(x)|, x \in [0,1]\}$

$= \cancel{\sup\{|x^n - f(x)|, x \in [0,1]\}}$

$= \sup\{|x^n - 0|, x \in [0,1)\}$ → as at $x=1$ $f(x) = 1$ & $f_n(x) = 1$ so, $|f_n(x) - f(x)| = 0$

$= \sup\{x^n, 0 \le x < 1\}$

$= 1$

Here $M_n \not\to 0$ as $n \to \infty$

so, $f_n \not\to f$ uniformly.

② Suppose a sequence of fun^n's $\{f_n\}$ is defined by $f_n(x) = \frac{nx}{1+n^2x^2}, x \in [0,1]$.

For $x=0$, $f_n(0) = 0$

For $0 < x \le 1$, $f_n(x) = \frac{nx}{1+n^2x^2} = \frac{x/n}{1/n^2 + x^2}$

$\therefore \lim_{n\to\infty} f_n(x) = 0$.

Therefore $f(x)$ is a limit fun^n, then $f(x) = 0 \ \forall x \in [0,1]$.

So, $\{f_n\}$ is boint wise convergence on $[0,1]$.

Now, $M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)|$

$= \sup_{x\in[0,1]} \frac{nx}{1+n^2x^2}$

$= \frac{n \cdot 1/n}{1 + n^2 \cdot 1/n^2} = 1/2$

$\therefore \lim_{n\to\infty} M_n = 1/2 \ne 0$

so, $\{f_n\}$ is not uniformly conregent.

$g(x) = \frac{nx}{1+n^2x^2}$

$g'(x) = n\frac{1-n^2x^2}{(1+n^2x^2)^2}$

$g'(x) = 0$

$\Rightarrow 1 - n^2x^2 = 0 \Rightarrow x = \pm 1/n$

Since $x \in [0,1]$, $x = 1/n$.

also $g''(x) < 0$

or, take nx, $1/nx$ apply A.M. $\ge$ G.M.

(2) $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ on $\mathbb{R}$

(3) Let the sequence of func^{n}s are given by $f_n(x) = nx\,e^{-nx^2}$, $x \geq 0$.

(4) $f_n(x) = \frac{x}{n}$ on $\mathbb{R}$, $n \in \mathbb{N}$

Sol^n:- $\lim_{n\to\infty} f_n(x) = 0$

$|f_n(x) - f(x)| = |\frac{x}{n}| < \varepsilon$

$|\frac{x}{n}| < \varepsilon$

$\Rightarrow n > \frac{|x|}{\varepsilon} = k$

$n_0 = [\frac{x}{\varepsilon}] + 1$

~~k is~~ n_0 is dependent on x.

So, is not convergent uniformly.

(5) $f_n(x) = x^n$ on $[0, \frac{1}{2}]$

$\lim_{n\to\infty} f_n(x) = 0$

$|f_n(x) - f(x)| = |x^n| \leq (\frac{1}{2})^n$

$M_n \to 0$ as $n \to \infty$

$\Rightarrow$ uniformly converges to 0.

(6) $f_n(x) = \frac{x}{1+nx^2}$, $x \in [0,1]$, for each $n \in \mathbb{N}$

(7) $f_n(x) = x^n e^{-nx}$, $x \geq 0$, $n \geq 1$.

⑧ Show that the sequence $\{f_n\}$, where

$f_n(x) = x^n$

is uniformly convergent on $[0, k]$, $k < 1$

and only pointwise convergent on $[0, 1]$.

Sol^n:- $\therefore f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$

Let $\varepsilon > 0$.

we have $|f_n(x) - f(x)| = x^n < \varepsilon$

If $\left(\frac{1}{x}\right)^n > 1/\varepsilon$

or if $n > \overbrace{\log(1/\varepsilon)/\log(1/x)}^{n_0(x)}$

Now $\max\limits_{0 \le x \le k < 1} \left\{ \log(1/\varepsilon)/\log(1/x) \right\}$

$= \log(1/\varepsilon)/\log(1/k)$

[Since as x increases $\log(1/\varepsilon)/\log(1/x)$ increases]

$x \not< k$

~~$\Rightarrow \log x > \log k$~~

$1/x \not> 1/k$

$\Rightarrow \log(1/x) > \log(1/k)$

$\Rightarrow \log(1/\varepsilon)/\log(1/x) < \frac{\log(1/\varepsilon)}{\log(1/k)}$

Let $N = [\log(1/\varepsilon)/\log(1/k)] + 1$

$\therefore |f_n(x) - f(x)| < \varepsilon \quad \forall n \ge N, \ 0 < x < 1$

Again at $x = 0$

$|f_n(x) - f(x)| = 0 < \varepsilon \quad \forall n \ge 1$.

Thus for any $\varepsilon > 0$, $\exists$ n such that $\forall x \in [0, k]$, $k < 1$.

$|f_n(x) - f(x)| < \varepsilon \quad \forall n \ge N$.

Therefore the sequence is uniformly convergent on $[0, k]$, $k < 1$, However the number $\log(1/\varepsilon)/\log(1/x) \to \infty$ as $x \to 1$, so it's not possible to find any integer N such that $|f_n(x) - f(x)| < \varepsilon \quad \forall n \geq N$ and all x in $[0,1]$.

Hence the sequence is not uniformly convergent on any interval containing 1 and in particular on $[0,1]$.

9. Show that the sequence $\{f_n\}$, where $f_n(x) = \frac{1}{x+n}$ is uniformly convergent in any interval $[0,b]$, $b>0$.

Sol^n:- The limit fun^n is
$$f(x) = \lim_{n\to\infty} \frac{1}{x+n} = 0 \quad \forall x \in [0,b]$$
So, the sequence converges pt.wise to 0.

For any $\varepsilon > 0$,
$$|f_n(x) - f(x)| = \frac{1}{x+n} < \varepsilon$$
if $n > 1/\varepsilon - x$, which decreases with x, the maxm value being $1/\varepsilon$.

