UG CBCS 2ND Sem.
Subject: Mathematics (Hons.)
Paper: C4T. / Unit- IV.

---

Name of the teacher:

Dr. Sangita Chakraborty
Associate Professor
Deptt. of Mathematics
Kharagpur College
WhatsUP No. 943455417

---

If any student have queries, he/she can contact me through my WhatsUP No. (given above).



Scanned with CamScanner

## Vector function :-

If to each value of a scalar variable $t$, there corresponds a unique vector $\bar{F}$, determined by any given rule, then $\bar{F}(t)$ is called a vector function.

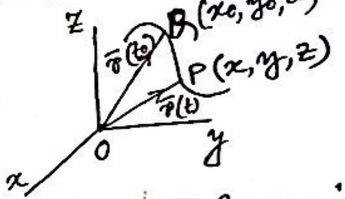
$$\bar{F}(t) = F_1(t)\hat{i} + F_2(t)\hat{j} + F_3(t)\hat{k}\text{, where}$$

$F_1(t)$, $F_2(t)$, $F_3(t)$ are scalar functions of $t$.

$\bar{F} = (F_1, F_2, F_3)$. $\left\{ \hat{i}, \hat{j}, \hat{k} \right.$ be three unit vectors along three mutually perpendicular fixed directions.

## Example:

Let $\bar{r} = \overline{OP}$ be the vector from a fixed origin $O$ to a variable point $P$ on a space curve. for a fixed value of the scalar variable $t$, say $t_0$, $\exists$ a unique vector $\bar{r}(t_0)$.




$$\bar{r} = \bar{r}(t)$$

$\bar{r}$ is a vector function of position.

As $t$ changes, the terminal point of $\bar{r}$ describes a space curve.

$\bar{r}(t)$ is the position vector of the variable point $P(x, y, z)$ on the space curve. Then

$$\bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}.$$

Parametric equations of the space curve:

$$x = x(t)$$
$$y = y(t)\text{,} \quad t \text{ is a parameter.}$$
$$z = z(t)\text{,}$$



Scanned with CamScanner

2

# VECTOR CALCULUS. CC-4/2nd sem.

## Limit of a vector function:

A vector function $\bar{F}(t)$ is said to tend to a limit $\bar{A}$ (constant vector), as $t$ approaches $t_0$, if, to any preassigned +ve number $\varepsilon$, however small, there corresponds a +ve number $\delta$, such that

$$|\bar{F}(t)-\bar{A}|<\varepsilon \text{ when } 0<|t-t_0|<\delta.$$

Then

$$\lim_{t\to t_0} \bar{F}(t)=\bar{A}$$

Roughly speaking, the length of the vector $\bar{F}(t)-\bar{A}$ approaches zero.

$\lim_{t\to t_0} \bar{F}(t)=\bar{A}$ implies $\lim_{t\to t_0} F_K(t)=A_K$, for $K=1,2,3$.

where $\bar{F}(t)=F_1(t)\hat{i}+F_2(t)\hat{j}+F_3(t)\hat{k}$, $\bar{A}=A_1\hat{i}+A_2\hat{j}+A_3\hat{k}$.

## PROPERTIES:

(i) $\lim_{t\to t_0}[\bar{F}(t)\pm\bar{f}(t)]=\lim_{t\to t_0}\bar{F}(t)\pm\lim_{t\to t_0}\bar{f}(t)$

(ii) $\lim_{t\to t_0}[\bar{F}(t)\overset{\cdot}{\times}\bar{f}(t)]=\lim_{t\to t_0}\bar{F}(t)\overset{\cdot}{\times}\lim_{t\to t_0}\bar{f}(t)$

(iii) $\lim_{t\to t_0}[\phi(t)\bar{F}(t)]=\lim_{t\to t_0}\phi(t)\lim_{t\to t_0}\bar{F}(t)$

## Continuity of a vector function

A vector function $\bar{F}(t)$ is said to be continuous at $t=t_0$ if, given any (+ve) number $\varepsilon$, it is possible to find another (+ve) number $\delta$ such that

$$|\bar{F}(t)-\bar{F}(t_0)|<\varepsilon \text{ whenever } |t-t_0|<\delta.$$

Also $\bar{F}(t)$ is said to be continuous at $t=t_0$, if $F_1(t), F_2(t), F_3(t)$ are all " " " "

$\bar{F}(t)$ is continuous at $t$ if $|\bar{F}(t+\Delta t)-\bar{F}(t)|<\varepsilon$, when $|\Delta t|<\delta$.

$\bar{F}(t)$ " " in $a\le t\le b$ if $\bar{F}(t)$ is continuous for every value of $t$ in this interval.

## PROPERTIES:-

Suppose, $\bar{F}(t), \bar{f}(t), \phi(t)$ are continuous at $t=t_0$.

(i) $\lim_{t\to t_0}[\bar{F}(t)\pm\bar{f}(t)]=\bar{F}(t_0)\pm\bar{f}(t_0) \Rightarrow \bar{F}(t)\pm\bar{f}(t)$ are continuous at $t=t_0$.

(ii) $\lim_{t\to t_0}[\bar{F}(t)\overset{\cdot}{\times}\bar{f}(t)]=\bar{F}(t_0)\overset{\cdot}{\times}\bar{f}(t_0)$

(iii) $\lim_{t\to t_0}[\phi(t)\bar{F}(t)]=\phi(t_0)\bar{F}(t_0)$



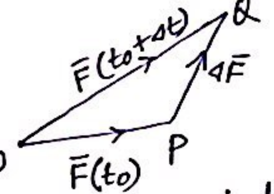
Scanned with CamScanner

## DERIVATIVE OF VECTORS

Suppose that $\bar{F}(t)$ is a single-valued vector function of a scalar variable $t$.

Let $\overline{OP} = \bar{F}(t_0)$ w.r.t. fixed origin $O$.

& $\overline{OQ} = \bar{F}(t_0 + \Delta t)$.




$\overline{PQ} = \overline{OQ} - \overline{OP} \Rightarrow \overline{PQ} = \bar{F}(t_0+\Delta t) - \bar{F}(t_0) = \Delta\bar{F}$

$\therefore \quad \dfrac{\overline{PQ}}{\Delta t} = \dfrac{\Delta\bar{F}}{\Delta t}$ ; as $t \to 0$, $\dfrac{\Delta\bar{F}}{\Delta t} \to \bar{F}'(t_0)$, if the limit exists.

$$\lim_{\Delta t \to 0} \frac{\Delta\bar{F}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\bar{F}(t_0+\Delta t) - \bar{F}(t_0)}{\Delta t} = \frac{d\bar{F}}{dt} = \bar{F}'(t_0) = \bar{F}'(t)\Big|_{at\ t=t_0}.$$

If the derived function $\bar{F}'(t)$ is also derivable, i.e., if $\displaystyle\lim_{\Delta t \to 0} \frac{\bar{F}'(t+\Delta t) - \bar{F}'(t)}{\Delta t}$ exists, then $= \bar{F}''(t) = \dfrac{d^2\bar{F}}{dt^2}$.

Higher order derivatives $\dfrac{d^3\bar{F}}{dt^3}, \dfrac{d^4\bar{F}}{dt^4}$, etc. are similarly defined like in scalar calculus.

### Relation between continuity and derivability →

**Theorem**: If $\bar{F}'(t)$ exists at $t = t_0$, then $\bar{F}(t)$ is continuous at $t = 0$. [Every derivable function is continuous].

