UG CBCS 2ND Sem.
Subject: Mathematics (Hons.)
Paper: C4T. / Unit- IV.

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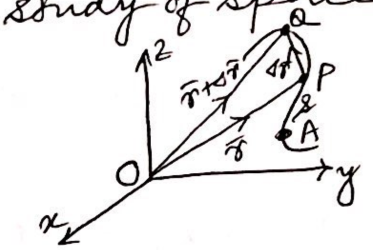
Vector Calculus.

Differential Geometry involves a study of space curves and space surfaces.



Let $\bar{r} = \bar{f}(s)$ be a space curve.

<u>Unit tangent vector</u>: $\hat{t} = \frac{d\bar{r}}{ds}$.

<u>Unit Principal normal</u>: Since $\hat{t}\cdot\hat{t} = 1$, hence

$$\frac{d}{ds}(\hat{t}\cdot\hat{t}) = 0 \Rightarrow 2\hat{t}\cdot\frac{d\hat{t}}{ds} = 0 \Rightarrow \hat{t}\cdot\frac{d\hat{t}}{ds} = 0$$

i.e., $\frac{d\hat{t}}{ds}$, if not zero, must be perpendicular to $\hat{t}$.

If $\hat{n}$ is a unit vector in the direction of $\frac{d\hat{t}}{ds}$, then $\frac{d\hat{t}}{ds}$ is parallel to $\hat{n}$, i.e., $\frac{d\hat{t}}{ds} = K\hat{n}$, $K(\geq 0)$ is said to be the curvature of the curve at P.

$\hat{n}$ is called unit principal normal.

<u>Unit Binormal</u>: Let $\hat{b} = \hat{t}\times\hat{n}$ such that $\hat{t}, \hat{n}, \hat{b}$ form a right-handed system of orthogonal unit vectors. A directed line through the point P in the direction of $\hat{b}$ is called binormal to the curve.

$$\text{Tangent Vector at } P = \frac{d\bar{r}}{dt} \quad [\text{If } \bar{r} = \bar{f}(t) \text{ be the space curve}]$$

$$\text{Unit tangent vector at } P = \frac{d\bar{r}}{ds} \equiv \hat{t} = \frac{\frac{d\bar{r}}{dt}}{\left|\frac{d\bar{r}}{dt}\right|}$$

$$\text{Principal Normal at } P = \frac{d\hat{t}}{ds}$$

$$\text{Unit Principal Normal at } P = \frac{\frac{d\hat{t}}{ds}}{\left|\frac{d\hat{t}}{ds}\right|} = \frac{\frac{d\hat{t}}{ds}}{K} = \hat{n}.$$

$$\text{Unit Binormal at } P = \hat{b} = \hat{t}\times\hat{n}.$$

Eqn. of the tangent line at $P = (\bar{R} - \bar{r})\times\frac{d\bar{r}}{ds} = \bar{0}$

" " " normal " at $P = (\bar{R} - \bar{r})\times\hat{n} = \bar{0}$

" " " binormal " at $P = (\bar{R} - \bar{r})\times\hat{b} = 0$

where $\bar{R}$ is the current vector on the respective lines.

Osculating Plane: Containing $\hat{t}$ and $\hat{n}$.

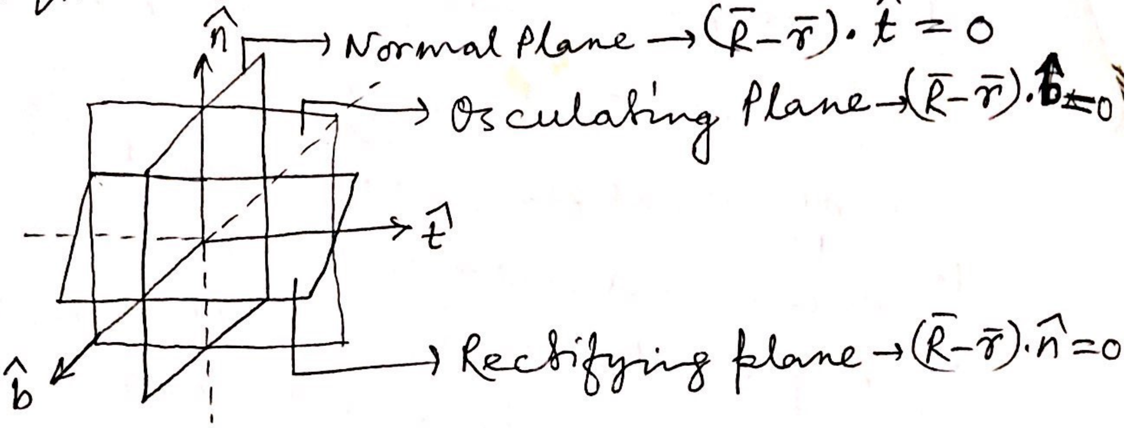
$\therefore$ Eqn is $(\bar{R}-\bar{r})\cdot\hat{b}=0$

Normal Plane: Containing $\hat{n}$ and $\hat{b}$.

Eqn is $(\bar{R}-\bar{r})\cdot\hat{t}=0$.

Rectifying plane: containing $\hat{t}$ and $\hat{b}$.

Eqn. is $(\bar{R}-\bar{r})\cdot\hat{n}=0$



Prove the Frenet-Serret formulae:

(i) $\frac{d\hat{t}}{ds}=\kappa\hat{n}$, (ii) $\frac{d\hat{b}}{ds}=-\tau\hat{n}$, (iii) $\frac{d\hat{n}}{ds}=\tau\hat{b}-\kappa\hat{t}$.

Proof: (i) Since $\hat{t}\cdot\hat{t}=1 \Rightarrow \frac{d}{ds}(\hat{t}\cdot\hat{t})=0 \Rightarrow \hat{t}\cdot\frac{d\hat{t}}{ds}=0$

$\Rightarrow \frac{d\hat{t}}{ds}$ is $\perp^r$ to $\hat{t}$.

If $\hat{n}$ is a unit vector in the direction $\frac{d\hat{t}}{ds}$, then

$\frac{d\hat{t}}{ds}=\kappa\hat{n}$, κ is the curvature, $\rho=1/\kappa$, radius of curvature

(ii) Let $\hat{b}=\hat{t}\times\hat{n}$, so that $\frac{d\hat{b}}{ds}=\hat{t}\times\frac{d\hat{n}}{ds}+\frac{d\hat{t}}{ds}\times\hat{n}$

$=\hat{t}\times\frac{d\hat{n}}{ds}+\kappa\hat{n}\times\hat{n}$

$\therefore \frac{d\hat{b}}{ds}=\hat{t}\times\frac{d\hat{n}}{ds}$

Ex

Now, $\hat{t}\cdot\frac{d\hat{b}}{ds}=\hat{t}\cdot\hat{t}\times\frac{d\hat{n}}{ds}=0 \Rightarrow \hat{t}$ is $\perp^r$ to $\frac{d\hat{b}}{ds}$.

Again from $\hat{b}\cdot\hat{b}=1$, we have $\hat{b}\cdot\frac{d\hat{b}}{ds}=0$.

Ex

$\Rightarrow \frac{d\hat{b}}{ds}$ is $\perp^r$ to $\hat{b}$ and thus in the plane of $\hat{t}$ and $\hat{n}$. Also $\frac{d\hat{b}}{ds}$ is $\perp^r$ to $\hat{t}$, then it must be $\|^l$ to $\hat{n}$. Then $\frac{d\hat{b}}{ds}=\tau\hat{n}$, τ the torsion

$\sigma=1/\tau$, the radius of torsion

) Since $\hat{t}, \hat{n}, \hat{b}$ form a right-handed system, so do $\hat{n}, \hat{b}, \hat{t}$, i.e., $\hat{n} = \hat{b} \times \hat{t}$

Then $\frac{d\hat{n}}{ds} = \hat{b} \times \frac{d\hat{t}}{ds} + \frac{d\hat{b}}{ds} \times \hat{t} = \hat{b} \times \kappa\hat{n} - \tau\hat{n} \times \hat{t}$
$= \underline{-\kappa\hat{t} + \tau\hat{b}}$ proved.