Let $N = [1/\varepsilon] + 1$

So, for every $\varepsilon > 0 \; \exists \; N$ s.t.
$$|f_n(x) - f(x)| < \varepsilon, \quad \forall n \geq N.$$

(9) ~~Show that~~ 0 is a point of ~~non-uniform~~

(10) Show that the sequence $\{f_n\}$, where $f_n(x) = nxe^{-nx^2}, x \geq 0$ is not uniformly convergent on $[0, k]$, $k > 0$.

(11) Show that the sequence $\{f_n\}$, where $f_n(x) = \frac{x}{1+nx^2}$, x is real converges uniformly on any closed interval I.

$\Rightarrow f(x) = \lim_{n\to\infty} f_n(x) = 0 \ \forall x$

$$M_n = \sup_{x\in I} |f_n(x) - f(x)| = \sup_{x\in I} \left|\frac{x}{1+nx^2}\right|$$

$$= \frac{1/\sqrt{n}}{1+n\cdot\frac{1}{n}} = \frac{1}{2\sqrt{n}} \to 0$$

as $n \to \infty$

$$u(x) = \frac{x}{1+nx^2}$$

$$u'(x) = \frac{1-nx^2}{(1+nx^2)^2} = 0 \Rightarrow x = \frac{1}{\sqrt{n}}$$

Hence $\{f_n\}$ converges uniformly on I.

(12) Show that the sequence $\{f_n\}$, where $f_n(x) = x^{n-1}(1-x)$ converges uniformly in the interval $[0,1]$.

(13) Show that $f_n : E \to \mathbb{R}$, $E = (0,1)$ is defined by $f_n(x) = \frac{n}{nx+1}$ is not uniformly converge however it is convergent uniformly on every interval $[a,1)$ with $0 < a < 1$.

(15) Check whether $\{f_n\}$, where $f_n(x) = 1 - \frac{x^n}{n}$, $x \in [0,1]$ is uniformly convergent or not.

(16) $f_n(x) = \frac{x}{1+nx^2}$, $x \in [0,1]$

(17) $f_n(x) = nx(1-x)^n$, $x \in [0,1]$

(18) $f_n(x) = n^2 e^{-nx}$, $x \in [0,\infty)$

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(21) $f_n(x) = x^2 e^{-nx}$, $x \in [0,\infty)$

(22) $f_n(x) = x e^{-nx}$, $x \in [0,\infty)$

(23) $f_n(x) = \frac{n^2x}{1+n^4x^2}$, $x \in [0,1]$

(24) $f_n(x) = \frac{nx}{1+n^3x}$, $x \in [0,1)$

(25) For each $n \in \mathbb{N}$, let

$$f_n(x) = \begin{cases} nx^2, & 0 \le x \le 1/n \\ x, & 1/n < x \le 1 \end{cases}$$

~~(i)~~ Show that the sequence is uniform convergen on $[0,1]$.

(26) $f_n(x) = n^2x(1-x^2)^n$, $0 \le x \le 1$

(27) ~~$f_n(x) = \log(n^2+x^2)$, $x \in \mathbb{R}$~~

(27) $f_n(x) = x + \frac{x}{n}$, $x \in \mathbb{R}$

(28) $f_n(x) = e^{-nx}$, $x \ge 0$.

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Sequence and Series of Functions

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Sequence is a function whose domain is the set of all natural numbers and codomain can be anything. When the codomain is a real number we call it sequence of real numbers.

If it is a complex it will be sequence of complex numbers. So, similarly you can talk of ~~complex~~ sequence of anything, sequence of matrices, sequence of vectors and also sequence of functions.

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Now suppose for each $x \in E$. $\{f_n(x)\}$ converges and suppose it converges to some real number. Then that will define a new function from E to $\mathbb{R}$, because for each x you have a new number which is limit of $\{f_n(x)\}$.

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We say $f_n \to f$ pointwise ~~means~~
$\equiv$ for each x, $\{f_n(x)\}$ of real numbers converges to the real number $f(x)$
$\equiv f_n(x) \to f(x) \;\; \forall x \in E$

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Ex1:- $E = [0,1]$, $f_n(x) = x^n$, $0 \le x \le 1$.

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Whether sequence converges pointwise depends on what is the set E.

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For $-1 < x < 1$, $\lim_{n \to \infty} f_n(x) = 0$

$x = 1$, $\Rightarrow \{1, 1, \cdots\} \to$ converges to 1

But for $x = -1$ the sequence becomes $\{-1, 1, -1, 1, \cdots\}$, which is a divergent sequence.

What are the kinds of ~~that~~ problems that we are interested. The problems are as follow:

What we want to know is that given a sequence, suppose each f_n has some property, then does the limit also have that property?

Usual properties, for example, continuity, differentiability and integribility.

Suppose our question is each f_n is continuous does it follow that f is continuous.

Similarly about the differentiability - suppose each f_n is differentiable does it follow that f is differentiable and suppose each f_n is integrable on E, can we say that f is integrable on E, etc.

Now when we start analysing these questions it turns out that this pointwise convergence is not a very satisfied thing.

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$$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$$

Here each $f_n(x)$ is continuous in $\boxed{[0,1]}$ but $f(x)$ is not continuous at $x=1$

If each f_n is continuous on E and $f_n \to f$ pointwise. Does it follow f is continuous?

f is continuous at $x \in E$

$$\equiv \lim_{t \to x} f(t) = f(x)$$

[as f is a pointwise limit of $f_n(t)$]

$$\equiv \lim_{t \to x} \lim_{n \to \infty} f_n(t) = f(x) = \lim_{n \to \infty} f_n(x) \quad \text{—} \; *$$

But we have assumed that each $f_n(x)$ is continuous. So, we can replace $f_n(x)$ by $\lim_{t \to x} f_n(t)$.

$$\text{So, } * \equiv \overbrace{\lim_{t \to x} \lim_{n \to \infty}}^{\text{should be same}} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t)$$

So, to say that f is continuous at x where f is pt.wise limit of f_n, it's same as saying whether you can interchange these two limits.

So, asking whether f is continuous at x is basically asking whether these two limiting process can be interchanged.

Each of these questions (differentiability, Integrability) basically can be transformed into question like this, whether certain limiting process can be interchanged.