**Proof**: $$\lim_{\Delta t \to 0} \{\bar{F}(t_0+\Delta t) - \bar{F}(t_0)\} = \lim_{\Delta t \to 0} \left[\Delta t \left\{\frac{\bar{F}(t_0+\Delta t) - \bar{F}(t_0)}{\Delta t}\right\}\right]$$

$$= \left\{\lim_{\Delta t \to 0} \Delta t\right\} \left\{\lim_{\Delta t \to 0} \frac{\bar{F}(t_0+\Delta t) - \bar{F}(t_0)}{\Delta t}\right\} = 0 \cdot \bar{F}'(t_0) = \bar{0}$$

$$\therefore \lim_{\Delta t \to 0} \bar{F}(t_0 + \Delta t) = \bar{F}(t_0).$$

### Converse is NOT True always →

Let us take $\bar{F}(t) = |t|\hat{i}$. $\therefore \bar{F}(0) = \bar{0}$

$|\bar{F}(t) - \bar{F}(0)| = \left||t|\hat{i} - \bar{0}\right| = |t|$

$\Rightarrow \displaystyle\lim_{t \to 0} \bar{F}(t) = \bar{0} = \bar{F}(0)$, $\therefore \bar{F}(t) = |t|\hat{i}$ is continuous at $t=0$.

But, $$\frac{\bar{F}(t) - \bar{F}(0)}{t - 0} = \frac{|t|\hat{i} - \bar{0}}{t} = \frac{|t|}{t}\hat{i} \to \begin{cases} \hat{i}, & \text{as } t \to 0 \text{ through (+ve) values} \\ -\hat{i}, & \text{as } t \to 0 \text{ through (-ve) values.} \end{cases}$$

So, the limit is not unique, hence $\bar{F}'(0)$ does not exist.



Scanned with CamScanner

## Derivatives of sums and products

① If $\bar{F}(t) = F_1(t)\hat{i} + F_2(t)\hat{j} + F_3(t)\hat{k}$ is derivable function of $t$, then $F_1, F_2, F_3$ are also derivable scalar functions of $t$.

Also $\dfrac{d\bar{F}}{dt} = \dfrac{dF_1}{dt}\hat{i} + \dfrac{dF_2}{dt}\hat{j} + \dfrac{dF_3}{dt}\hat{k}$, and conversely.

__Proof:__ $\dfrac{\bar{F}(t+\Delta t) - \bar{F}(t)}{\Delta t} = \dfrac{F_1(t+\Delta t) - F_1(t)}{\Delta t}\hat{i} + \dfrac{F_2(t+\Delta t) - F_2(t)}{\Delta t}\hat{j} + \dfrac{F_3(t+\Delta t) - F_3(t)}{\Delta t}\hat{k}$

The limit as $\Delta t \to 0$ on the L.H.S. exists iff the three limits of R.H.S. all exist.

$\therefore \quad \bar{F}'(t) = F_1'(t)\hat{i} + F_2'(t)\hat{j} + F_3'(t)\hat{k}$.

② The derivative of any constant vector $\bar{c}$ is zero and conversely.

__Proof:__ For any constant vector $\bar{c}$, $\Delta\bar{c} = 0$, so that

$$\frac{\Delta\bar{c}}{\Delta t} = 0 \text{ and } \lim_{\Delta t \to 0} \frac{\Delta\bar{c}}{\Delta t} = \frac{d\bar{c}}{dt} = \bar{0}.$$

Conversely,

let $\bar{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, $\therefore \dfrac{d\bar{c}}{dt} = \dfrac{dc_1}{dt}\hat{i} + \dfrac{dc_2}{dt}\hat{j} + \dfrac{dc_3}{dt}\hat{k}$.

Now $\dfrac{d\bar{c}}{dt} = \bar{0} \Rightarrow \dfrac{dc_1}{dt}\hat{i} + \dfrac{dc_2}{dt}\hat{j} + \dfrac{dc_3}{dt}\hat{k} = 0\hat{i} + 0\hat{j} + 0\hat{k}$

$\Rightarrow \dfrac{dc_1}{dt} = 0,\ \dfrac{dc_2}{dt} = 0,\ \dfrac{dc_3}{dt} = 0$

$\Rightarrow c_1, c_2, c_3$ are constants

$\Rightarrow \bar{c}$ is a constant vector.

③ $\dfrac{d}{dt}(\bar{u} + \bar{v}) = \dfrac{d\bar{u}}{dt} + \dfrac{d\bar{v}}{dt}$.

__Proof:__ $\Delta(\bar{u} \pm \bar{v}) = (\bar{u} + \Delta\bar{u} \pm \bar{v} + \Delta\bar{v}) - (\bar{u} + \bar{v}) = \Delta\bar{u} \pm \Delta\bar{v}$

or, $\displaystyle\lim_{\Delta t \to 0} \frac{\Delta(\bar{u} \pm \bar{v})}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\bar{u}}{\Delta t} \pm \lim_{\Delta t \to 0} \frac{\Delta\bar{v}}{\Delta t}$

or, $\dfrac{d}{dt}(\bar{u} \pm \bar{v}) = \dfrac{d\bar{u}}{dt} \pm \dfrac{d\bar{v}}{dt}$. $\left[\text{Infact, } \dfrac{d}{dt}\left(\displaystyle\sum_{i=1}^{n} \bar{u}_i\right) = \displaystyle\sum_{i=1}^{n} \dfrac{d\bar{u}_i}{dt}\right]$

④ $\dfrac{d}{dt}(f\bar{u}) = f\dfrac{d\bar{u}}{dt} + \dfrac{df}{dt}\bar{u}$

__Proof:__ $\Delta(f\bar{u}) = (f + \Delta f)(\bar{u} + \Delta\bar{u}) - f\bar{u} = f\Delta\bar{u} + \Delta f\,\bar{u} + \Delta f\,\Delta\bar{u}$.

or, $\dfrac{\Delta(f\bar{u})}{\Delta t} = f\dfrac{\Delta\bar{u}}{\Delta t} + \dfrac{\Delta f}{\Delta t}\bar{u} + \dfrac{\Delta f}{\Delta t}\cdot\dfrac{\Delta\bar{u}}{\Delta t}\cdot\Delta t$

or, $\displaystyle\lim_{\Delta t \to 0} \frac{\Delta(f\bar{u})}{\Delta t} = f\left(\lim_{\Delta t \to 0} \frac{\Delta\bar{u}}{\Delta t}\right) + \left(\lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t}\right)\bar{u} + \left(\lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t}\right)\left(\lim_{\Delta t \to 0} \frac{\Delta\bar{u}}{\Delta t}\right)\left(\lim_{\Delta t \to 0} \Delta t\right)$

or, $\dfrac{d}{dt}(f\bar{u}) = f\dfrac{d\bar{u}}{dt} + \dfrac{df}{dt}\bar{u} + \dfrac{df}{dt}\dfrac{d\bar{u}}{dt}\cdot 0 = f\dfrac{d\bar{u}}{dt} + \dfrac{df}{dt}\bar{u}$.