Let the parametric eqn of the space curve be given as:
$\bar{r} = \bar{f}(t) \longrightarrow (1)$, where $x = x(t)$, $y = y(t)$, $z = z(t)$

Differentiating w.r.t. t, we get

$$\dot{\bar{r}} = \frac{d\bar{r}}{dt} = \frac{d\bar{r}}{ds} \cdot \frac{ds}{dt} = \hat{t}\dot{s} \longrightarrow (2)$$

It follows, that $|\dot{\bar{r}}| = |\dot{s}| \longrightarrow (3)$

Again, differentiating (2) w.r.t. t ~~to (2)~~, we obtain,

$$\ddot{\bar{r}} = \frac{d\hat{t}}{dt}\dot{s} + \hat{t}\ddot{s} = \frac{d\hat{t}}{ds} \cdot \dot{s}^2 + \hat{t}\ddot{s} = \kappa\hat{n}\dot{s}^2 + \hat{t}\ddot{s} \longrightarrow (4)$$

And finally differentiating (4) w.r.t. t, we get

$$\dddot{\bar{r}} = \dot{\kappa}\hat{n}\dot{s}^2 + \kappa\frac{d\hat{n}}{ds}\dot{s}^3 + 2\kappa\hat{n}\dot{s}\ddot{s} + \frac{d\hat{t}}{ds}\dot{s}\ddot{s} + \hat{t}\dddot{s}$$
$$= \dot{\kappa}\hat{n}\dot{s}^2 + \kappa(\tau\hat{b} - \kappa\hat{t})\dot{s}^3 + 2\kappa\hat{n}\dot{s}\ddot{s} + \kappa\hat{n}\dot{s}\ddot{s} + \hat{t}\dddot{s}$$
$$= \dddot{s}\hat{t} + (3\dot{s}\ddot{s}\kappa + \dot{\kappa}\dot{s}^2)\hat{n} + \dot{s}^3\kappa(+\tau\hat{b} - \kappa\hat{t})$$
$$= \dddot{s}\hat{t} + (3\dot{s}\ddot{s}\kappa + \dot{s}^2\dot{\kappa})\hat{n} + \dot{s}^3\kappa\tau\hat{b} \longrightarrow (5)$$
$$= (\dddot{s} - \dot{s}^3\kappa^2)\hat{t} + (3\dot{s}\ddot{s}\kappa + \dot{s}^2\dot{\kappa})\hat{n} + \dot{s}^3\kappa\tau\hat{b}$$

Hence, we obtain,

$$\dot{\bar{r}} \times \ddot{\bar{r}} = \kappa\dot{s}^3\hat{b} \longrightarrow (6) \Rightarrow \kappa = \frac{|\dot{\bar{r}} \times \ddot{\bar{r}}|}{\dot{s}^3} = \frac{|\dot{\bar{r}} \times \ddot{\bar{r}}|}{|\dot{\bar{r}}|^3}$$

and $\dot{\bar{r}} \times \ddot{\bar{r}} \cdot \dddot{\bar{r}} = [\dot{\bar{r}}\,\ddot{\bar{r}}\,\dddot{\bar{r}}] = \kappa^2\dot{s}^6\tau \longrightarrow (7) \Rightarrow \tau = \frac{[\dot{\bar{r}}\,\ddot{\bar{r}}\,\dddot{\bar{r}}]}{|\dot{\bar{r}} \times \ddot{\bar{r}}|^2}$

Eqn. of Osculating plane : $(\bar{R} - \bar{r}) \cdot \dot{\bar{r}} \times \ddot{\bar{r}} = 0$

" " Normal plane : $(\bar{R} - \bar{r}) \cdot \dot{\bar{r}} = 0$

" " Rectifying plane : $(\bar{R} - \bar{r}) \cdot (\dot{\bar{r}} \times \ddot{\bar{r}}) \times \dot{\bar{r}} = 0$

For a plane curve : $x = x(t)$, $y = y(t)$

$$\bar{r} = x\hat{i} + y\hat{j}, \quad \dot{\bar{r}} = \dot{x}\hat{i} + \dot{y}\hat{j}, \quad \ddot{\bar{r}} = \ddot{x}\hat{i} + \ddot{y}\hat{j}$$

$$\kappa = \frac{|\dot{\bar{r}} \times \ddot{\bar{r}}|}{|\dot{\bar{r}}|^3} = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}, \quad \tau = 0.$$

In particular, if the curve has the cartesian eqn $y = f(x)$, we take $x = t$, $y = f(t)$ as parametric form $\therefore \kappa = \frac{|\ddot{y}|}{(1 + \dot{y}^2)^{3/2}}$

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▦ Three fundamental planes associated with every ordinary point(P) of a smooth curve are :

(i) Osculating plane → A plain containing the tangent and principal normal at P.

(ii) Normal Plane → A plane containing principal normal and the binormal at P.

(iii) Rectifying plane → A plane containing the tangent and the binormal at P.

These 3 planes are mutually perpendicular.

If $\bar{r}$ be the p.v. of any point on the plane and $\bar{r}_0$ be the p.v. of the specified point(P) of the smooth curve, then

The equation of Osculating plane : $(\bar{r}-\bar{r}_0)\cdot\hat{b}=0 \longrightarrow (1)$

" " " Normal " : $(\bar{r}-\bar{r}_0)\cdot\hat{t}=0 \longrightarrow (2)$

" " " Rectifying " : $(\bar{r}-\bar{r}_0)\cdot\hat{n}=0 \longrightarrow (3)$.

▦ Find the expressions for curvature (κ) and torsion(τ) of the curve whose vector equation is given by

$$\bar{r}=\bar{r}(t) \longrightarrow (1)$$

$$\dot{\bar{r}}=\bar{r}'\dot{s}=\hat{t}\dot{s} \to (2), \quad |\dot{\bar{r}}|=|\dot{s}| \longrightarrow (3), \text{ where } \dot{\bar{r}}=\frac{d\bar{r}}{dt}, \bar{r}'=\frac{d\bar{r}}{ds}.$$

$$\ddot{\bar{r}}=\frac{d\hat{t}}{dt}\dot{s}+\hat{t}\ddot{s}=\frac{d\hat{t}}{ds}\dot{s}^2+\hat{t}\ddot{s}=\kappa\hat{n}\dot{s}^2+\hat{t}\ddot{s}$$

$$\therefore \ddot{\bar{r}}=\kappa\dot{s}^2\hat{n}+\ddot{s}\hat{t} \longrightarrow (4)$$

$$\dddot{\bar{r}}=\dot{\kappa}\dot{s}^2\hat{n}+2\kappa\dot{s}\ddot{s}\hat{n}+\kappa\dot{s}^3\frac{d\hat{n}}{ds}+\dddot{s}\hat{t}+\ddot{s}\dot{s}\frac{d\hat{t}}{ds}$$

$$=\dot{\kappa}\dot{s}^2\hat{n}+2\kappa\dot{s}\ddot{s}\hat{n}+\kappa\dot{s}^3(\tau\hat{b}-\kappa\hat{t})+\dddot{s}\hat{t}+\ddot{s}\dot{s}\kappa\hat{n}$$

$$=\dot{\kappa}\dot{s}^2\hat{n}+2\kappa\dot{s}\ddot{s}\hat{n}+\kappa\dot{s}^3\tau\hat{b}... \longrightarrow (5)$$

$$\text{or, } \dddot{\bar{r}}=(\dddot{s}-\kappa^2\dot{s}^3)\hat{t}+(3\kappa\dot{s}\ddot{s}+\dot{\kappa}\dot{s}^2)\hat{n}+\kappa\tau\dot{s}^3\hat{b}$$

$$\text{Now } \dot{\bar{r}}\times\ddot{\bar{r}}=\kappa\dot{s}^3\hat{b} \longrightarrow (6)$$

$$\text{and } \dot{\bar{r}}\times\ddot{\bar{r}}\cdot\dddot{\bar{r}}=\kappa^2\tau\dot{s}^6 \longrightarrow (7)$$

$$\therefore \kappa=\frac{|\dot{\bar{r}}\times\ddot{\bar{r}}|}{|\dot{s}^3|}=\frac{|\dot{\bar{r}}\times\ddot{\bar{r}}|}{|\dot{\bar{r}}|^3} \longrightarrow (8)$$

$$\text{and } \tau=\frac{\dot{\bar{r}}\times\ddot{\bar{r}}\cdot\dddot{\bar{r}}}{\kappa^2\dot{s}^6}=\frac{[\dot{\bar{r}}\,\ddot{\bar{r}}\,\dddot{\bar{r}}]}{|\dot{\bar{r}}\times\ddot{\bar{r}}|^2} \longrightarrow (9)$$

<u>Note</u>: $\hat{t}$ is in the direction of $\dot{\bar{r}}$

$\hat{n}$ " " " " $(\dot{\bar{r}}\times\ddot{\bar{r}})\times\dot{\bar{r}}$

$\hat{b}$ " " " " $\dot{\bar{r}}\times\ddot{\bar{r}}$

Therefore equations (1), (2), (3) of 3 fundamental planes become : $(\bar{r}-\bar{r}_0)\cdot\dot{\bar{r}}\times\ddot{\bar{r}}=0 \longrightarrow$ Osculating plane

$(\bar{r}-\bar{r}_0)\cdot\dot{\bar{r}}=0 \longrightarrow$ Normal Plane

$(\bar{r}-\bar{r}_0)\cdot(\dot{\bar{r}}\times\ddot{\bar{r}})\times\dot{\bar{r}}=0 \longrightarrow$ Rectifying plane.