Unless you put some additional restrictions ~~on them~~ we can't in general interchange the way in which we take the limit.

Suppose we take the example

$$\lim_{m\to\infty} \frac{m}{m+n} = \lim_{m\to\infty} \frac{1}{1+\frac{n}{m}} = 1$$

$$\lim_{n\to\infty} \lim_{m\to\infty} \frac{m}{m+n} = 1$$

$$\lim_{n\to\infty} \frac{m}{m+n} = 0$$, independent of m

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So, for the same funⁿ, if you take the limits in two different orders you will get / you may get different limits

Of course, in some cases the limits may be the same, but you need conditions for that.

So, what are the extra conditions?
We will talk of some different kind of convergence which is supposed to be stronger than the pointwise convergence and under which these properties are preserved.
That is if each f_n is continuous, then f is continuous etc.
And that is one way of dealing, which is that you strengthen the concept of convergence.

Exercise on Pointwise convergence

① $f_n(x) = nx(1-x^2)^n, \quad 0 \le x \le 1$

② $f_n(x) = \left(1 - \frac{nx}{n+1}\right)^{n/2}, \quad n \geq 1$

$E = (-\infty, 1]$

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Uniform Convergence

Let $f_n : E \to \mathbb{R}$ be a sequence of fun's and $f : E \to \mathbb{R}$ is also a funn. We say that $f_n \to f$ uniformly if for every $\varepsilon > 0$ $\exists$ $n_0 \in \mathbb{N}$ s.t.

$$n \geq n_0, \ |f_n(x) - f(x)| < \varepsilon, \ \forall x \in E$$

Since $f_n(x)$ converges to $f(x)$ uniformly $\forall x \in E$ means for every $x \in E$, $f_n(x)$ converges to $f(x)$

$\Rightarrow$ Therefore uniform convergence imply pointwise convergence.

<u>Theorem:</u> If $f_n \to f$ uniformly, then $f_n \to f$ pointwise.

Therefore uniform convergence is stronger that pointwise convergence.

Is the converse true?

If that were true both the notions of convergence coincide, that is not a fact. So, the converse is false.

As an example $f_n(x) = x^n$, $0 \leq x \leq 1$ is converges pointwise to the funn

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

But $\{f_n\}$ is not ~~not~~ uniformly convergent.

What is the difference?

$f_n \to f$ pointwise $\equiv f_n(x) \to f(x)$ $\forall x \in E$

$\equiv \forall x \in E$, $\forall \epsilon > 0$ $\exists n_0 \in \mathbb{N}$ s.t.

$\forall n \geq n_0$, $|f_n(x) - f(x)| < \epsilon$.

We start with a fixed x, and we look at the convergence only at that point. Though it happens at each point, n_0 may depend on x, because given $x \in E$ for this ϵ we say that some n_0 exist. But this n_0 in general depend on this x. So, we can write it as $n_0(x)$.

~~So, we can~~

But in case of uniform convergence, we are saying that ~~this~~ n_0, that ~~n0~~ should work for every $x \in E$, that is the main difference.

~~What~~

In case of pointwise convergence, we take different n_0 for different values of x. Suppose it is possible to take max^m of all those n_0's, then that max^m will work for all x.

But this may or may not be possible. If it is possible then that sequence converges uniformly. Otherwise, if it is not possible means there is no maximum. That is if you take $n_0(x)$ for different values of x, it's max^m is infinity. That is what can happen in case of point wise convergence.

let us consider a set

$M_n = \{ |f_n(x) - f(x)| : x \in E \} \to$ set of non-negative real numbers

each are $< \varepsilon$,

that means this is a bounded set.

So, we can look at it's supremum or least upper bound.

let $M_n = \sup \{ |f_n(x) - f(x)| : x \in E \}$

so, $M_n < \varepsilon \quad \forall n \geq n_0$

or, for every $\varepsilon > 0 \ \exists n_0 \in \mathbb{N}$ s.t. $M_n < \varepsilon$

sequence of real numbers

$\equiv M_n \to 0$ as $n \to \infty$.

So, saying $f_n \to f$ uniformly is equivalent to saying that the sequence of real numbers M_n converges to 0.

This is a very useful criteria for deciding whether a certain sequence converges uniformly or not.

So, $f_n \to f$ uniformly $\equiv M_n \to 0$ as $n \to \infty$

→ this will be useful in many practical problems.

suppose you are given a sequence $\{f_n\}$ and suppose you are asked to check whether that sequence converges uniformly or not. Then the best thing to do is to try to find out M_n. If not M_n, something which

is bounded by M_n, some estimate of M_n and then try to conclude from that whether the sequence converges uniformly or not. In most of the cases you will be able to find out M_n.

<u>Theorem</u>:- Let $E \subseteq \mathbb{R}$ and $\{f_n\}$ be a sequence of function pointwise converges to a fun^n f on E. Suppose $M_n = \sup_{x \in E} |f_n(x) - f(x)|$ then $\{f_n\}$ is uniformly convergent on E iff $\lim_{n\to\infty} M_n = 0$.

<u>Proof</u>:- Suppose $\{f_n\}$ converges uniformly to f on E.

Let $\varepsilon > 0$ be given.