__Note:__ Law ① can be deduced by Law ④



Scanned with CamScanner

⑤ $\frac{d}{dt}(\bar{u}\cdot\bar{v}) = \bar{u}\cdot\frac{d\bar{v}}{dt} + \frac{d\bar{u}}{dt}\cdot\bar{v}$

Proof: $\Delta(\bar{u}\cdot\bar{v}) = \{(\bar{u}+\Delta\bar{u})\cdot(\bar{v}+\Delta\bar{v})\} - (\bar{u}\cdot\bar{v})$

$= \bar{u}\cdot\Delta\bar{v} + \Delta\bar{u}\cdot\bar{v} + \Delta\bar{u}\cdot\Delta\bar{v}$

$$\frac{\Delta(\bar{u}\cdot\bar{v})}{\Delta t} = \bar{u}\cdot\frac{\Delta\bar{v}}{\Delta t} + \frac{\Delta\bar{u}}{\Delta t}\cdot\bar{v} + \left(\frac{\Delta\bar{u}}{\Delta t}\cdot\frac{\Delta\bar{v}}{\Delta t}\right)\Delta t$$

$$\lim_{\Delta t\to 0}\frac{\Delta(\bar{u}\cdot\bar{v})}{\Delta t} = \bar{u}\left(\lim_{\Delta t\to 0}\frac{\Delta\bar{v}}{\Delta t}\right) + \left(\lim_{\Delta t\to 0}\frac{\Delta\bar{u}}{\Delta t}\right)\cdot\bar{v} + \left(\lim_{\Delta t\to 0}\frac{\Delta\bar{u}}{\Delta t}\right)\cdot\left(\lim_{\Delta t\to 0}\frac{\Delta\bar{v}}{\Delta t}\right)\lim_{\Delta t\to 0}\Delta t$$

$$= \bar{u}\cdot\frac{d\bar{v}}{dt} + \frac{d\bar{u}}{dt}\cdot\bar{v} + \left(\frac{d\bar{u}}{dt}\cdot\frac{d\bar{v}}{dt}\right)0 = \bar{u}\cdot\frac{d\bar{v}}{dt} + \frac{d\bar{u}}{dt}\cdot\bar{v}$$

or, $\frac{d}{dt}(\bar{u}\cdot\bar{v}) = \bar{u}\cdot\frac{d\bar{v}}{dt} + \frac{d\bar{u}}{dt}\cdot\bar{v}$

⑥ $\frac{d}{dt}(\bar{u}\times\bar{v}) = \bar{u}\times\frac{d\bar{v}}{dt} + \frac{d\bar{u}}{dt}\times\bar{v}$

Proof. $\Delta(\bar{u}\times\bar{v}) = \{(\bar{u}+\Delta\bar{u})\times(\bar{v}+\Delta\bar{v})\} - (\bar{u}\times\bar{v})$

$= \bar{u}\times\Delta\bar{v} + \Delta\bar{u}\times\bar{v} + \Delta\bar{u}\times\Delta\bar{v}$

or, $\frac{\Delta(\bar{u}\times\bar{v})}{\Delta t} = \bar{u}\times\frac{\Delta\bar{v}}{\Delta t} + \frac{\Delta\bar{u}}{\Delta t}\times\bar{v} + \left(\frac{\Delta\bar{u}}{\Delta t}\times\frac{\Delta\bar{v}}{\Delta t}\right)\Delta t$

$$\lim_{\Delta t\to 0}\frac{\Delta(\bar{u}\times\bar{v})}{\Delta t} = \bar{u}\times\left(\lim_{\Delta t\to 0}\frac{\Delta\bar{v}}{\Delta t}\right) + \left(\lim_{\Delta t\to 0}\frac{\Delta\bar{u}}{\Delta t}\right)\times\bar{v} + \left(\lim_{\Delta t\to 0}\frac{\Delta\bar{u}}{\Delta t}\right)\times\left(\lim_{\Delta t\to 0}\frac{\Delta\bar{v}}{\Delta t}\right)\lim_{\Delta t\to 0}\Delta t$$

$$= \bar{u}\times\frac{d\bar{v}}{dt} + \frac{d\bar{u}}{dt}\times\bar{v} + \left(\frac{d\bar{u}}{dt}\times\frac{d\bar{v}}{dt}\right)0 = \bar{u}\times\frac{d\bar{v}}{dt} + \frac{d\bar{u}}{dt}\times\bar{v}$$

or, $\frac{d}{dt}(\bar{u}\times\bar{v}) = \bar{u}\times\frac{d\bar{v}}{dt} + \frac{d\bar{u}}{dt}\times\bar{v}$

⑦ If $\bar{u} = \bar{u}(s)$ and $s = s(t)$, then $\frac{d\bar{u}}{dt} = \frac{d\bar{u}}{ds}\frac{ds}{dt}$

Now, $\Delta\bar{u} = (\bar{u}+\Delta\bar{u}) - \bar{u}$. $\Rightarrow \frac{\Delta\bar{u}}{\Delta t} = \left\{\frac{(\bar{u}+\Delta\bar{u})-\bar{u}}{(s+\Delta s)-s}\right\}\left\{\frac{(s+\Delta s)-s}{(t+\Delta t)-t}\right\} = \frac{\Delta\bar{u}}{\Delta s}\cdot\frac{\Delta s}{\Delta t}$

$\lim_{\Delta t\to 0}\frac{\Delta\bar{u}}{\Delta t} = \lim_{\Delta s\to 0}\left(\frac{\Delta\bar{u}}{\Delta s}\right)\lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t}$ [Since $\Delta t\to 0 \Rightarrow \Delta s\to 0$]

$\therefore$ $\boxed{\frac{d\bar{u}}{dt} = \frac{d\bar{u}}{ds}\frac{ds}{dt}}$.

Theorem:

A necessary and sufficient condition that a proper vector $\bar{u}$

(i) has a constant length is that $\bar{u}\cdot\frac{d\bar{u}}{dt} = 0$

(ii) always remains parallel to a fixed line is that $\bar{u}\times\frac{d\bar{u}}{dt} = \bar{0}$.

Proof: (i) Let $\bar{u}(\neq\bar{0})$ have a constant length $[=|\bar{u}|]$

i.e., $\frac{d}{dt}(|\bar{u}|^2) = \frac{d}{dt}(\bar{u}^2) = \frac{d}{dt}(\bar{u}\cdot\bar{u}) = 0$

$\Rightarrow \bar{u}\cdot\frac{d\bar{u}}{dt} + \frac{d\bar{u}}{dt}\cdot\bar{u} = 0 \Rightarrow 2\bar{u}\cdot\frac{d\bar{u}}{dt} = 0$

$\Rightarrow \bar{u}\cdot\frac{d\bar{u}}{dt} = 0$

Conversely, let $\bar{u}\cdot\frac{d\bar{u}}{dt} = 0 \Rightarrow 2\bar{u}\cdot\frac{d\bar{u}}{dt} = 0 \Rightarrow \frac{d}{dt}(\bar{u}\cdot\bar{u}) = 0$

$\Rightarrow \frac{d}{dt}(|\bar{u}|^2) = 0 \Rightarrow |\bar{u}| = $ constant.



Scanned with CamScanner

(ii) Let $\bar{u}$ be always parallel to a fixed line.
Let $\bar{u} = u\hat{u}$, where $\hat{u}$ is the unit vector along $\bar{u}$ and $u$ be its length, i.e., $u = |\bar{u}|$.

$$\bar{u} \times \frac{d\bar{u}}{dt} = (u\hat{u}) \times \frac{d}{dt}(u\hat{u}) = (u\hat{u}) \times \left(u\frac{d\hat{u}}{dt} + \frac{du}{dt}\hat{u}\right)$$

$$= u^2\left(\hat{u} \times \frac{d\hat{u}}{dt}\right) + u\frac{du}{dt}(\hat{u} \times \hat{u}) = u^2\left(\hat{u} \times \frac{d\hat{u}}{dt}\right) \quad [\because \hat{u} \times \hat{u} = \bar{0}]$$

When $\bar{u}$ remains parallel to a fixed line, $\hat{u}$ is constant and $\frac{d\hat{u}}{dt} = \bar{0}$

$\therefore \quad \hat{u} \times \frac{d\hat{u}}{dt} = \bar{0}$

$\therefore \quad \underline{\bar{u} \times \frac{d\bar{u}}{dt} = \bar{0}}$.

Conversely, let $\bar{u} \times \frac{d\bar{u}}{dt} = \bar{0}$, where $\bar{u} \neq \bar{0}$

$\therefore \quad \bar{u} \times \frac{d\bar{u}}{dt} = u^2\left(\hat{u} \times \frac{d\hat{u}}{dt}\right) \Rightarrow \hat{u} \times \frac{d\hat{u}}{dt} = \bar{0} \longrightarrow (1)$.