▦ Prove the relations: $\kappa = |\bar{r}''|$, $\bar{r}' \times \bar{r}'' = \kappa \hat{b}$, $[\bar{r}'\bar{r}''\bar{r}'''] = \kappa^2 \tau$, $\bar{r}''' = \kappa(\tau\hat{b} - \kappa\hat{t}) + \kappa'\hat{n}$; for the space curve $\bar{r} = \bar{r}(s)$, s being the arc-length of the point (P) from some fixed point of the curve.

$$\bar{r}' = \frac{d\bar{r}}{ds} = \hat{t}, \quad \bar{r}'' = \frac{d\hat{t}}{ds} = \kappa\hat{n} \Rightarrow \kappa = |\bar{r}''|.$$

Now $\bar{r}' \times \bar{r}'' = \hat{t} \times \kappa\hat{n} = \kappa\hat{b}$

Also $\bar{r}''' = \kappa'\hat{n} + \kappa\frac{d\hat{n}}{ds} = \kappa(\tau\hat{b} - \kappa\hat{t}) + \kappa'\hat{n}$.

$\bar{r}' \times \bar{r}'' \cdot \bar{r}''' = \kappa^2\tau \Rightarrow [\bar{r}'\bar{r}''\bar{r}'''] = \kappa^2\tau$.

$\therefore \kappa = |\bar{r}''|$, $\tau = \dfrac{[\bar{r}'\bar{r}''\bar{r}''']}{|\bar{r}''|^2}$, $\rho = \dfrac{1}{\kappa} \rightarrow$ radius of curvature

$\sigma = \dfrac{1}{\tau} \rightarrow$ " " torsion

▦ For a plane curve: $x = x(t)$, $y = y(t)$, obtain κ and τ.

$\bar{r} = x\hat{i} + y\hat{j}$; $\dot{\bar{r}} = \dot{x}\hat{i} + \dot{y}\hat{j}$; $\ddot{\bar{r}} = \ddot{x}\hat{i} + \ddot{y}\hat{j}$; $|\dot{\bar{r}} \times \ddot{\bar{r}}| = |\dot{x}\ddot{y} - \ddot{x}\dot{y}|$

$$\kappa = \frac{|\dot{\bar{r}} \times \ddot{\bar{r}}|}{|\dot{\bar{r}}|^3} = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

$\dddot{\bar{r}} = \dddot{x}\hat{i} + \dddot{y}\hat{j}$, $\dot{\bar{r}} \times \ddot{\bar{r}} \cdot \dddot{\bar{r}} = (\dot{x}\ddot{y} - \ddot{x}\dot{y})\hat{k} \cdot (\dddot{x}\hat{i} + \dddot{y}\hat{j})$
$= 0$

$$\therefore \tau = \frac{[\dot{\bar{r}}\ddot{\bar{r}}\dddot{\bar{r}}]}{|\dot{\bar{r}} \times \ddot{\bar{r}}|^2} = 0.$$

In particular, if $y = f(x)$; $\left.\begin{matrix} x = t \\ y = f(t) \end{matrix}\right\}$, then

$$\kappa = \frac{|\ddot{y}|}{(1 + \dot{y}^2)^{3/2}}.$$

Equation of the tangent line: $(\bar{r} - \bar{r}_0) \times \dot{\bar{r}} = \bar{0}$, (at $\bar{r}_0$).

or, $(\bar{r} - \bar{r}_0) \times \hat{t} = \bar{0}$, (at $\bar{r}_0$).

Equation of the normal plane: $(\bar{r} - \bar{r}_0) \cdot \dot{\bar{r}} = 0$ (at $\bar{r}_0$).

Ex (53). Show that the Frenet-Serret formulae can be written in the form $\frac{d\hat{t}}{ds} = \bar{\omega}\times\hat{t}$, $\frac{d\hat{n}}{ds} = \bar{\omega}\times\hat{n}$, $\frac{d\hat{b}}{ds} = \bar{\omega}\times\hat{b}$ and determine $\bar{\omega}$.

Frenet-Serret formulae:

$$\frac{d\hat{t}}{ds} = \kappa\hat{n} \longrightarrow ①, \quad \frac{d\hat{n}}{ds} = \tau\hat{b} - \kappa\hat{t} \longrightarrow ②, \quad \frac{d\hat{b}}{ds} = -\tau\hat{n}.$$

From ②, $\frac{d\hat{n}}{ds} = \tau\hat{b} - \kappa\hat{t} = \tau(\hat{t}\times\hat{n}) + \kappa(\hat{b}\times\hat{n}) = (\tau\hat{t}+\kappa\hat{b})\times\hat{n}$

$= \bar{\omega}\times\hat{n}$, where $\bar{\omega} = \tau\hat{t}+\kappa\hat{b}$.

From ①, $\frac{d\hat{t}}{ds} = \kappa\hat{n} = \kappa(\hat{b}\times\hat{t}) = (\kappa\hat{b}+\tau\hat{t})\times\hat{t}$ $[\because \hat{t}\times\hat{t} = \bar{0}]$

$= \bar{\omega}\times\hat{t}$, where $\bar{\omega} = \tau\hat{t}+\kappa\hat{b}$.

From ③, $\frac{d\hat{b}}{ds} = -\tau\hat{n} = \tau(\hat{t}\times\hat{b}) = (\tau\hat{t}+\kappa\hat{b})\times\hat{b}$ $[\because \hat{b}\times\hat{b} = \bar{0}]$

$= \bar{\omega}\times\hat{b}$.

$\therefore$ $\underline{\bar{\omega} = \tau\hat{t}+\kappa\hat{b}}$.

Ex. 66. A particle moves along a space curve $\bar{r}=\bar{r}(t)$ with velocity $\bar{v}$. Show that its accl^n $\bar{a}$ is given by $\bar{a} = \frac{dv}{dt}\hat{t} + \frac{v^2}{\rho}\hat{n}$. ρ is the radius of curvature.

Let $\bar{r}$ be the p.v. of the particle at any time t.

$\therefore$ $\bar{v} = \frac{d\bar{r}}{dt} = \frac{d\bar{r}}{ds}\frac{ds}{dt} = \hat{t}v$

$\bar{a} = \frac{d}{dt}(\bar{v}) = \frac{d}{dt}(\hat{t}v) = \frac{d\hat{t}}{dt}v + \hat{t}\frac{dv}{dt}$

$= \hat{t}\frac{dv}{dt} + \frac{d\hat{t}}{ds}\cdot\frac{ds}{dt}\cdot v = \hat{t}\frac{dv}{dt} + v^2\kappa\hat{n}$

$= \hat{t}\frac{dv}{dt} + \frac{v^2}{\rho}\hat{n}$.

Ex. 66. A particle moves along the curve $\bar{r} = (t^3-4t)\hat{i} + (t^2+4t)\hat{j} + (8t^2-3t^3)\hat{k}$, t is the time. Find the magnitudes of the tangential and normal components of its accl^n when $t=2$.

$\bar{a} = \hat{t}\frac{dv}{dt} + \frac{v^2}{\rho}\hat{n}$. tangential component $= \frac{dv}{dt}$

normal " $= v^2/\rho$.

$\bar{v} = \frac{d\bar{r}}{dt} = (3t^2-4)\hat{i} + (2t+4)\hat{j} + (16t-9t^2)\hat{k}$.

$v = |\bar{v}| = \left[(3t^2-4)^2 + (2t+4)^2 + (16t-9t^2)^2\right]^{1/2} = 12$, at $t=$

$\frac{dv}{dt} = \frac{1}{2v}\left\{2(3t^2-4)\times 6t + 2(2t+4)\times 2 + 2(16t-9t^2)\times(16-18t)\right\}$

$= \frac{1}{2\times 12}\{24\times 8 + 4\times 8 + 2\times(-4)\times(-20)\} = \frac{4\times 8\times 12}{2\times 12} = \underline{16}$

Now $\rho = \frac{|\dot{\bar{r}}|^3}{|\dot{\bar{r}}\times\ddot{\bar{r}}|} = \frac{v^3}{|\dot{\bar{r}}\times\ddot{\bar{r}}|}$

a, $\frac{v^2}{\rho} = \frac{|\dot{\bar{r}}\times\ddot{\bar{r}}|}{v}$.