Then $\exists\, n_0 \in \mathbb{N}$ such that

$|f_n(x) - f(x)| < \varepsilon/2 \quad \forall n \geq n_0, \forall x \in E$

So, $\sup |f_n(x) - f(x)| \leq \varepsilon/2 < \varepsilon \quad \forall n \geq n_0$

$\Rightarrow M_n < \varepsilon \quad \forall n \geq n_0$

$\Rightarrow |M_n - 0| < \varepsilon \quad \forall n \geq n_0$

i.e., $\lim_{n\to\infty} M_n = 0$

Conversely, suppose $\lim_{n\to\infty} M_n = 0$.

then for each $\varepsilon > 0$ $\exists\, n_0 \in \mathbb{N}$ such that

$|M_n - 0| < \varepsilon \quad \forall n \geq n_0$

$\Rightarrow M_n < \varepsilon \quad \forall n \geq n_0$

$\Rightarrow |f_n(x) - f(x)| < \varepsilon \quad \forall n \geq n_0 \; \forall x \in E$

$\Rightarrow |f_n(x) - f(x)| < \varepsilon \quad \forall n \geq n_0 \quad \forall x \in E$

Hence $\{f_n\}$ converges uniformly to f on E

① $f_n(x) = x^n,\ 0 \le x \le 1$

$$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$$

$M_n = \sup\{|f_n(x) - f(x)|, x \in [0,1]\}$

$= \cancel{\sup\{|x^n - f(x)|, x \in [0,1]\}}$

$= \sup\{|x^n - 0|, x \in [0,1]\}$ → as at $x=1$ $f(x)=1$ & $f_n(x)=1$ so, $|f_n(x)-f(x)| = 0$

$= \sup\{x^n, 0 \le x <\}$

$= 1$

Here $M_n \not\to 0$ as $n \to \infty$

so, $f_n \not\to f$ uniformly.

② Suppose a sequence of fun^{n}s $\{f_n\}$ is defined by $f_n(x) = \frac{nx}{1+n^2x^2}, x \in [0,1]$.

For $x=0$, $f_n(0) = 0$

For $0 < x \le 1$, $f_n(x) = \frac{nx}{1+n^2x^2} = \frac{x/n}{1/n^2 + x^2}$

$\therefore \lim_{n\to\infty} f_n(x) = 0$.

Therefore $f(x)$ is a limit fun^n, then

$f(x) = 0\ \forall x \in [0,1]$.

so, $\{f_n\}$ is point wise convergence on $[0,1]$.

Now, $M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)|$

$= \sup_{x \in [0,1]} \frac{nx}{1+n^2x^2}$

$= \frac{n \cdot 1/n}{1+n^2 \cdot 1/n^2} = 1/2$

$\therefore \lim_{n\to\infty} M_n = 1/2 \ne 0$.

so, $\{f_n\}$ is not uniformly convergent.

$g(x) = \frac{nx}{1+n^2x^2}$

$g'(x) = n\frac{1-n^2x^2}{(1+n^2x^2)^2}$

$g'(x) = 0$

$\Rightarrow 1 - n^2x^2 = 0 \Rightarrow x = \pm 1/n$

Since $x \in [0,1]$, $x = 1/n$.

also $g''(x) < 0$

or, ~~take~~ nx, $1/nx$ apply $\frac{nx+\frac{1}{nx}}{}$ A.M. $\ge$ G.M.

(2) $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ on $\mathbb{R}$

(3) Let the sequence of fun^{n}s are given by $f_n(x) = nx\,e^{-nx^2}$, $x \geq 0$.

(4) $f_n(x) = \frac{x}{n}$ on $\mathbb{R}$, $n \in \mathbb{N}$

Sol^n:- $\lim_{n\to\infty} f_n(x) = 0$

$|f_n(x) - f(x)| = |\frac{x}{n}| < \epsilon$

$|\frac{x}{n}| < \epsilon$

$\Rightarrow n > \frac{|x|}{\epsilon} \neq k$

$n_0 = [\frac{x}{\epsilon}] + 1$

~~k is~~ n_0 is dependent on x.

So, is not convergent uniformly.

(5) $f_n(x) = x^n$ on $[0, \frac{1}{2}]$

$\lim_{n\to\infty} f_n(x) = 0$

$|f_n(x) - f(x)| = |x^n| \leq (\frac{1}{2})^n$

$M_n \to 0$ as $n \to \infty$

$\Rightarrow$ uniformly converges to 0.

(6) $f_n(x) = \frac{x}{1+nx^2}$, $x \in [0,1]$, for each $n \in \mathbb{N}$

(7) $f_n(x) = x^n e^{-nx}$, $x \geq 0$, $n \geq 1$.

(8) Show that the sequence $\{f_n\}$, where

$f_n(x) = x^n$

is uniformly convergent on $[0,k]$, $k<1$ and only pointwise convergent on $[0,1]$.

Sol^n:- $\therefore f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$

Let $\epsilon > 0$.

We have $|f_n(x) - f(x)| = x^n < \epsilon$

If $\left(\frac{1}{x}\right)^n > 1/\epsilon$

or if $n > \underbrace{\log(1/\epsilon)/\log(1/x)}_{n_0(x)}$

Now $\max_{0 \le x \le k < 1} \left\{ \log(1/\epsilon)/\log(1/x) \right\}$

$= \log(1/\epsilon) / \log(1/k)$

[Since as x increases $\log(1/\epsilon)/\log(1/x)$ increases]

$x \ge k$
~~$\Rightarrow \log x > \log k$~~
$1/x \ge 1/k$
$\Rightarrow \log(1/x) > \log(1/k)$
$\Rightarrow \log(1/\epsilon)/\log(1/x) < \frac{\log(1/\epsilon)}{\log(1/k)}$

Let $N = [\log(1/\epsilon)/\log(1/k)] + 1$

$\therefore |f_n(x) - f(x)| < \epsilon \quad \forall n \ge N,\ 0 < x < 1$

Again at $x = 0$

$|f_n(x) - f(x)| = 0 < \epsilon \quad \forall n \ge 1.$

Thus for any $\epsilon > 0$, $\exists\, n$ such that $\forall x \in [0,k]$, $k<1$.

$|f_n(x) - f(x)| < \epsilon \quad \forall n \ge N$.

Therefore the sequence is uniformly convergent on $[0,k]$, $k<1$, However the number $\log(1/\varepsilon)/\log(1/x) \to \infty$ as $x \to 1$, so it's not possible to find any integer N such that $|f_n(x) - f(x)| < \varepsilon \quad \forall n \geq N$ and all x in $[0,1]$.

Hence the sequence is not uniformly convergent on any interval containing 1 and in particular on $[0,1]$.

9. Show that the sequence $\{f_n\}$, where $f_n(x) = \frac{1}{x+n}$ is uniformly convergent in any interval $[0,b]$, $b>0$.