Since $\hat{u}$ is a vector of constant length 1, then

$$\frac{d}{dt}(\hat{u} \cdot \hat{u}) = 0 \Rightarrow 2\hat{u} \times \frac{d\hat{u}}{dt} = 0 \Rightarrow \hat{u} \cdot \frac{d\hat{u}}{dt} = 0 \longrightarrow (2).$$

Equations (1) & (2) are contradictory unless

$$\frac{d\hat{u}}{dt} = \bar{0} \Rightarrow \hat{u} \text{ is } \underline{\text{constant}} \text{ unit } \underline{\text{vector}}.$$

$\therefore$ $\bar{u}$ is parallel to a fixed line.

**Derivatives of Triple products:-**

(1) $\frac{d}{dt}[\bar{a} \cdot (\bar{b} \times \bar{c})] = \frac{d\bar{u}}{dt} \cdot (\bar{v} \times \bar{w}) + \bar{u} \cdot \left(\frac{d\bar{v}}{dt} \times \bar{w}\right) + \bar{u} \cdot \left(\bar{v} \times \frac{d\bar{w}}{dt}\right)$.

(2) $\frac{d}{dt}[\bar{u} \times (\bar{v} \times \bar{w})] = \frac{d\bar{u}}{dt} \times (\bar{v} \times \bar{w}) + \bar{u} \times \left(\frac{d\bar{v}}{dt} \times \bar{w}\right) + \bar{u} \times \left(\bar{v} \times \frac{d\bar{w}}{dt}\right)$.

---

9 Solve: $\frac{d^2\bar{r}}{dt^2} + 4\bar{r} = \bar{0}$ ; $\frac{d^2\bar{r}}{dt^2} - 7\frac{d\bar{r}}{dt} + 12\bar{r} = \bar{0}$

Considering $\bar{r}$ as scalar variable,

$\frac{d^2r}{dt^2} + 4r = 0$. Let $r = Ae^{mt}$ be a trial soln.

A.E. is $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

complete soln is given by $r = c_1\cos 2t + c_2\sin 2t$.

Transfering $r$ as $\bar{r}$, $c_1$ as $\bar{c_1}$, $c_2$ as $\bar{c_2}$, $\bar{c_1}, \bar{c_2}$ are arbitrary vector consts.

Req. Soln: $\underline{\bar{r} = \bar{c_1}\cos 2t + \bar{c_2}\sin 2t}$,

Similary for $\frac{d^2\bar{r}}{dt^2} - 7\frac{d\bar{r}}{dt} + 12\bar{r} = \bar{0}$, we get the complete soln as: $\bar{r} = \bar{c_1}e^{3t} + \bar{c_2}e^{4t}$.



Scanned with CamScanner

# Vector Functions of several scalar variables

$\overline{V} = \overline{f}(x_1, x_2, \ldots, x_n)$ means that a vector $\overline{f}$ is uniquely determined when $x_1, x_2, \ldots, x_n$ are known real values within their ranges.

In particular, $\overline{F} = \overline{F}(x,y)$ in a region $R$

$\Rightarrow$ $\overline{F}$ is a function of $x$ and $y$ for all points of the region $R$

$\Rightarrow$ to each point $(x,y) \in R$, a number and a direction can be assigned to $\overline{F}$

$$\overline{F} = F_1(x,y)\,\hat{i} + F_2(x,y)\,\hat{j} + F_3(x,y)\,\hat{k};$$ $F_1, F_2, F_3$ are all defined in $R$.

## PARTIAL DERIVATIVES:

Let us consider a vector function $\overline{F}(x,y)$.

$$\overline{F}_x = \frac{\partial \overline{F}}{\partial x} = \lim_{\Delta x \to 0} \frac{\overline{F}(x+\Delta x, y) - \overline{F}(x,y)}{\Delta x}, \text{ if limit exists.}$$

$$\overline{F}_y = \frac{\partial \overline{F}}{\partial y} = \lim_{\Delta y \to 0} \frac{\overline{F}(x, y+\Delta y) - \overline{F}(x,y)}{\Delta y}, \quad " \quad " \quad "$$

If $\overline{F}(x,y) = F_1(x,y)\,\hat{i} + F_2(x,y)\,\hat{j} + F_3(x,y)\,\hat{k}$

$$\frac{\partial \overline{F}}{\partial x} = \frac{\partial F_1}{\partial x}\hat{i} + \frac{\partial F_2}{\partial x}\hat{j} + \frac{\partial F_3}{\partial x}\hat{k}, \text{ etc.}$$

Total differential: $d\overline{F} = dF_1\,\hat{i} + dF_2\,\hat{j} + dF_3\,\hat{k}$, where

$$dF_1 = \frac{\partial F_1}{\partial x}dx + \frac{\partial F_1}{\partial y}dy, \text{ etc.}$$

$$d\overline{F} = \left(\frac{\partial F_1}{\partial x}dx + \frac{\partial F_2}{\partial x}dx + \frac{\partial F_3}{\partial x}dx\right)\hat{i} + \cdots$$

$$= dF_1\,\hat{i} + dF_2\,\hat{j} + dF_3\,\hat{k}.$$

**Some Results:**

1. $\frac{\partial}{\partial x}(\overline{F} + \overline{G}) = \frac{\partial \overline{F}}{\partial x} + \frac{\partial \overline{G}}{\partial x}$
2. $\frac{\partial}{\partial x}(f\overline{F}) = f\frac{\partial \overline{F}}{\partial x} + \frac{\partial f}{\partial x}\overline{F}$
3. $\frac{\partial}{\partial x}(\overline{F}\cdot\overline{G}) = \frac{\partial \overline{F}}{\partial x}\cdot\overline{G} + \overline{F}\cdot\frac{\partial \overline{G}}{\partial x}$.
4. $\frac{\partial}{\partial x}(\overline{F}\times\overline{G}) = \frac{\partial \overline{F}}{\partial x}\times\overline{G} + \overline{F}\times\frac{\partial \overline{G}}{\partial x}$

③ $$\frac{\partial}{\partial x}(\overline{F}\cdot\overline{G}) = \lim_{\Delta x \to 0} \frac{\overline{F}(x+\Delta x,y)\cdot\overline{G}(x+\Delta x,y) - \overline{F}(x,y)\cdot\overline{G}(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\overline{F}(x+\Delta x,y)\cdot\overline{G}(x+\Delta x,y) - \overline{F}(x,y)\cdot\overline{G}(x+\Delta x,y) + \overline{F}(x,y)\cdot\overline{G}(x+\Delta x,y) - \overline{F}(x,y)\cdot\overline{G}(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\{\overline{F}(x+\Delta x,y) - \overline{F}(x,y)\}}{\Delta x}\cdot\overline{G}(x+\Delta x,y) + \lim_{\Delta x \to 0} \frac{\overline{F}(x,y)\cdot\{\overline{G}(x+\Delta x,y) - \overline{G}(x,y)\}}{\Delta x}$$

$$= \frac{\partial \overline{F}}{\partial x}\cdot\overline{G}(x,y) + \overline{F}(x,y)\cdot\frac{\partial \overline{G}}{\partial x}.$$

**Example:** ① If $\overline{F} = e^{xy}\,\hat{i} + (2x-y)\,\hat{j} + y\sin x\,\hat{k}$, calculate $\overline{F}_x, \overline{F}_{xx}, \overline{F}_{xy}, \overline{F}_{yy}, \overline{F}_x \times \overline{F}_y$.