Now $\dot{\bar{r}} = (3t^2-4,\ 2t+4,\ 16t-9t^2) = (8,8,-4)$ at $t=2$

$\ddot{\bar{r}} = (6t,\ 2,\ 16-18t) = (12,2,-20)$ at $t=2$.

$\therefore\ \dot{\bar{r}}\times\ddot{\bar{r}} = (-152, 112, -80) = 8(-19, 14, -10)$

a, $|\dot{\bar{r}}\times\ddot{\bar{r}}| = 8\sqrt{19^2+14^2+10^2} = 8\times\sqrt{361+196+100}$

$= 8\sqrt{657}$

a, $\frac{v^2}{\rho} = \frac{\cancel{8}^{2}\sqrt{657}}{\cancel{12}_{3}} = \frac{2\times\cancel{3}\sqrt{73}}{\cancel{3}} = 2\sqrt{73}$.

69. Prove that the accleration vector of a particle moving along a space curve always lies in the osculating plane.

We have $\bar{a} = \frac{dv}{dt}\hat{t} + \frac{v^2}{\rho}\hat{n}$

a, $\bar{a}\cdot\hat{t}\times\hat{n} = \frac{dv}{dt}\hat{t}\cdot\hat{t}\times\hat{n} + \frac{v^2}{\rho}\hat{n}\cdot\hat{t}\times\hat{n}$

$= 0$

$\therefore\ \bar{a}, \hat{t}, \hat{n}$ lie on the same plane

i.e, $\bar{a}$ lies in the plane containing $\hat{t}$ & $\hat{n}$, called the osculating plane.

64. Find equations for the tangent plane & normal line to the surface $4z = x^2-y^2$ at $(3,1,2)$.

~~Let $\phi(x,y,z) = x^2-y^2-4z = 0$~~

~~$\therefore\ \nabla\phi$~~

Let $x=u,\ y=v,\ z=\frac{u^2-v^2}{4}$; $u=3,\ v=1$.

$\therefore\ \bar{r} = \bar{r}(u,v) = u\hat{i} + v\hat{j} + \frac{u^2-v^2}{4}\hat{k}$

$\frac{\partial\bar{r}}{\partial u}\times\frac{\partial\bar{r}}{\partial v} = \left(\hat{i}+\frac{u}{2}\hat{k}\right)\times\left(\hat{j}-\frac{v}{2}\hat{k}\right) = \hat{k} + \frac{u}{2}(-\hat{i}) + \frac{v}{2}\hat{j}$

$= \left(-\frac{u}{2}, \frac{v}{2}, 1\right) = \left(-\frac{3}{2}, \frac{1}{2}, 1\right)$ at $(3,1,2)$

$\bar{n} = \left(-\frac{3}{2}, \frac{1}{2}, 1\right)$

Eqn of tangent plane: $(x-3)\left(-\frac{3}{2}\right) + (y-1)\frac{1}{2} + (z-2)\times 1 = 0$

a, $-3x+y+2z+4=0$

Space curves

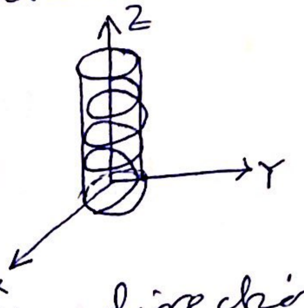
Parametric eqns of a space curve:

$$x = x(t),\ y = y(t),\ z = z(t), \quad a \le t \le b.$$

Vector form of a space curve:

$$\bar{r} = \bar{r}(t) \equiv \bar{f}(t)$$

① **Circular helix:-** The parametric eqns
$x = a\cos t,\ y = a\sin t,\ z = bt\ (b \neq 0)$.
represent a curve on the surface of a circular cylinder and cutting the generators at a constant angle. This curve is known as a circular helix.



This curve is twisted, i.e., does not lie in a plane.
This curve is described by a moving point P which is subjected to a constant rotation together with a translation in the direction of the axis of rotation.

Verify for circular helix, $\kappa = \dfrac{a}{a^2+b^2}$, $\tau = \dfrac{b}{a^2+b^2}$.

$x = a\cos t,\quad y = a\sin t,\quad z = bt.$

$$\kappa = \frac{|\dot{\bar{r}} \times \ddot{\bar{r}}|}{|\dot{\bar{r}}|^3}, \quad \tau = \frac{[\dot{\bar{r}}\ \ddot{\bar{r}}\ \dddot{\bar{r}}]}{|\dot{\bar{r}} \times \ddot{\bar{r}}|^2}$$

$$\bar{r}(t) = x\hat{i} + y\hat{j} + z\hat{k} = a\cos t\ \hat{i} + a\sin t\ \hat{j} + bt\ \hat{k}$$

$$\dot{\bar{r}} = -a\sin t\ \hat{i} + a\cos t\ \hat{j} + b\hat{k}$$

$$\ddot{\bar{r}} = -a\cos t\ \hat{i} - a\sin t\ \hat{j}$$

$$\dddot{\bar{r}} = a\sin t\ \hat{i} - a\cos t\ \hat{j}$$

$$\kappa = \frac{|\dot{\bar{r}} \times \ddot{\bar{r}}|}{|\dot{\bar{r}}|^3} = \frac{\sqrt{a^2(a^2+b^2)}}{(a^2+b^2)^{3/2}}$$

$$\kappa = \frac{a}{a^2+b^2}$$

$$\tau = \frac{[\dot{\bar{r}}\ \ddot{\bar{r}}\ \dddot{\bar{r}}]}{|\dot{\bar{r}} \times \ddot{\bar{r}}|^2} = \frac{a^2 b}{a^2(a^2+b^2)} = \frac{b}{a^2+b^2}$$

$$\dot{\bar{r}} \times \ddot{\bar{r}} = a^2\sin^2 t\ \hat{k} + a^2\cos^2 t\ \hat{k} - ab\cos t\ \hat{j} + ab\sin t\ \hat{i}$$

$$= ab\sin t\ \hat{i} - ab\cos t\ \hat{j} + a^2\hat{k}$$

$$|\dot{\bar{r}} \times \ddot{\bar{r}}|^2 = a^2b^2 + a^4 = a^2(a^2+b^2)$$

$$|\dot{\bar{r}}| = \sqrt{a^2+b^2}$$

$$[\dot{\bar{r}}\ \ddot{\bar{r}}\ \dddot{\bar{r}}] = \dot{\bar{r}} \times \ddot{\bar{r}} \cdot \dddot{\bar{r}}$$

$$= a^2 b\sin^2 t + a^2 b\cos^2 t$$

$$= a^2 b.$$

Also find the length of the curve measured from $t=0$ to some point t is $t\sqrt{a^2+b^2}$.

We have $\dot{\bar{r}} = \dfrac{d\bar{r}}{dt} = \dfrac{d\bar{r}}{ds}\cdot\dfrac{ds}{dt} = \hat{t}\dfrac{ds}{dt} \Rightarrow |\dot{\bar{r}}| = \left|\dfrac{ds}{dt}\right|$.

$$\therefore \left|\frac{ds}{dt}\right| = \sqrt{a^2+b^2} \Rightarrow \int_0 ds = \int_0 \sqrt{a^2+b^2}\, dt \Rightarrow s = t\sqrt{a^2+b^2}$$

Ex: If the tangent to a curve makes a constant angle α with a fixed line, then $\kappa = \pm \tau \tan\alpha$, and conversely show that if κ/τ is constant the tangent makes a constant angle with a fixed direction.

Let $\hat{e}$ be the unit vector parallel to the fixed line so that by the given condition.

$$\hat{t}\cdot\hat{e} = \cos\alpha \longrightarrow (1)$$

Differentiating w.r.t. s (arc-length), we get

$$\frac{d\hat{t}}{ds}\cdot\hat{e} = 0 \longrightarrow (2) \quad \left[\text{Since the direction of } \hat{e} \text{ is fixed}\right].$$

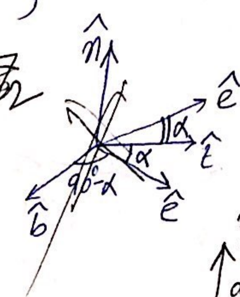
By Frenet-Serret's formula, we have

$$\kappa\hat{n}\cdot\hat{e} = 0 \left[\text{using (2) \& } \frac{d\hat{t}}{ds} = \kappa\hat{n}\right]$$

a, $\hat{n}\cdot\hat{e} = 0 \Rightarrow \hat{n}$ is $\perp$ to $\hat{e}$

and since $\hat{n} = \hat{b}\times\hat{t}$, & $\hat{n}$ is $\perp$ to $\hat{e}$, therefore $\hat{b}, \hat{t}, \hat{e}$ are coplanar.