Sol^n:- The limit fun^n is

$$f(x) = \lim_{n\to\infty} \frac{1}{x+n} = 0 \quad \forall x \in [0,b]$$

So, the sequence converges pt.wise to 0.

For any $\varepsilon > 0$,

$$|f_n(x) - f(x)| = \frac{1}{x+n} < \varepsilon$$

if $n > 1/\varepsilon - x$, which decreases with x, the maxm value being $1/\varepsilon$,

let $N = [1/\varepsilon] + 1$

So, for every $\varepsilon > 0 \ \exists N$ s.t.

$$|f_n(x) - f(x)| < \varepsilon, \ \forall n \geq N.$$

⑨ ~~Show that~~ 0 is a point of ~~non-uniform~~

(10) Show that the sequence $\{f_n\}$, where $f_n(x) = nxe^{-nx^2}$, $x \geq 0$ is not uniformly convergent on $[0, k]$, $k > 0$.

(11) Show that the sequence $\{f_n\}$, where $f_n(x) = \frac{x}{1+nx^2}$, x is real converges uniformly on any closed interval I.

$\Rightarrow f(x) = \lim_{n\to\infty} f_n(x) = 0 \ \forall x$

$$M_n = \sup_{x \in I} |f_n(x) - f(x)| = \sup_{x \in I} \left|\frac{x}{1+nx^2}\right|$$

$$= \frac{1/\sqrt{n}}{1+n\cdot\frac{1}{n}} = \frac{1}{2\sqrt{n}} \to 0$$

as $n \to \infty$

$$u(x) = \frac{x}{1+nx^2}$$

$$u'(x) = \frac{1-nx^2}{(1+nx^2)^2} = 0 \Rightarrow x = \frac{1}{\sqrt{n}}$$

Hence $\{f_n\}$ converges uniformly on I.

(12) Show that the sequence $\{f_n\}$, where $f_n(x) = x^{n-1}(1-x)$ converges uniformly in the interval $[0, 1]$.

(13) Show that $f_n : E \to \mathbb{R}$, $E = (0, 1)$ is defined by $f_n(x) = \frac{x}{nx+1}$ is not uniformly convergent however it is convergent uniformly on every interval $[a, 1)$ with $0 < a < 1$.

(15) Check whether $\{f_n\}$, where $f_n(x) = 1 - \frac{x^n}{n}$, $x \in [0,1]$ is uniform convergent or not.

(16) $f_n(x) = \frac{x}{1+nx^2}$, $x \in [0,1]$

(17) $f_n(x) = nx(1-x)^n$, $x \in [0,1]$

(18) $f_n(x) = n^2 e^{-nx}$, $x \in [0,\infty)$

(19) $f_n(x) = \frac{x}{n+x^2}$, $x \in [0,1]$

(20) $f_n(x) = n^2x^2 e^{-nx}$, $x \in [0,\infty)$

(21) $f_n(x) = x^2 e^{-nx}$, $x \in [0,\infty)$

(22) $f_n(x) = x e^{-nx}$, $x \in [0,\infty)$

(23) $f_n(x) = \frac{n^2x}{1+n^4x^2}$, $x \in [0,1]$

(24) $f_n(x) = \frac{nx}{1+n^3x}$, $x \in [0,1)$

(25) For each $n \in \mathbb{N}$, let

$$f_n(x) = \begin{cases} nx^2, & 0 \le x \le 1/n \\ x, & 1/n < x \le 1 \end{cases}$$

~~(i)~~ show that the sequence is uniform convergen on $[0,1]$.

(26) $f_n(x) = n^2x(1-x^2)^n$, $0 \le x \le 1$

(27) ~~$f_n(x) = \log(x^2+n^2)$, $x \in \mathbb{R}$~~

(27) $f_n(x) = x + \frac{x}{n}$, $x \in \mathbb{R}$

(28) $f_n(x) = e^{-nx}$, $x \ge 0$.

Sequence and Series of Functions

You are already familiar with sequence and series of numbers. So, all that we do here is that replace the numbers by functions.

Sequence is a function whose domain is the set of all natural numbers and codomain can be anything. When the codomain is a real number we call it sequence of real numbers.

If it is a complex it will be sequence of complex numbers. So, similarly you can talk of ~~complex~~ sequence of anything, sequence of matrices, sequence of vectors and also sequence of functions.

Each functions defined on different sets, but with that kind of thing we will not be able to do anything further.

So, we shall just fix one set.

Suppose E is a non-empty set, $E \neq \phi$ and we shall assume that all functions are defined on this set E.

In some cases E will be subset of some metric space.

As we are studying in real line, in most of the application it will be some interval in the real line, closed and bounded interval, $I = [a,b]$, or $[a,b)$, $(a,b]$, (a,b).

So, for each $n \in \mathbb{N}$ we have real valued function $f_n : E \to \mathbb{R}$, $\forall n \in \mathbb{N}$.

Then $\{f_n\}$ is a sequence of functions.

Once you have a sequence of fun's $\{f_n\}$ then if you take any $x \in E$, then $f_n(x)$ is a real number.

Then $\{f_n(x)\}$ is a sequence of real numbers.

Now suppose for each $x \in E$. $\{f_n(x)\}$ converges and suppose it converges to some real number. Then that will define a new function from E to $\mathbb{R}$, because for each x you have a new number which is limit of $\{f_n(x)\}$.

<u>Defn</u>:- Let $f_n : E \to \mathbb{R}$ be a function for each $n \in \mathbb{N}$. Then sequence of functions $\{f_n\}$ is said to be converges pointwise if for each $x \in E$, $\{f_n(x)\}$ converges.

We can define a funⁿ $f : E \to \mathbb{R}$ s.t.

$$f(x) = \lim_{n \to \infty} f_n(x), \quad x \in E.$$

When this happen we say $\{f_n\}$ converges to the function f pointwise.

We say $f_n \to f$ pointwise ~~means~~

$\equiv$ for each x, $\{f_n(x)\}$ of real numbers converges to the real number $f(x)$

$\equiv f_n(x) \to f(x) \quad \forall x \in E$

Ex1:- $E = [0,1]$, $f_n(x) = x^n$, $0 \le x \le 1$.

suppose we take,

$$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$$

then $f_n \to f$ pointwise.