② $\overline{F} = xyz\,\hat{i} + xz^2\,\hat{j} - y^3\,\hat{k}$, $\overline{G} = x^3\,\hat{i} - xyz\,\hat{j} + x^2z\,\hat{k}$

Prove that (i) $\overline{F}_{xy}(0,0) = \overline{0}$, (ii) $\overline{F}_{yy} \times \overline{G}_{xx}$ at $(1,1,0) = -36\,\hat{j}$



Scanned with CamScanner

# Integration of Vectors.

**Differentials:** If $\bar{F}(t) = F_1(t)\hat{i} + F_2(t)\hat{j} + F_3(t)\hat{k}$, and if $\bar{F}'(t)$ exists, then $d\bar{F} = \bar{F}'(t)\,dt = dF_1\hat{i} + dF_2\hat{j} + dF_3\hat{k}$.

If $\bar{F}(t) = \bar{c}$ (constant), then $d\bar{c} = \bar{0}$.

Also, $d(\bar{F}\cdot\bar{f}) = \bar{F}\cdot d\bar{f} + d\bar{F}\cdot\bar{f}$

$d(\bar{F}\times\bar{f}) = \bar{F}\times d\bar{f} + d\bar{F}\times\bar{f}$

**Indefinite integration:**

If $\bar{F}(t) = \frac{d}{dt}\bar{f}(t)$, then $\bar{F}(t)\,dt = d\bar{f}(t)$, and

$\int \bar{F}(t)\,dt = \int d\bar{f}(t) = \bar{f}(t) \rightarrow$ indefinite integral of $\bar{F}(t)$.

<u>Theorem</u>: If $\bar{f}(t)$ be an indefinite integral of $\bar{F}(t)$, then $\bar{f}(t)+\bar{c}$, where $\bar{c}$ is a constant vector, is also the integral of $\bar{F}(t)$.

<u>Proof</u>: By the given condition, $d\{\bar{f}(t)\} = \bar{F}(t)\,dt$

Also, $d\{\bar{f}(t)+\bar{c}\} = d\{\bar{f}(t)\} + d\bar{c} = \bar{F}(t)\,dt$ $[\because d\bar{c} = \bar{0}]$

Hence, by the definition, $\bar{f}(t)+\bar{c}$ is also an integral of $\bar{F}(t)$.

**Important Formulae:** Let $\bar{r}(t)$ and $\bar{s}(t)$ be two derivable functions, denoted by $\bar{r}$, $\bar{s}$ resp.

① $\int\left(\bar{r}\cdot\frac{d\bar{s}}{dt} + \bar{s}\cdot\frac{d\bar{r}}{dt}\right)dt = \int\frac{d}{dt}(\bar{r}\cdot\bar{s})\,dt = \bar{r}\cdot\bar{s} + c$

② $\int 2\bar{r}\cdot\frac{d\bar{r}}{dt}\,dt = \int\frac{d}{dt}(\bar{r}\cdot\bar{r})\,dt = \bar{r}\cdot\bar{r} + \bar{c} = \bar{r}^2 + c$

③ $\int 2\frac{d\bar{r}}{dt}\cdot\frac{d^2\bar{r}}{dt^2}\,dt = \int\frac{d}{dt}\left(\frac{d\bar{r}}{dt}\cdot\frac{d\bar{r}}{dt}\right)dt = \frac{d\bar{r}}{dt}\cdot\frac{d\bar{r}}{dt} + c = \left(\frac{d\bar{r}}{dt}\right)^2 + c$

④ $\int\left(\bar{r}\times\frac{d^2\bar{r}}{dt^2}\right)dt = \int\frac{d}{dt}\left(\bar{r}\times\frac{d\bar{r}}{dt}\right)dt = \bar{r}\times\frac{d\bar{r}}{dt} + \bar{c}$

⑤ $\int\left(\frac{1}{r}\frac{d\bar{r}}{dt} - \frac{dr}{dt}\frac{\bar{r}}{r^2}\right)dt = \int\frac{d}{dt}\left(\frac{1}{r}\bar{r}\right)dt = \frac{\bar{r}}{r} + \bar{c}$

⑥ $\int\left(\bar{a}\times\frac{d\bar{r}}{dt}\right)dt = \int\frac{d}{dt}(\bar{a}\times\bar{r})\,dt = \bar{a}\times\bar{r} + \bar{c}$ [$\bar{a}$ is a given constant vector]

⑦ If $\bar{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$, then

$\int\bar{f}(t)\,dt = \hat{i}\int f_1(t)\,dt + \hat{j}\int f_2(t)\,dt + \hat{k}\int f_3(t)\,dt$.

<u>Note</u>: The constant of integration is a scalar or a vector according as the integrand is a scalar or a vector function.



Scanned with CamScanner

## Definite Integral as the limit of a sum:

Let $\bar{F}(t)$ be a vector function, which is finite and continuous in $a \le t \le b$.

Let us divide $[a, b]$ into a finite number of sub-intervals, say, $n$ which corresponds to increments $\Delta t_1, \Delta t_2, \ldots, \Delta t_n$ of the scalar variable $t$

Let us take, some value $t_1$ for $t$ in the subinterval $\Delta t_1$

" " $t_2$ " $t$ " " " $\Delta t_2$

- - - - - - - - - - - - - - - - - - - -

" " $t_n$ " $t$ " " " " $\Delta t_n$

Then the corresponding functions are respectively $\bar{F}(t_1), \bar{F}(t_2), \ldots, \bar{F}(t_n)$

Let us form the sum

$$\bar{S}_n = \Delta t_1 \bar{F}(t_1) + \Delta t_2 \bar{F}(t_2) + \cdots + \Delta t_n \bar{F}(t_n).$$

Let the number of subintervals, $n \to \infty$ so that the greatest of $\Delta t \to 0$, then $\bar{S}_n \to$ a unique finite limit which is independent of the choice of the values of $t_1, t_2, \ldots, t_n$, and also independent of the mode of subdivision of the range $[a, b]$.

$$\boxed{\lim_{n\to\infty} \bar{S}_n = \bar{f}(b) - \bar{f}(a), \text{ where } \bar{f}(t) = \int \bar{F}(t)\,dt}$$

This limiting value is called the definite integral of the function $\bar{F}(t)$ between $a$ to $b$, and is given by

$$\int_a^b \bar{F}(t)\,dt = \lim_{n\to\infty} \sum_{r=1}^{n} \Delta t_r \bar{F}(t_r) = \bar{f}(b) - \bar{f}(a),$$

$= \lim_{n\to\infty} \bar{S}_n$

where $\bar{f}(t) = \int \bar{F}(t)\,dt$.

9. Solve for $\bar{r}$:

(i) $\frac{d^2\bar{r}}{dt^2} + 4\bar{r} = \bar{0}$ (ii) $\frac{d^2\bar{r}}{dt^2} - 7\frac{d\bar{r}}{dt} + 12\bar{r} = 0.$ (iii) $\frac{d^2\bar{r}}{dt^2} - 6\frac{d\bar{r}}{dt} + 9\bar{r} = \bar{0}$

~~Integrating w.r.t. t,~~

$m^2 - 7m + 12 = 0$

$(m-4)(m-3) = 0$

$r = e^{4t}, \ r = e^{3t}$

$2\frac{d\bar{r}}{dt}\cdot\frac{d^2\bar{r}}{dt^2} + 4\bar{r}\cdot\frac{d\bar{r}}{dt} = \bar{0} \Rightarrow \frac{d}{dt}\left(\frac{d\bar{r}}{dt}\right)^2 + 4\bar{r}\cdot\frac{d\bar{r}}{dt} = \bar{0}$

Integrating w.r.t. $t$, $\left(\frac{d\bar{r}}{dt}\right)^2 + 2\bar{r}^2 = c$

(i) $\bar{r} = \bar{a}\cos 2t + \bar{b}\sin 2t$

(ii) $\bar{r} = \bar{a}e^{4t} + \bar{b}e^{3t}$

(iii) $\bar{r} = (\bar{a} + \bar{b}t)e^{3t}$

Exercise.