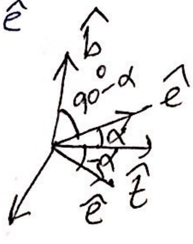
so that $\hat{b}\cdot\hat{e} = \cos(90^\circ \pm \alpha)$
$= \pm\sin\alpha$



$\hat{n}\cdot\hat{e} = 0$, differentiating w.r.t. s,

$$\frac{d\hat{n}}{ds}\cdot\hat{e} = 0$$

a, $(\tau\hat{b} - \kappa\hat{t})\cdot\hat{e} = 0 \quad \Rightarrow \tau\hat{b}\cdot\hat{e} - \kappa\hat{t}\cdot\hat{e} = 0$
$\Rightarrow \tau(\pm\sin\alpha) - \kappa\cos\alpha = 0.$
$\Rightarrow \kappa = \pm\tau\tan\alpha.$



64

Conversely, let $\kappa = c\tau$, where c is some scalar constant

Now, $\frac{d\hat{t}}{ds} = \kappa\hat{n} = \left(-\frac{\kappa}{\tau}\right)\frac{d\hat{b}}{ds} = (-c)\frac{d\hat{b}}{ds}$

Integrating we get, $\hat{t} = -c\hat{b} + \bar{d}$, where d is constant vector of integration,

~~Now, $\hat{t}\cdot\bar{d} = c\hat{b}\cdot\bar{d} + \bar{d}\cdot\bar{d}$~~

Now, $\hat{t}\cdot\hat{t} = \hat{t}\cdot(-c\hat{b} + \bar{d}) = c(\hat{t}\cdot\hat{b}) + \hat{t}\cdot\bar{d}$

a, $1 = c\cdot 0 + \hat{t}\cdot\bar{d} \Rightarrow \hat{t}\cdot\bar{d} = 1$

Hence the tangent makes a constant angle with the direction of the fixed vector $\bar{d}$.

proved.

Ex: Prove that a necessary and sufficient condition that a curve, $\bar{r} = \bar{f}(s)$ be a helix is that $\bar{r}'' \cdot \bar{r}''' \times \bar{r}'''' = 0$.

Soln: $\bar{r} = f(s) \longrightarrow (1)$

$\therefore \bar{r}' = \frac{d\bar{r}}{ds} = \hat{t} \longrightarrow (2)$

Now, $\frac{d\hat{t}}{ds} = \frac{d^2\bar{r}}{ds^2} = \bar{r}'' = \kappa\hat{n}$, [by Frenet-Serret's formula] $\longrightarrow (3)$.

$\bar{r}''' = \frac{d\kappa}{ds}\hat{n} + \kappa\frac{d\hat{n}}{ds} = \frac{d\kappa}{ds}\hat{n} + \kappa(\tau\hat{b} - \kappa\hat{t}) \longrightarrow (4)$.

$$\bar{r}'''' = \frac{d^2\kappa}{ds^2}\hat{n} + \frac{d\kappa}{ds}\frac{d\hat{n}}{ds} + \kappa\tau\frac{d\hat{b}}{ds} + \frac{d\kappa}{ds}\tau\hat{b} + \kappa\frac{d\tau}{ds}\hat{b} - 2\kappa\frac{d\kappa}{ds}\hat{t} - \kappa^2\frac{d\hat{t}}{ds}$$

$$= \frac{d^2\kappa}{ds^2}\hat{n} + \frac{d\kappa}{ds}(\tau\hat{b} - \kappa\hat{t}) + \kappa\tau(-\tau\hat{n}) + \frac{d\kappa}{ds}\tau\hat{b} + \kappa\frac{d\tau}{ds}\hat{b} - 2\kappa\frac{d\kappa}{ds}\hat{t} - \kappa^2(\kappa\hat{n}).$$

$$= -3\hat{t}\,\kappa\frac{d\kappa}{ds} + \left(\frac{d^2\kappa}{ds^2} - \kappa^3 - \kappa\tau^2\right)\hat{n} + \left(2\tau\frac{d\kappa}{ds} + \kappa\frac{d\tau}{ds}\right)\hat{b}.$$

$$\therefore \bar{r}''' \times \bar{r}'''' = +3\kappa\left(\frac{d\kappa}{ds}\right)^2\hat{b} + \left[2\tau\left(\frac{d\kappa}{ds}\right)^2 + \kappa\frac{d\kappa}{ds}\frac{d\tau}{ds}\right]\hat{t} - 3\kappa^2\tau\frac{d\kappa}{ds}\hat{n} - \left(\kappa\tau\frac{d^2\kappa}{ds^2} - \kappa^4\tau - \kappa^2\tau^3\right)\hat{t}$$
$$- \left(\kappa^2\frac{d^2\kappa}{ds^2} + \kappa^5 - \kappa^3\tau^2\right)\hat{b} + \kappa^2\left(2\tau\frac{d\kappa}{ds} + \kappa\frac{d\tau}{ds}\right)\hat{n}$$

$$\therefore \bar{r}'' \cdot \bar{r}''' \times \bar{r}'''' = -3\kappa^3\tau\frac{d\kappa}{ds} + 2\kappa^3\tau\frac{d\kappa}{ds} + \kappa^4\frac{d\tau}{ds}$$
$$= \left(-\tau\frac{d\kappa}{ds} + \kappa\frac{d\tau}{ds}\right)\kappa^3 = 0$$

or, $\frac{\kappa}{\tau} = \frac{d\kappa/ds}{\frac{d\tau}{ds}} = \frac{d\kappa}{d\tau}$

or, $\frac{d\left(\frac{\kappa}{\tau}\right)}{\left(\kappa/\tau\right)} = 0 \Rightarrow \log\left(\frac{\kappa}{\tau}\right) = \log c$.

$\Rightarrow \frac{\kappa}{\tau} = \text{constant}$.

$\therefore$ If, $\bar{r}'' \cdot \bar{r}''' \times \bar{r}'''' = 0$, then $\frac{\kappa}{\tau} = \text{Constant} \Rightarrow \bar{r} = \bar{f}(s)$ be a helix.

Now, if $\frac{\kappa}{\tau} = \text{Constant}$, then $\frac{d}{ds}\left(\frac{\kappa}{\tau}\right) = 0$

or, $\frac{\tau\frac{d\kappa}{ds} - \kappa\frac{d\tau}{ds}}{\tau^2} = 0$.

or, $\tau\frac{d\kappa}{ds} - \kappa\frac{\partial\tau}{ds} = 0$.

$\therefore \bar{r}'' \cdot \bar{r}''' \times \bar{r}'''' = \left(-\tau\frac{d\kappa}{ds} + \kappa\frac{\partial\tau}{ds}\right)\kappa^3 = 0$.

Ex: The displacement vector $\bar{r}$ at any time t of a moving point is

$$\bar{r} = \bar{a}\cos\omega t + \bar{b}\sin\omega t,$$

where $\bar{a}$ and $\bar{b}$ are constant vectors and ω is a scalar constant.

(a) Find the velocity $\vec{v}$ and show that $\bar{r} \times \bar{v}$ = a constant vector.

(b) Show that the acceleration is directed towards the origin and is proportional to the displacement.

(c) What is the physical interpretation of this motion?

Soln: (a) $\bar{r} = \bar{a}\cos\omega t + \bar{b}\sin\omega t$.

$$\bar{v} = \frac{d\bar{r}}{dt} = (-\bar{a}\sin\omega t + \bar{b}\cos\omega t)\,\omega$$

$$\bar{r} \times \bar{v} = (\bar{a}\cos\omega t + \bar{b}\sin\omega t) \times (-\bar{a}\sin\omega t + \bar{b}\cos\omega t)\,\omega$$

$$= \omega(\bar{a} \times \bar{b})\{\cos^2\omega t + \cancel{\bar{a}\times\bar{b}}\sin^2\omega t\}$$

$$\bar{r} \times \bar{v} = \omega(\bar{a} \times \bar{b}) \longrightarrow \text{a constant vector.}$$

(b) $$\bar{a} = \frac{d\bar{v}}{dt} = \frac{d^2\bar{r}}{dt^2} = \omega^2(-\bar{a}\cos\omega t - \bar{b}\sin\omega t) = -\omega^2\bar{r}$$

acceleration $\bar{a}$ is opposite to the direction of $\bar{r}$.
i.e, $\bar{a}$ is directed towards the origin.
and $\bar{a} \propto \bar{r}$

(c) Physical interpretation:
The motion is that of a particle moving on the circumference of a circle with constant angular speed ω. The force, directed towards the centre of the circle is the centripetal force.