If we take $E = [0,2]$, then for $x > 1$, sequence does not converge.

That means it is not necessary that given any sequence it should converge to some function f.

Whether sequence converges pointwise depends on what is the set E.

2. $E = [-1,1]$, $f_n(x) = x^n$

For $-1 < x < 1$, $\lim_{n \to \infty} f_n(x) = 0$

$x = 1$, $\Rightarrow \{1, 1, \cdots\} \to$ converges to 1

But for $x = -1$ the sequence becomes $\{-1, 1, -1, 1, \cdots\}$, which is a divergent sequence.

What are the kinds of ~~the~~ problems that we are interested. The problems are as follow:

What we want to know is that given a sequence, suppose each f_n has some property, then does the limit also have that property?

Usual properties, for example, continuity, differentiability and integribility.

Suppose our question is each f_n is continuous does it follow that f is continuous.

Similarly about the differentiability - suppose each f_n is differentiable does it follow that f is differentiable and suppose each f_n is integrable on E, can we say that f is integrable on E, etc.

Now when we start analysing these questions it turns out that this pointwise convergence is not a very satisfied thing.

Let $f_n(x) = x^n$, $x \in [0,1]$

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

Here each $f_n(x)$ is continuous in $\boxed{[0,1]}$ but $f(x)$ is not continuous at $x=1$

If each f_n is continuous on E and $f_n \to f$ pointwise. Does it follow f is continuous?

f is continuous at $x \in E$

$\equiv \lim_{t \to x} f(t) = f(x)$

[as f is a pointwise limit of $f_n(t)$]

$\equiv \lim_{t \to x} \lim_{n \to \infty} f_n(t) = f(x) = \lim_{n \to \infty} f_n(x)$ — ⊛

But we have assumed that each $f_n(x)$ is continuous. So, we can replace $f_n(x)$ by $\lim_{t \to x} f_n(t)$.

$$\text{So, } ⊛ \equiv \underbrace{\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t)}_{\text{should be same}}$$

So, to say that f is continuous at x where f is pt.wise limit of f_n, it's same as saying whether you can interchange these two limits.

So, asking whether f is continuous at x is basically asking whether these two limiting process can be interchanged.

Each of these questions (differentiability, Integrability) basically can be transformed into question like this, whether certain limiting process can be interchanged.

Unless you put some additional restrictions ~~on them~~ we can't in general interchange the way in which we take the limit.

Suppose we take the example

$$\lim_{m\to\infty} \frac{m}{m+n} = \lim_{m\to\infty} \frac{1}{1+\frac{n}{m}} = 1$$

$$\lim_{n\to\infty} \lim_{m\to\infty} \frac{m}{m+n} = 1$$

$$\lim_{n\to\infty} \frac{m}{n+n} = 0$$, independent of m

$$\lim_{m\to\infty} \lim_{n\to\infty} \frac{m}{m+n} = 0$$

So, for the same funⁿ, if you take the limits in two different orders you will get / you may get different limits

Of course, in some cases the limits may be the same, but you need conditions for that.

So, what are the extra conditions?
We will talk of some different kind of convergence which is supposed to be stronger than the pointwise convergence and under which these properties are preserved. That is if each f_n is continuous, then f is continuous etc.

And that is one way of dealing, which is that you strengthen the concept of convergence.

<u>Exercise on Pointwise convergence</u>

① $f_n(x) = nx(1-x^2)^n$, $0 \leq x \leq 1$

② $f_n(x) = \left(1 - \frac{nx}{n+1}\right)^{n/2}$, $n \geq 1$

$E = (-\infty, 1]$

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Let $f_n : E \to \mathbb{R}$ be a sequence of fun's and $f : E \to \mathbb{R}$ is also a funⁿ. We say that $f_n \to f$ uniformly if for every $\varepsilon > 0$ $\exists$ $n_0 \in \mathbb{N}$ s.t.

$$n \geq n_0, \ |f_n(x) - f(x)| < \varepsilon, \ \forall x \in E$$

Since $f_n(x)$ converges to $f(x)$ uniformly $\forall x \in E$ means for every $x \in E$, $f_n(x)$ converges to $f(x)$

$\Rightarrow$ Therefore uniform convergence imply pointwise convergence.

Theorem: If $f_n \to f$ uniformly, then $f_n \to f$ pointwise.

Therefore uniform convergence is stronger that pointwise convergence.

Is the converse true?

If that were true both the notions of convergence coincide, that is not a fact. So, the converse is false.

As an example $f_n(x) = x^n$, $0 \leq x \leq 1$ is converges pointwise to the funⁿ

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

But $\{f_n\}$ is not ~~not~~ uniformly convergent.

What is the difference?

$f_n \to f$ pointwise $\equiv f_n(x) \to f(x)$ $\forall x \in E$

$\equiv \forall x \in E, \forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N}$ s.t.

$\forall n \geq n_0, \ |f_n(x) - f(x)| < \varepsilon$.

We start with a fixed x, and we look at the convergence only at that point. Though it happens at each point, n_0 may depend on x, because given $x \in E$ for this ε we say that some n_0 exist. But this n_0 in general depend on this x. So, we can write it as $n_0(x)$.

~~So, we can~~

But in case of uniform convergence, we are saying that ~~this~~ n_0, that ~~n₀~~ should work for every $x \in E$, that is the main difference.

~~What~~

In case of pointwise convergence, we take different n_0 for different values of x. Suppose it is possible to take max^m of all those n_0's, then that max^m will work for all x.

But this may or may not be possible. If it is possible then that sequence converges uniformly. Otherwise, if it is not possible means there is no maximum. That is if you take $n_0(x)$ for different values of x, it's max^m is infinity. That is what can happen in case of pointwise convergence.

let us consider a set

$$M_n = \{ |f_n(x) - f(x)| : x \in E \} \to \text{set of non-negative real numbers}$$

each are $< \varepsilon$,

that means this is a bounded set.