1. If $\bar{r}(t) = (3t^2-1)\hat{i} + (2-6t)\hat{j} - 4t\hat{k}$, show that
$\int \bar{r}(t)\,dt = (t^3 - \frac{1}{2}t^2)\hat{i} + (2t - 3t^2)\hat{j} - 2t^2\hat{k} + \bar{c}$,
and $\int_1^2 \bar{r}(t)\,dt = \frac{11}{2}\hat{i} - 7\hat{j} - 6\hat{k}$. $\int \bar{r}\,dt = \bar{f}(t) + \bar{c}$
$\int_1^2 \bar{r}\,dt = \bar{f}(2) - \bar{f}(1)$

3. If $\bar{r}(t) = (t, -t^2, t-1)$, $\bar{s}(t) = (2t^2, 0, 6t)$, show that
$\int_0^2 \bar{r}\cdot\bar{s}\,dt = 12$ and $\int_0^2 \bar{r}\times\bar{s}\,dt = \left(-24, -\frac{40}{3}, \frac{65}{5}\right)$.

Soln. $\bar{r}\cdot\bar{s} = 2t^3 + 6t^2 - 6t$, $\bar{r}\times\bar{s} = -6t^2\hat{j} + 2t^4\hat{k} - 6t^3\hat{i} + (2t^3 - 2\hat{i})$
$= -6t^3\hat{i} + (2t^3 - 8t^2)\hat{j} + 2t^4\hat{k}$

5. (i) Prove that $\int_1^2 \bar{r}\times\frac{d^2\bar{r}}{dt^2}\,dt = -14\hat{i} + 75\hat{j} - 15\hat{k}$, where
$\bar{r}(t) = (5t^2, t, -t^3)$, $\dot{\bar{r}} = (10t, 1, -3t^2)$, ~~$\ddot{\bar{r}} = (10, 0, -6t)$~~
~~$\bar{r}\times\ddot{\bar{r}} = (5t^2, t, -t^3)\times(10, 0, -6t) = 30t^3\hat{j} - 10t\hat{k} - 6t^2\hat{i} - 10t^3\hat{j} = (-6t^2, 20t^3, -10t)$~~

$\int \bar{r}\times\frac{d^2\bar{r}}{dt^2}\,dt = \int \frac{d}{dt}\left(\bar{r}\times\frac{d\bar{r}}{dt}\right)dt = \int d\left(\bar{r}\times\frac{d\bar{r}}{dt}\right) = \bar{r}\times\frac{d\bar{r}}{dt} + \bar{c}$

If $\bar{f}(t) = \bar{r}\times\frac{d\bar{r}}{dt}$, then $\int_1^2 \bar{r}\times\frac{d^2\bar{r}}{dt^2}\,dt = \bar{f}(2) - \bar{f}(1)$

(ii) $\int_1^2 \left(\frac{1}{r}\frac{d\bar{r}}{dt} - \frac{dr}{dt}\frac{\bar{r}}{r^2}\right)dt = \int_1^2 \frac{d}{dt}\left(\frac{\bar{r}}{r}\right)dt = \left[\frac{\bar{r}}{r}\right]_{t=1}^2$.
$= \bar{f}(2) - \bar{f}(1)$, where $\bar{f}(t) = \frac{\bar{r}}{r}$.

(iii) Given that $\bar{r}(t) = \begin{cases} 2i - j + 2k, & \text{when } t = 2 \\ 4i - 2j + 3k, & \text{〃 } t = 3 \end{cases}$

Show that $\int_2^3 \bar{r}\cdot\frac{d\bar{r}}{dt}\,dt = 10$
$= \frac{1}{2}\int_2^3 \frac{d}{dt}(\bar{r}\cdot\bar{r})\,dt = \frac{1}{2}\left[\bar{r}^2\right]_{t=2}^3$

6 (i) The acceleration at any instant t of a moving particle is given by $\frac{d^2\bar{r}}{dt^2} = 12\cos 2t\, i - 8\sin 2t\, j + 16t\, k$
Find the velocity $\bar{V}$ and displacement $\bar{r}$ at any time t, if at $t=0$, $\bar{V} = \bar{0}$ and $\bar{r} = \bar{0}$

Ans: $\bar{V} = \frac{d\bar{r}}{dt} = 6\sin 2t\, i + 4\cos 2t\, j + 8t^2 k - 4j$
$\bar{r} = (3 - 3\cos 2t)\, i + (2\sin 2t - 4t)\, j + \frac{8}{3}t^3 k$



Scanned with CamScanner

# Exercises – V(A) [Maity & Ghosh].

① If $\bar{r} = t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}$, find at $t=0$ the values of $\frac{d\bar{r}}{dt}, \frac{d^2\bar{r}}{dt^2}, \frac{d^3\bar{r}}{dt^3}, \left|\frac{d\bar{r}}{dt}\right|, \left|\frac{d^2\bar{r}}{dt^2}\right|$.

② If $\bar{r} = (5t^2, t, -t^3)$ and $\bar{s} = (\sin t, -\cos t, 0)$, find the values of $\frac{d}{dt}(\bar{r}\cdot\bar{s}), \frac{d}{dt}(\bar{r}\times\bar{s}), \frac{d}{dt}(\bar{r}\cdot\bar{r})$.

③ ⓐ If $\bar{r}_1 = (\sin\theta, \cos\theta, \theta)$, $\bar{r}_2 = (\cos\theta, -\sin\theta, -3)$ and $\bar{r}_3 = (2,3,-1)$, find $\frac{d}{d\theta}\{\bar{r}_1\times(\bar{r}_2\times\bar{r}_3)\}$, at $\theta = 0$.

ⓑ Prove that $\frac{d}{dt}\left(\bar{r}\cdot\frac{d\bar{r}}{dt}\right) = \left(\frac{d\bar{r}}{dt}\right)^2 + \bar{r}\cdot\frac{d^2\bar{r}}{dt^2}$, and
$\frac{d}{dt}\left(\bar{r}\times\frac{d\bar{r}}{dt}\right) = \bar{r}\times\frac{d^2\bar{r}}{dt^2}$.

$$\frac{d}{dt}\left(\bar{r}\cdot\frac{d\bar{r}}{dt}\right) = \frac{d\bar{r}}{dt}\cdot\frac{d\bar{r}}{dt} + \bar{r}\cdot\frac{d^2\bar{r}}{dt^2} = \left(\frac{d\bar{r}}{dt}\right)^2 + \bar{r}\cdot\frac{d^2\bar{r}}{dt^2}.$$

$$\frac{d}{dt}\left(\bar{r}\times\frac{d\bar{r}}{dt}\right) = \frac{d\bar{r}}{dt}\times\frac{d\bar{r}}{dt} + \bar{r}\times\frac{d^2\bar{r}}{dt^2} = \bar{0} + \bar{r}\times\frac{d^2\bar{r}}{dt^2} = \bar{r}\times\frac{d^2\bar{r}}{dt^2}.$$

④ Evaluate the following derivaties :

ⓐ $$\frac{d}{dt}\left\{\bar{r}\cdot\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2}\right\} = \frac{d\bar{r}}{dt}\cdot\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2} + \bar{r}\cdot\frac{d^2\bar{r}}{dt^2}\times\frac{d^2\bar{r}}{dt^2} + \bar{r}\cdot\frac{d\bar{r}}{dt}\times\frac{d^3\bar{r}}{dt^3}$$
$$= 0 + 0 + \left[\bar{r}\ \frac{d\bar{r}}{dt}\ \frac{d^3\bar{r}}{dt^3}\right] = \left[\bar{r}\ \frac{d\bar{r}}{dt}\ \frac{d^3\bar{r}}{dt^3}\right]$$