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Q: Show that the acceleration $\bar{a}$ of a particle which travels along a space curve with vel. $\bar{v}$ is given by $\bar{a} = \frac{dv}{dt}\hat{t} + \frac{v^2}{\rho}\hat{n}$, where $\hat{t}$ is the unit tangent vector to the space curve, $\hat{n}$ is its unit principal normal, ρ is the radius of curvature $\rho = \frac{v^3}{|\bar{v}\times\bar{a}|}$

Mechanics :- A particle moves along the curve $\bar{r} = (t^3-4t)\hat{i} + (t^2+4t)\hat{j} + (8t^2-3t^3)\hat{k}$, where t is the time. Find the magnitudes of the tangential and normal components of its acceleration when $t=2$.

Solution : Velocity $\bar{v} = v\hat{t}$, $v = |\bar{v}|$.

Differentiating w.r.t. t, we get,

$$\bar{a} = \frac{d\bar{v}}{dt} = \frac{d}{dt}(v\hat{t}) = \frac{dv}{dt}\hat{t} + v\frac{d\hat{t}}{ds}\cdot\frac{ds}{dt} = \frac{dv}{dt}\hat{t} + v^2\cdot\kappa\hat{n}$$

$\therefore$ $\bar{a} = \frac{dv}{dt}\hat{t} + \frac{v^2}{\rho}\hat{n}$.

Magnitude of the tangential component of its acceleration $= \boxed{\frac{dv}{dt}}$

Magnitude of " normal " $= \boxed{\frac{v^2}{\rho}}$

Now, $|\bar{v}\times\bar{a}| = \left|v\frac{dv}{dt}\hat{t}\times\hat{t} + \frac{v^3}{\rho}\hat{t}\times\hat{n}\right| = \left|\frac{v^3}{\rho}\hat{b}\right| = \frac{v^3}{\rho}$.

$\therefore$ $\rho = v^3/|\bar{v}\times\bar{a}|$.

$$\bar{r} = (t^3-4t)\hat{i} + (t^2+4t)\hat{j} + (8t^2-3t^3)\hat{k}$$

$$\frac{d\bar{r}}{dt} = \dot{\bar{r}} = (3t^2-4)\hat{i} + (2t+4)\hat{j} + (16t-9t^2)\hat{k} = 8\hat{i} + 8\hat{j} - 4\hat{k}$$

Now, $\frac{d\bar{r}}{dt} = \dot{\bar{r}} = \frac{d\bar{r}}{ds}\cdot\frac{ds}{dt} = v\hat{t}$,

$\therefore$ $v = |\dot{\bar{r}}| = \sqrt{8^2+8^2+4^2} = \sqrt{144} = 12$.

Again $v = \sqrt{(3t^2-4)^2 + (2t+4)^2 + (16t-9t^2)^2}$

$\therefore$ $\frac{dv}{dt} = \frac{1}{2}\left\{(3t^2-4)^2 + (2t+4)^2 + (16t-9t^2)^2\right\}^{-1/2} \times \left\{2(3t^2-4)\times 6t + 2(2t+4)\cdot\right.$ $\left. + 2(16t-9t^2)\times(16-18t)\right\}$

$= \frac{1}{2\times 12}\cdot\left[~~24+8~~\; 12\times 8\times 2 + 4\times 8 + 2\times(-4)\times(-20)\right]$

$= \frac{1}{2}\times\frac{1}{12}(192+32+160) = \frac{1}{2}\times 16\times 2 = 16$

$\therefore$ Tangential component $= 16 \left[= \frac{dv}{dt}\right]$

Now, $\ddot{\bar{r}} = 6t\hat{i} + 2\hat{j} + (16-18t)\hat{k} = 12\hat{i} + 2\hat{j} - 20\hat{k}$

Now, $\kappa = \frac{|\dot{\bar{r}}\times\ddot{\bar{r}}|}{|\dot{\bar{r}}|^3}$, $\therefore$ $\dot{\bar{r}}\times\ddot{\bar{r}} = 16\hat{k} + 160\hat{j} - 96\hat{k} - 160\hat{i} - 48\hat{j} + 8\hat{i} = -152\hat{i} + 112\hat{j} - 80\hat{k}$

$= \frac{\sqrt{(152)^2 + (112)^2 + (80)^2}}{(12)^3}$

$\therefore$ $\frac{v^2}{\rho} = v^2\kappa = (12)^2 \times \frac{\sqrt{(152)^2 + (112)^2 + (80)^2}}{(12)^3} = \frac{1}{12}\times 4\sqrt{(38)^2 + (28)^2 + (20)^2} = \frac{1}{12}\times 4\times 2\left(\sqrt{(19)^2 + (14)^2 + \dots}\right) = \frac{1}{12}\times 4\times 2\times 3\sqrt{73}$

$= 2\sqrt{73}$.

14, 14, 56, 14, 71, 149, 361, 150, 73, 73, 657

Ex: Show that the radius of curvature of a plane curve $y = f(x),\ z = 0$ is given by $\rho = \dfrac{(1+y'^2)^{3/2}}{|y''|}$.
And the torsion $\tau = 0$ for this plane curve.

For a plane curve, ie, a curve in the xy-plane,
$\bar{r} = x\hat{i} + y\hat{j}$, where $y = f(x)$. Let $x = t$, $y = f(t)$ then,

$$\dot{\bar{r}} = \dot{x}i + \dot{y}\hat{j} = \hat{i} + \dot{y}\hat{j}$$

$$\ddot{\bar{r}} = \ddot{y}\hat{j}, \quad \dddot{\bar{r}} = \dddot{y}\hat{j}$$

$$\therefore K = \frac{|\dot{\bar{r}} \times \ddot{\bar{r}}|}{|\dot{\bar{r}}|^3} = \frac{|\ddot{y}\hat{k}|}{(1+\dot{y}^2)^{3/2}} = \frac{|\ddot{y}|}{(1+\dot{y}^2)^{3/2}}$$

Now, $\dot{y} = \dfrac{dy}{dt} = \dfrac{dy}{dx}\cdot\dfrac{dx}{dt} = y'\dot{x} = y'$ $[\because \dot{x} = 1]$

$$\ddot{y} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}(y'\dot{x}) = y''\dot{x}^2 + y'\ddot{x} = y'' \quad [\because \ddot{x} = 0]$$

$$\therefore \rho = \frac{1}{K} = \frac{(1+y'^2)^{3/2}}{|y''|}.$$

$$\tau = \frac{[\dot{\bar{r}}\ \ddot{\bar{r}}\ \dddot{\bar{r}}]}{|\dot{\bar{r}} \times \ddot{\bar{r}}|^2} = \frac{0}{\ddot{y}^2} = 0$$

Ex: Prove that for the space curve $\bar{r} = \bar{r}(s)$, $\tau = \dfrac{\frac{d\bar{r}}{ds}\cdot\frac{d^2\bar{r}}{ds^2}\times\frac{d^3\bar{r}}{ds^3}}{(d^2\bar{r}/ds^2)^2}$
where $\tau = \dfrac{[\dot{\bar{r}}\ \ddot{\bar{r}}\ \dddot{\bar{r}}]}{|\dot{\bar{r}} \times \ddot{\bar{r}}|^2}$ for the space curve $\bar{r} = \bar{r}(t)$.