So, we can look at it's supremum or least upper bound.

let $M_n = \sup \{ |f_n(x) - f(x)| : x \in E \}$

So, $M_n < \varepsilon \quad \forall n \geq n_0$

or, for every $\varepsilon > 0 \ \exists \ n_0 \in \mathbb{N}$ s.t. $M_n < \varepsilon$ — sequence of real numbers

$\equiv M_n \to 0$ as $n \to \infty$.

So, saying $f_n \to f$ uniformly is equivalent to saying that the sequence of real numbers M_n converges to 0.

This is a very useful criteria for deciding whether a certain sequence converges uniformly or not.

So, $f_n \to f$ uniformly $\equiv M_n \to 0$ as $n \to \infty$

→ this will be useful in many practical problems.

suppose you are given a sequence $\{f_n\}$ and suppose you are asked to check whether that sequence converges uniformly or not. Then the best thing to do is to try to find out M_n. If not M_n, something which

is bounded by M_n, some estimate of M_n and then try to conclude from that whether the sequence converges uniformly or not. In most of the cases you will be able to find out M_n.

Theorem:- Let $E \subseteq \mathbb{R}$ and $\{f_n\}$ be a sequence of function pointwise converges to a func f on E. Suppose $M_n = \sup_{x \in D} |f_n(x) - f(x)|$ then $\{f_n\}$ is uniformly convergent on E iff $\lim_{n\to\infty} M_n = 0$.

Proof:- suppose $\{f_n\}$ converges uniformly to f on E.

Let $\epsilon > 0$ be given.

Then $\exists\, n_0 \in \mathbb{N}$ such that

$|f_n(x) - f(x)| < \epsilon/2 \quad \forall n \geq n_0, \forall x \in E$

So, $\sup |f_n(x) - f(x)| \leq \epsilon/2 < \epsilon \quad \forall n \geq n_0$

$\Rightarrow M_n < \epsilon \quad \forall n \geq n_0$

$\Rightarrow |M_n - 0| < \epsilon \quad \forall n \geq n_0$

i.e., $\lim_{n\to\infty} M_n = 0$

Conversely, suppose $\lim_{n\to\infty} M_n = 0$.

then for each $\epsilon > 0$ $\exists\, n_0 \in \mathbb{N}$ such that

$|M_n - 0| < \epsilon \quad \forall n \geq n_0$

$\Rightarrow M_n < \epsilon \quad \forall n \geq n_0$

$\Rightarrow |f_n(x) - f(x)| < \epsilon \quad \forall n \geq n_0 \;\forall x \in E$

$\Rightarrow |f_n(x) - f(x)| < \epsilon \quad \forall n \geq n_0 \quad \forall x \in E$

Hence $\{f_n\}$ converges uniformly to f on E

① $f_n(x) = x^n, \quad 0 \le x \le 1$

$$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$$

$M_n = \sup \{ |f_n(x) - f(x)|, x \in [0,1] \}$

$= \cancel{\sup \{ |x^n - f(x)|, x \in [0,1] \}}$

$= \sup \{ |x^n - 0|, x \in [0,1] \}$ → as at $x=1$ $f(x) = 1$ & $f_n(x) = 1$ so, $|f_n(x) - f(x)| = 0$

$= \sup \{ x^n, 0 \le x < 1 \}$

$= 1$

Here $M_n \not\to 0$ as $n \to \infty$

So, $f_n \not\to f$ uniformly.

② Suppose a sequence of funn's $\{f_n\}$ is defined by $f_n(x) = \frac{nx}{1+n^2x^2}, x \in [0,1]$.

For $x = 0$, $f_n(0) = 0$

For $0 < x \le 1$, $f_n(x) = \frac{nx}{1+n^2x^2} = \frac{x/n}{1/n^2 + x^2}$

$\therefore \lim_{n \to \infty} f_n(x) = 0$.

Therefore $f(x)$ is a limit funn, then $f(x) = 0 \; \forall x \in [0,1]$.

So, $\{f_n\}$ is point wise convergence on $[0,1]$.

Now, $M_n = \sup_{x \in [0,1]} |f_n(x) - f(x)|$

$= \sup_{x \in [0,1]} \frac{nx}{1+n^2x^2}$

$= \frac{n \cdot 1/n}{1 + n^2 \cdot 1/n^2} = 1/2$

$\therefore \lim_{n \to \infty} M_n = 1/2 \ne 0$

So, $\{f_n\}$ is not uniformly convergent.

$g(x) = \frac{nx}{1+n^2x^2}$

$g'(x) = n \frac{1 - n^2x^2}{(1+n^2x^2)^2}$

$g'(x) = 0$

$\Rightarrow 1 - n^2x^2 = 0 \Rightarrow x = \pm 1/n$

Since $x \in [0,1]$, $x = 1/n$.

also $g''(x) < 0$

or, ~~take~~ nx, $1/nx$ apply A.M. $\ge$ G.M

~~$nx+1$~~

(2) $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ on $\mathbb{R}$

(3) Let the sequence of fun^{n}s are given by $f_n(x) = nx e^{-nx^2}$, $x \geq 0$.

(4) $f_n(x) = \frac{x}{n}$ on $\mathbb{R}$, $n \in \mathbb{N}$

Sol^n:- $\lim_{n\to\infty} f_n(x) = 0$

$|f_n(x) - f(x)| = |\frac{x}{n}| < \varepsilon$

$|\frac{x}{n}| < \varepsilon$

$\Rightarrow n > \frac{|x|}{\varepsilon} = k$

$n_0 = [\frac{x}{\varepsilon}] + 1$

~~k is~~ n_0 is dependent on x.
So, is not convergent uniformly.

(5) $f_n(x) = x^n$ on $[0, 1/2]$

$\lim_{n\to\infty} f_n(x) = 0$

$|f_n(x) - f(x)| = |x^n| \leq (1/2)^n$

$M_n \to 0$ as $n \to \infty$

$\Rightarrow$ uniformly converges to 0.

(6) $f_n(x) = \frac{x}{1+nx^2}$, $x \in [0,1]$, for each $n \in \mathbb{N}$

(7) $f_n(x) = x^n e^{-nx}$, $x \geq 0$, $n \geq 1$.