ⓑ $$\frac{d^2}{dt^2}\left\{\bar{r}\cdot\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2}\right\} = \frac{d}{dt}\left\{\bar{r}\cdot\frac{d\bar{r}}{dt}\times\frac{d^3\bar{r}}{dt^3}\right\}$$
$$= \frac{d\bar{r}}{dt}\cdot\frac{d\bar{r}}{dt}\times\frac{d^3\bar{r}}{dt^3} + \bar{r}\cdot\frac{d^2\bar{r}}{dt^2}\times\frac{d^3\bar{r}}{dt^3} + \bar{r}\cdot\frac{d\bar{r}}{dt}\times\frac{d^4\bar{r}}{dt^4}$$
$$= 0 + \left[\bar{r}\ \frac{d^2\bar{r}}{dt^2}\ \frac{d^3\bar{r}}{dt^3}\right] + \left[\bar{r}\ \frac{d\bar{r}}{dt}\ \frac{d^4\bar{r}}{dt^4}\right].$$

⑤ Find the derivatives w.r.t. $t$ of the following :

ⓐ $$\frac{d}{dt}\left\{r^2\bar{r} + (\bar{a}\cdot\bar{r})\bar{b}\right\} = \left\{\frac{d}{dt}(r^2)\right\}\bar{r} + r^2\frac{d\bar{r}}{dt} + \left(\bar{a}\cdot\frac{d\bar{r}}{dt}\right)\bar{b}$$
$$= 2r\frac{dr}{dt}\bar{r} + r^2\frac{d\bar{r}}{dt} + \left(\bar{a}\cdot\frac{d\bar{r}}{dt}\right)\bar{b} = 2rr'\bar{r} + r^2\bar{r}' + (\bar{a}\cdot\bar{r}')\bar{b}$$

ⓑ $$\frac{d}{dt}\left\{(a\bar{r} + r\bar{b})^2\right\} = \cancel{2(a\bar{r}+r\bar{b})}\ 2(a\bar{r}+r\bar{b})\frac{d}{dt}(a\bar{r}+r\bar{b})$$
$$= 2(a\bar{r}+r\bar{b})(a\bar{r}' + r'\bar{b}).$$

ⓒ $$\frac{d}{dt}\left\{\frac{\bar{r}}{r}\right\} = \frac{\bar{r}'}{r} - \frac{1}{r^2}\bar{r}r' = \frac{1}{r^2}(r\bar{r}' - \bar{r}r').$$

ⓓ $$\frac{d}{dt}\left\{r^n\bar{r}\right\} = nr^{n-1}r'\bar{r} + r^n\bar{r}'.$$

ⓔ $$\frac{d}{dt}\left\{r^3\bar{r} + \bar{a}\times\frac{d\bar{r}}{dt}\right\} = 3r^2r'\bar{r} + r^3\bar{r}' + \bar{a}\times\bar{r}''.$$

ⓕ $$\frac{d}{dt}\left\{\frac{\bar{r}\times\bar{a}}{\bar{r}\cdot\bar{a}}\right\} = \frac{\bar{r}'\times\bar{a}}{\bar{r}\cdot\bar{a}} - \frac{(\bar{r}\times\bar{a})}{(\bar{r}\cdot\bar{a})^2}(\bar{r}'\cdot\bar{a}).$$



Scanned with CamScanner

And the acceleration vector is given by $\frac{d^2\bar{r}}{dt^2}$.

<u>Problem</u>: At any instant $t$ the vector from origin to a moving point is $\bar{r} = \bar{a}\cos\omega t + \bar{b}\sin\omega t$; $\bar{a}, \bar{b}$ are constant vectors and $\omega$ is a scalar constant.

(a) Find the velocity $\vec{V}$ and show that $\bar{r} \times \bar{V}$ = a constant vector.

(b) Show that the acceleration is directed towards the origin and is proportional to the displacement.

(c) What is the physical interpretation of this motion?

(a) $\bar{r} = \bar{a}\cos\omega t + \bar{b}\sin\omega t$, $\bar{r}' = -\bar{a}\omega\sin\omega t + \bar{b}\omega\cos\omega t$

& $\bar{r}'' = -\bar{a}\omega^2\cos\omega t - \bar{b}\omega^2\sin\omega t = -\omega^2\bar{r}$

Velocity vector $\bar{V} = \frac{d\bar{r}}{dt} = \bar{r}' = \omega(-\bar{a}\sin\omega t + \bar{b}\cos\omega t)$

Now $\bar{r} \times \bar{V} = (\bar{a}\cos\omega t + \bar{b}\sin\omega t) \times \{\omega(-\bar{a}\sin\omega t + \bar{b}\cos\omega t)\} = \omega(\bar{a}\times\bar{b})$

$= \omega\{(\bar{a}\times\bar{b})\cos^2\omega t + (\bar{a}\times\bar{b})\sin^2\omega t\}$

= a constant vector.

(b) Acceleration is $\frac{d^2\bar{r}}{dt^2} = \bar{r}'' = -\omega^2\bar{r}$, ~~which is directed towards the origin for its (-ve) sign and is proportional to the displacement vector $\bar{r}$~~.

Acceleration is opposite to the direction of $\bar{r}$, i.e., it is directed towards the origin.

$\left|\frac{d^2\bar{r}}{dt^2}\right| = \omega^2|\bar{r}|$, magnitude of acceleration is proportional to the displacement $|\bar{r}|$ which is the distance from the origin.

(c) <u>Physical interpretation</u>:

The motion is that of a particle moving on the circumference of a circle with constant angular speed $\omega$. The acceleration (force), directed towards the centre of the circle is the centripetal acceleration (force).

(9) Verify that $\bar{r} = \bar{a}e^{m_1 t} + \bar{b}e^{m_2 t}$ satisfies the differential equation $\frac{d^2\bar{r}}{dt^2} + K\frac{d\bar{r}}{dt} + l\bar{r} = \bar{0}$ ($K, l$ are scalar constants) and $m_1, m_2$ are two unequal roots of the equation $m^2 + Km + l = 0$ and $\bar{a}, \bar{b}$ are abitrary constant vectors.

<u>Solution</u>: $\bar{r}' = m_1\bar{a}e^{m_1 t} + m_2\bar{b}e^{m_2 t}$, $\bar{r}'' = m_1^2\bar{a}e^{m_1 t} + m_2^2\bar{b}e^{m_2 t}$

$\bar{r}'' + K\bar{r}' + l\bar{r} = (m_1^2 + Km_1 + l)\bar{a}e^{m_1 t} + (m_2^2 + Km_2 + l)\bar{b}e^{m_2 t}$

$= 0(\bar{a}e^{m_1 t}) + 0(\bar{b}e^{m_2 t}) = \bar{0}$.