Soln:
$$\dot{\bar{r}} = \frac{d\bar{r}}{dt} = \frac{d\bar{r}}{ds}\cdot\frac{ds}{dt} = \dot{s}\frac{d\bar{r}}{ds}$$

$$\ddot{\bar{r}} = \frac{d^2\bar{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\bar{r}}{dt}\right) = \frac{d}{dt}\left(\dot{s}\frac{d\bar{r}}{ds}\right) = \ddot{s}\frac{d\bar{r}}{ds} + \dot{s}^2\frac{d^2\bar{r}}{ds^2}.$$

$$\dddot{\bar{r}} = \frac{d}{dt}(\ddot{\bar{r}}) = \frac{d}{dt}\left(\ddot{s}\frac{d\bar{r}}{ds} + \dot{s}^2\frac{d^2\bar{r}}{ds^2}\right) = \dddot{s}\frac{d\bar{r}}{ds} + \ddot{s}\dot{s}\frac{d^2\bar{r}}{ds^2} + 2\dot{s}\ddot{s}\frac{d^2\bar{r}}{ds^2} + \dot{s}^3\frac{d^3\bar{r}}{ds^3}.$$

$$= \dddot{s}\frac{d\bar{r}}{ds} + 3\dot{s}\ddot{s}\frac{d^2\bar{r}}{ds^2} + \dot{s}^3\frac{d^3\bar{r}}{ds^3}$$

$$\dot{\bar{r}} \times \ddot{\bar{r}} = \dot{s}\ddot{s}\frac{d\bar{r}}{ds}\times\frac{d\bar{r}}{ds} + \dot{s}^3\frac{d\bar{r}}{ds}\times\frac{d^2\bar{r}}{ds^2} = \dot{s}^3\frac{d\bar{r}}{ds}\times\frac{d^2\bar{r}}{ds^2}.$$

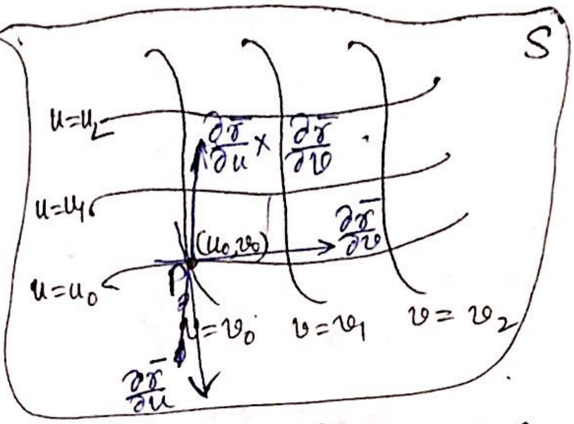
$$[\dot{\bar{r}}\ \ddot{\bar{r}}\ \dddot{\bar{r}}] = \dot{s}^3\dddot{s}\frac{d\bar{r}}{ds}\cdot\frac{d\bar{r}}{ds}\times\frac{d^2\bar{r}}{ds^2} + 3\dot{s}^4\ddot{s}\frac{d^2\bar{r}}{ds^2}\cdot\frac{d\bar{r}}{ds}\times\frac{d^2\bar{r}}{ds^2} + \dot{s}^6\frac{d^3\bar{r}}{ds^3}\cdot\frac{d\bar{r}}{ds}\times\frac{d^2\bar{r}}{ds^2}$$

$$= \dot{s}^6\frac{d\bar{r}}{ds}\cdot\frac{d^2\bar{r}}{ds^2}\times\frac{d^3\bar{r}}{ds^3}$$

$$\tau = \frac{[\dot{\bar{r}}\ \ddot{\bar{r}}\ \dddot{\bar{r}}]}{|\dot{\bar{r}} \times \ddot{\bar{r}}|^2} = \frac{\frac{d\bar{r}}{ds}\cdot\frac{d^2\bar{r}}{ds^2}\times\frac{d^3\bar{r}}{ds^3}}{\left(\frac{d^2\bar{r}}{ds^2}\right)^2}.$$

(b) Show that the eqn. $\bar{r} = \bar{r}(u, v)$ represents a surface.
Show that $\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}$ represents a vector normal to the surf

(a) if u have a fixed value, say $u_0, u_1, u_2, \cdots$ then $\bar{r} = \bar{r}(u_0, v)$, $\bar{r} = \bar{r}(u_1, v)$, $\bar{r} = \bar{r}(u_2, v), \cdots$ represent curves denoted by respectively, $u = u_0$, $u = u_1$, $u = u_2, \cdots$



As v varies, $\bar{r} = \bar{r}(u, v)$ represents a curve which moves in space and generates a surface.

Similarly $v = v_0$, $v = v_1$, $v = v_2 \cdots$ represent curves on the surface.

(u, v) defines the curvilinear coordinates on the surface. $\frac{\partial \bar{r}}{\partial u}, \frac{\partial \bar{r}}{\partial v}$ at P represents a tangent vector to the curves $v = v_0$ & $u = u_0$ resp.

$\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v} = \bar{n}$, the normal vector to the surface S at $P(u_0, v_0)$

Ex:- Find the eqns for the tangent plane and normal line to the surface $4z = x^2 - y^2$ at the point $(3,1,2)$

Soln; Let $x = u$, $y = v$, $z = \frac{u^2 - v^2}{4}$ be parametric eqns of the surface.

The p.v. to any point on the surface is

$$\bar{r} = u\hat{i} + v\hat{j} + \frac{u^2 - v^2}{4}\hat{k}.$$

$$\therefore \frac{\partial \bar{r}}{\partial u} = \hat{i} + \frac{u}{2}\hat{k}, \quad \frac{\partial \bar{r}}{\partial v} = \hat{j} - \frac{v}{2}\hat{k}.$$

Now, $\left.\frac{\partial \bar{r}}{\partial u}\right|_{(3,1,2)} = \hat{i} + \frac{3}{2}\hat{k}$, $\left.\frac{\partial \bar{r}}{\partial v}\right|_{(3,1,2)} = \hat{j} - \frac{1}{2}\hat{k}$.

$$\therefore \bar{n} = \left.\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}\right|_{(3,1,2)} = \left(\hat{i} + \frac{3}{2}\hat{k}\right) \times \left(\hat{j} - \frac{1}{2}\hat{k}\right) = \hat{k} + \frac{1}{2}\hat{j} - \frac{3}{2}\hat{i}$$

$\bar{n}$ is the normal to the surface $(3,1,2)$

The p.v. to the point $(3,1,2)$ is $\bar{R}_0 = 3\bar{i} + \bar{j} + 2\bar{k}$

" " " any " on the tangent plane is $\bar{R} = x\bar{i} + y\bar{j} + z\bar{k}$

$\therefore (\bar{R} - \bar{R}_0) \cdot \bar{n} = 0 \Rightarrow \{(x-3)\bar{i} + (y-1)\bar{j} + (z-2)\bar{k}\} \cdot \left(-\frac{3}{2}\bar{i} + \frac{1}{2}\bar{j} + \bar{k}\right) = 0$

$\Rightarrow -\frac{3}{2}(x-3) + \frac{1}{2}(y-1) + (z-2) = 0 \Rightarrow 3(x-3) - (y-1) - 2(z-2) = 0$

$\Rightarrow \boxed{3x - y - 2z = 4} \rightarrow$ eqn of the tangent plane at $(3,1,2)$

Eqn of the normal line $\rightarrow (\bar{R} - \bar{R}_0) \times \bar{n} = \bar{0}$; $x = 3t + 3$, $y = 1 - t$, $z = 2 - 2t$

Frenet-Serret Formulae:

$$\frac{d\bar{T}}{ds} = \kappa \bar{N}, \quad \frac{d\bar{N}}{ds} = \tau \bar{B} - \kappa \bar{T}, \quad \frac{d\bar{B}}{ds} = -\tau \bar{N}$$

Osculating Plane to a curve at a point P is the plane containing the tangent and principal normal at P.

Normal Plane is the plane through P perpendicular to the tangent.

Rectifying Plane is the plane through P perpendicular to the principal normal.

Ex: Find Eqns for the tangent, principal normal, binormal, and also the eqns for the osculating plane, normal plane and rectifying plane to the curve $x = 3t - t^3$, $y = 3t^2$, $z = 3t + t^3$ at the point where $t = 1$.