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(8) Show that the sequence $\{f_n\}$, where

$$f_n(x) = x^n$$

is uniformly convergent on $[0,k]$, $k<1$ and only pointwise convergent on $[0,1]$.

Sol^n:- $f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$

Let $\epsilon > 0$.

We have $|f_n(x) - f(x)| = x^n < \epsilon$

If $\left(\frac{1}{x}\right)^n > 1/\epsilon$

or if $n > \log(1/\epsilon)/\log(1/x) \rightarrow n_0(x)$

Now $\max_{0 \le x \le k < 1} \left\{ \log(1/\epsilon)/\log(1/x) \right\}$

$= \log(1/\epsilon)/\log(1/k)$

[Since as x increases $\log(1/\epsilon)/\log(1/x)$ increases]

$x \le k$

$\cancel{\Rightarrow \log x > \log k}$

$1/x \ge 1/k$

$\Rightarrow \log(1/x) > \log(1/k)$

$\Rightarrow \log(1/\epsilon)/\log(1/x) < \frac{\log(1/\epsilon)}{\log(1/k)}$

Let $N = [\log(1/\epsilon)/\log(1/k)] + 1$

$\therefore |f_n(x) - f(x)| < \epsilon \quad \forall n \ge N,\ 0 < x < 1$

Again at $x = 0$

$|f_n(x) - f(x)| = 0 < \epsilon \quad \forall n \ge 1$.

Thus for any $\epsilon > 0$, $\exists$ n such that $\forall x \in [0,k]$, $k<1$.

$|f_n(x) - f(x)| < \epsilon \quad \forall n \ge N$.

Therefore the sequence is uniformly convergent on $[0,k]$, $k<1$, However the number $\log(1/\varepsilon)/\log(1/x) \to \infty$

as $x \to 1$, so it's not possible to find any integer N such that

$|f_n(x) - f(x)| < \varepsilon \quad \forall n \geq N$ and all x in $[0,1]$.

Hence the sequence is not uniformly convergent on any interval containing 1 and in particular on $[0,1]$.

9. Show that the sequence $\{f_n\}$, where $f_n(x) = \frac{1}{x+n}$ is uniformly convergent in any interval $[0,b]$, $b>0$.

<u>Sol^n:-</u> The limit fun^n is

$$f(x) = \lim_{n\to\infty} \frac{1}{x+n} = 0 \quad \forall x \in [0,b]$$

So, the sequence converges pt.wise to 0.

For any $\varepsilon > 0$,

$$|f_n(x) - f(x)| = \frac{1}{x+n} < \varepsilon$$

if $n > 1/\varepsilon - x$, which decreases with x, the maxm value being $1/\varepsilon$.

Let $N = [1/\varepsilon] + 1$

So, for every $\varepsilon > 0 \ \exists N$ s.t.

$$|f_n(x) - f(x)| < \varepsilon, \ \forall n \geq N.$$

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(9) ~~Show that~~ 0 is a point of ~~non-uniform~~

(10) Show that the sequence $\{f_n\}$, where $f_n(x) = nx e^{-nx^2}$, $x \geq 0$ is not uniformly convergent on $[0, k]$, $k > 0$.

(11) Show that the sequence $\{f_n\}$, where $f_n(x) = \frac{x}{1+nx^2}$, x is real converges uniformly on any closed interval I.

$\Rightarrow f(x) = \lim_{n\to\infty} f_n(x) = 0 \ \forall x$

$$M_n = \sup_{x\in I} |f_n(x) - f(x)| = \sup_{x \in I} \left|\frac{x}{1+nx^2}\right|$$

$$= \frac{1/\sqrt{n}}{1+n/n} = \frac{1}{2\sqrt{n}} \to 0 \text{ as } n\to\infty$$

$$u(x) = \frac{x}{1+nx^2}$$

$$u'(x) = \frac{1-nx^2}{(1+nx^2)^2} = 0 \Rightarrow x = \frac{1}{\sqrt{n}}$$

Hence $\{f_n\}$ converges uniformly on I.

(12) Show that the sequence $\{f_n\}$, where $f_n(x) = x^{n-1}(1-x)$ converges uniformly in the interval $[0,1]$.

(13) Show that $f_n : E \to \mathbb{R}$, $E = (0,1)$ is defined by $f_n(x) = \frac{n}{nx+1}$ is not uniformly converge however it is convergent uniformly on every interval $[a, 1)$ with $0 < a < 1$.

(15) Check whether $\{f_n\}$, where $f_n(x) = 1 - \frac{x^n}{n}$, $x \in [0,1]$ is uniform convergent or not.

(16) $f_n(x) = \frac{x}{1+n x^2}$, $x \in [0,1]$

(17) $f_n(x) = n x (1-x)^n$, $x \in [0,1]$

(18) $f_n(x) = n^2 e^{-nx}$, $x \in [0,\infty)$

(19) $f_n(x) = \frac{x}{n+x^2}$, $x \in [0,1]$

(20) $f_n(x) = n^2 x^2 e^{-nx}$, $x \in [0,\infty)$

(21) $f_n(x) = x^2 e^{-nx}$, $x \in [0,\infty)$

(22) $f_n(x) = x e^{-nx}$, $x \in [0,\infty)$

(23) $f_n(x) = \frac{n^2 x}{1+n^4 x^2}$, $x \in [0,1]$

(24) $f_n(x) = \frac{nx}{1+n^3 x}$, $x \in [0,1]$

(25) For each $n \in \mathbb{N}$, let

$$f_n(x) = \begin{cases} nx^2, & 0 \le x \le 1/n \\ x, & 1/n < x \le 1 \end{cases}$$

~~(i)~~ show that the sequence is uniform convergen on $[0,1]$

(26) $f_n(x) = n^2 x (1-x^2)^n$, $0 \le x \le 1$

~~(27) $f_n(x) = \log(n^2+x^2)$, $x \in \mathbb{R}$~~

(27) $f_n(x) = x + \frac{x}{n}$, $x \in \mathbb{R}$

(28) $f_n(x) = e^{-nx}$, $x \ge 0$.