Soln: The p.v. is $\bar{r} = (3t - t^3)\bar{i} + 3t^2 \bar{j} + (3t + t^3)\bar{k}$

$$\frac{d\bar{r}}{dt} = (3 - 3t^2)\bar{i} + 6t\bar{j} + (3 + 3t^2)\bar{k}$$

$$\frac{ds}{dt} = \left|\frac{d\bar{r}}{dt}\right| = \sqrt{(3-3t^2)^2 + (6t)^2 + (3+3t^2)^2} = \sqrt{18 + 18t^4 + 36t^2} = 3\sqrt{2}(t^2+1)$$

$$\bar{T} = \frac{d\bar{r}}{ds} = \frac{d\bar{r}}{dt} \Big/ \frac{ds}{dt} = \frac{(1-t^2)\bar{i} + 2t\bar{j} + (1+t^2)\bar{k}}{\sqrt{2}(1+t^2)}$$

$$\frac{d\bar{T}}{dt} = \frac{\sqrt{2}(1+t^2).(-2t\bar{i} + 2\bar{j} + 2t\bar{k}) - 2\sqrt{2}t\{(1-t^2)\bar{i} + 2t\bar{j} + (1+t^2)\bar{k}\}}{2(1+t^2)^2}$$

$$= \frac{-4\sqrt{2}t\,\bar{i} + 2\sqrt{2}\bar{j}(1-t^2)}{2(1+t^2)^2} = \frac{-2\sqrt{2}t\,\bar{i} + \sqrt{2}\bar{j}(1-t^2)}{(1+t^2)^2}$$

$$\frac{d\bar{T}}{ds} = \frac{d\bar{T}}{dt} \Big/ \frac{ds}{dt} = \frac{-2t\,\bar{i} + \bar{j}(1-t^2)}{3(1+t^2)^3}$$

$$\frac{d\bar{T}}{ds} = \kappa \bar{N} \Rightarrow \kappa = \left|\frac{d\bar{T}}{ds}\right| = \frac{\sqrt{4t^2 + (1-t^2)^2}}{3(1+t^2)^3} \quad \frac{(1+t^2)}{3(1+t^2)^3} = \frac{1}{3(1+t^2)^2}.$$

$$\therefore \bar{N} = \frac{1}{\kappa}\frac{d\bar{T}}{ds} = \frac{-2t\,\bar{i} + \bar{j}(1-t^2)}{(1+t^2)}$$

Then $\bar{B} = \bar{T} \times \bar{N} = \dfrac{1}{\sqrt{2}(1+t^2)^2} \cdot \left\{(1-t^2)\bar{i} + 2t\bar{j} + (1+t^2)\bar{k}\right\} \times \left\{-2t\bar{i} + (1-t^2)\bar{j}\right\}$

$$= \frac{1}{\sqrt{2}(1+t^2)^2}\cdot\left\{(1-t^2)^2\hat{k} + 4t^2\hat{k} - 2t(1+t^2)\hat{j} - (1-t^4)\hat{i}\right\}$$

$$= \frac{1}{\sqrt{2}(1+t^2)^2}\cdot\left\{(t^2+1)^2\hat{k} - 2t(1+t^2)\hat{j} - (1-t^4)\hat{i}\right\}$$

$$\bar{T}_0 = \frac{\bar{j} + \bar{k}}{\sqrt{2}}, \quad \bar{N}_0 = -\bar{i}, \quad \bar{B}_0 = \frac{-\bar{j} + \bar{k}}{\sqrt{2}}.$$

If $\bar{A}$ denotes a given vector while $\bar{r}_0$ & $\bar{r}$ are the p.v.s of the initial point and arbitrary point of A, then $\bar{r}-\bar{r}_0$ is ll^{l} to $\bar{A}$, so the eqn. of $\bar{A}$ is $(\bar{r}-\bar{r}_0)\times\bar{A}=\bar{0}$

Eqn of tangent is $(\bar{r}-\bar{r}_0)\times\bar{T}_0=\bar{0}$

" " principal normal is $(\bar{r}-\bar{r}_0)\times\bar{N}_0=\bar{0}$

" " binormal is $(\bar{r}-\bar{r}_0)\times\bar{B}_0=\bar{0}$

$\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$
$\bar{r}_0=2\bar{i}+3\bar{j}+4\bar{k}$

Eqn of the osculating plane $=(\bar{r}-\bar{r}_0)\cdot\bar{B}_0=0$

or, $\left[(x-2)\bar{i}+(y-3)\bar{j}+(z-4)\bar{k}\right]\cdot\dfrac{(-\bar{j}+\bar{k})}{\sqrt{2}}=0$

or, $-y+3+z-4=0$ or, $\underline{y-z+1=0}$.

Eqn of the normal plane is $(\bar{r}-\bar{r}_0)\cdot\bar{T}_0=0$.

or, $\left\{(x-2)\bar{i}+(y-3)\bar{j}+(z-4)\bar{k}\right\}\cdot\left(\dfrac{\bar{j}+\bar{k}}{\sqrt{2}}\right)=0$

or, $y-3+z-4=0$ or, $\underline{y+z-7=0}$.

Eqn of the rectifying plane is $(\bar{r}-\bar{r}_0)\cdot\bar{N}_0=0$.

or, $(x-2)\cdot(-1)=0$ or, $\underline{x=2}$.

Mechanics:

Problem: Show that the acceleration $\bar{a}$ of a particle which travels along a space curve with velocity $\bar{v}$ given by $\bar{a}=\dfrac{dv}{dt}\bar{T}+\dfrac{v^2}{\rho}\bar{N}$,

$\bar{v}=v\bar{T}$

$\bar{a}=\dfrac{d\bar{v}}{dt}=\dfrac{dv}{dt}\bar{T}+v\dfrac{d\bar{T}}{dt}=\dfrac{dv}{dt}\bar{T}+\dfrac{ds}{dt}\dfrac{d\bar{T}}{ds}\cdot\dfrac{ds}{dt}=\dfrac{dv}{dt}\bar{T}+\dfrac{v^2}{\rho}\bar{N}$

$\bar{T}=\dfrac{(3t^2-4)\bar{i}+(2t+4)\bar{j}+(16t-9t^2)\bar{k}}{\frac{ds}{dt}}$

$\frac{ds}{dt}=\sqrt{(3t^2-4)^2+(2t+4)^2+(16t-9t^2)^2}$
$\frac{ds}{dt}=12$ at $t=2$

$\bar{T}\dfrac{ds}{dt}=(3t^2-4)\bar{i}+(2t+4)\bar{j}+(16t-9t^2)\bar{k}$

$\dfrac{d\bar{T}}{dt}\dfrac{ds}{dt}+\bar{T}\dfrac{d^2s}{dt^2}=6t\bar{i}+2\bar{j}+(16-18t)\bar{k}$

$12\dfrac{d\bar{T}}{dt}+\bar{T}\sqrt{144+4+400}=12\bar{i}+2\bar{j}-20\bar{k}$

or, $12\dfrac{d\bar{T}}{dt}+\dfrac{2\bar{i}+2\bar{j}-\bar{k}}{3}\sqrt{548}=12\bar{i}+2\bar{j}-20\bar{k}$

$\dfrac{1}{\rho}=\kappa=\left|\dfrac{d\bar{T}}{ds}\right|$

Ex. Find the unit normal to the surface

$$\bar{r} = a\cos u \sin v\,\hat{i} + a\sin u \sin v\,\hat{j} + a\cos v\,\hat{k} \quad (a>0)$$

How? $\begin{cases} \left|\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}\right| = \sqrt{a^4 \sin^2 v} = a^2 \sin v, \text{ if } \sin v > 0 \\ \qquad\qquad = -a^2 \sin v, \text{ if } \sin v < 0. \\ \hat{n} = \pm\{(\cos u \sin v)\hat{i} + (\sin u \sin v)\hat{j} + \cos v\,\hat{k}\}. \end{cases}$

$$\frac{\partial \bar{r}}{\partial u} = -a\sin u \sin v\,\hat{i} + a\cos u \sin v\,\hat{j}$$

$$\frac{\partial \bar{r}}{\partial v} = a\cos u \cos v\,\hat{i} + a\sin u \cos v\,\hat{j} - a\sin v\,\hat{k}$$

Then $\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ (-a\sin u \sin v) & (a\cos u \sin v) & 0 \\ (a\cos u \cos v) & (a\sin u \cos v) & (-a\sin v) \end{vmatrix}$

$$= -a^2 \cos u \sin^2 v\,\hat{i} - a^2 \sin u \sin^2 v\,\hat{j} - a^2 \sin v \cos v\,\hat{k}$$

represents a vector normal to the surface at any point (u, v).

Unit normal $\hat{n} = \dfrac{\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}}{\left|\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}\right|}$

Now $\left|\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}\right| = \sqrt{a^4 \cos^2 u \sin^4 v + a^4 \sin^2 u \sin^4 v + a^4 \sin^2 v \cos^2 v}$.

$$= \sqrt{a^4 \sin^2 v(\sin^2 v + \cos^2 v)} = \begin{cases} a^2 \sin v, & \text{if } \sin v > 0 \\ -a^2 \sin v, & \text{if } \sin v < 0. \end{cases}$$

$$\therefore \hat{n} = \frac{\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}}{\left|\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}\right|} = \pm(\cos u \sin v\,\hat{i} + \sin u \sin v\,\hat{j} + \cos v\,\hat{k}).$$