Scanned with CamScanner

(g) $\frac{d}{dt}\left\{\frac{\bar{r}}{\bar{a}\cdot\bar{r}} - \frac{\bar{r}\times\bar{a}}{r}\right\} = \frac{\bar{r}'}{(\bar{a}\cdot\bar{r})} - \frac{\bar{r}\,(\bar{a}\cdot\bar{r}')}{(\bar{a}\cdot\bar{r})^2} - \frac{\bar{r}'\times\bar{a}}{r} + \frac{1}{r^2}(\bar{r}\times\bar{a})r'$

$= \frac{(\bar{a}\cdot\bar{r})\bar{r}' - (\bar{a}\cdot\bar{r}')\bar{r}}{(\bar{a}\cdot\bar{r})^2} - \frac{r(\bar{r}'\times\bar{a}) - (\bar{r}\times\bar{a})\bar{r}}{r^2}.$

(h) $\frac{d}{dt}\left\{\frac{\bar{r}}{r^2} + \frac{r\bar{b}}{\bar{a}\cdot\bar{r}}\right\} = \frac{\bar{r}'}{r^2} - \frac{2\bar{r}}{r^3}r' + \frac{r'\bar{b}}{\bar{a}\cdot\bar{r}} - \frac{r\bar{b}}{(\bar{a}\cdot\bar{r})^2}(\bar{a}\cdot\bar{r}')$

(i) $\frac{d}{dt}\left\{\bar{r}^2 + \frac{1}{\bar{r}^2}\right\} = 2\bar{r}\bar{r}' - \frac{2}{\bar{r}^3}\bar{r}'$

(j) $\frac{d}{dt}\left\{r^2 + \frac{1}{r^2}\right\} = 2rr' - \frac{2}{r^3}r'$

✓(k) $\frac{d}{dt}\left\{\frac{1}{2}m\left(\frac{d\bar{r}}{dt}\right)^2\right\} = \frac{1}{2}m\,2\bar{r}'\bar{r}'' = m(\bar{r}'\cdot\bar{r}'')$

(l) $\frac{d}{dt}\left\{\bar{r}\times\bar{s}' - \bar{r}'\times\bar{s}\right\} = \cancel{(\bar{r}'\times\bar{s}')} + \bar{r}\times\bar{s}'' - \bar{r}''\times\bar{s} - \cancel{(\bar{r}'\times\bar{s}')} = \bar{r}\times\bar{s}'' - \bar{r}''\times\bar{s}$

(6) Differentiate $\frac{\bar{r}+\bar{b}}{\bar{r}^2+\bar{b}^2}$ w.r.t. t.

$\frac{d}{dt}\left\{\frac{\bar{r}+\bar{b}}{\bar{r}^2+\bar{b}^2}\right\} = \frac{\bar{r}'}{\bar{r}^2+\bar{b}^2} - \frac{(\bar{r}+\bar{b})}{(\bar{r}^2+\bar{b}^2)^2}\,2(\bar{r}\cdot\bar{r}') = \frac{(\bar{r}^2+\bar{b}^2)\bar{r}' - 2(\bar{r}\cdot\bar{r}')(\bar{r}+\bar{b})}{(\bar{r}^2+\bar{b}^2)^2}.$

(7) Given $\bar{r} = a\cos t\,\hat{i} + a\sin t\,\hat{j} + bt\,\hat{k}$, show that
(a) $(\bar{r}')^2 = a^2+b^2$ (b) $(\bar{r}'\times\bar{r}'')^2 = a^2(a^2+b^2)$ (c) $[\bar{r}'\bar{r}''\bar{r}'''] = a^2b$

(a) $\bar{r}' = -a\sin t\,\hat{i} + a\cos t\,\hat{j} + b\hat{k}$, $\bar{r}'' = -a\cos t\,\hat{i} - a\sin t\,\hat{j}$,
$\bar{r}''' = a\sin t\,\hat{i} - a\cos t\,\hat{j}$

Now, $(\bar{r}')^2 = a^2(\sin^2 t + \cos^2 t) + b^2 = a^2+b^2$

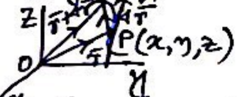
(b) $\bar{r}'\times\bar{r}'' = a^2\sin^2 t(\hat{i}\times\hat{j}) - a^2\cos^2 t(\hat{j}\times\hat{i}) - ab\cos t(\hat{k}\times\hat{i}) - ab\sin t(\hat{k}\times\hat{j})$
$= a^2(\sin^2 t + \cos^2 t)\hat{k} - ab\cos t\,\hat{j} + ab\sin t\,\hat{i}$
$= ab\sin t\,\hat{i} - ab\cos t\,\hat{j} + a^2\hat{k}.$

$(\bar{r}'\times\bar{r}'')^2 = a^2b^2 + a^4 = a^2(a^2+b^2).$

(c) $[\bar{r}'\bar{r}''\bar{r}'''] = (\bar{r}'\times\bar{r}'')\cdot\bar{r}''' = a^2b\sin^2 t + a^2b\cos^2 t = a^2b.$

(8) If $\bar{r}$ be the displacement (position) vector at any time t of a moving point P(x, y, z), then as t changes, the terminal point of $\bar{r}$ describes a space curve having parametric equations: $x = x(t), y = y(t), z = z(t)$,

$\bar{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$.




Then $\frac{\Delta\bar{r}}{\Delta t} = \frac{\bar{r}(t+\Delta t) - \bar{r}(t)}{\Delta t}$ is a vector in the direction of $\Delta\bar{r}$.

If $\lim_{\Delta t\to 0}\frac{\Delta\bar{r}}{\Delta t}$ exists, then the limit is a vector in the direction of the tangent to the space curve at (x, y, z) and is given by

$\frac{d\bar{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$, represents the velocity

$\vec{v}$ with which the terminal point of $\bar{r}$ describes the

⑤ $$\frac{d}{dt}(\bar{u}\cdot\bar{v}) = \bar{u}\cdot\frac{d\bar{v}}{dt} + \frac{d\bar{u}}{dt}\cdot\bar{v}$$

⑥ $$\frac{d}{dt}(\bar{u}\times\bar{v}) = \bar{u}\times\frac{d\bar{v}}{dt} + \frac{d\bar{u}}{dt}\times\bar{v}$$

⑦ $$\frac{d}{dt}(\bar{u}\cdot\bar{v}\times\bar{w}) = \bar{u}\cdot\frac{d\bar{v}}{dt}\times\bar{w} + \frac{d\bar{u}}{dt}\cdot\bar{v}\times\bar{w} + \bar{u}\cdot\bar{v}\times\frac{d\bar{w}}{dt}$$

⑧ $$\frac{d}{dt}(\bar{u}\times(\bar{v}\times\bar{w})) = \bar{u}\times\left(\bar{v}\times\frac{d\bar{w}}{dt}\right) + \bar{u}\times\left(\frac{d\bar{v}}{dt}\times\bar{w}\right) + \frac{d\bar{u}}{dt}\times(\bar{v}\times\bar{w}).$$

~~Th~~: In ⑥, put $\bar{v} = \frac{d\bar{u}}{dt}$, then

$$\frac{d}{dt}\left(\bar{u}\times\frac{d\bar{u}}{dt}\right) = \frac{d\bar{u}}{dt}\times\frac{d\bar{u}}{dt} + \bar{u}\times\frac{d^2\bar{u}}{dt^2} = \bar{u}\times\frac{d^2\bar{u}}{dt^2}$$

⑨ If $\bar{u}$ is a derivable vector fn. of a scalar variable $s$ and $s$ is again a derivable fn of another scalar variable $t$, then

$$\underline{\frac{d\bar{u}}{dt} = \frac{d\bar{u}}{ds}\frac{ds}{dt}}.$$

Ex: Find $\frac{d}{dt}[\bar{r}\,\dot{\bar{r}}\,\ddot{\bar{r}}]$, where $\bar{r}$ is a fn of $t$.

$$\frac{d}{dt}[\bar{r}\,\dot{\bar{r}}\,\ddot{\bar{r}}] = \frac{d}{dt}\{\bar{r}\cdot(\dot{\bar{r}}\times\ddot{\bar{r}})\} = \dot{\bar{r}}\cdot\dot{\bar{r}}\times\ddot{\bar{r}} + \bar{r}\cdot\ddot{\bar{r}}\times\ddot{\bar{r}} + \bar{r}\cdot\dot{\bar{r}}\times\dddot{\bar{r}}$$
$$= 0 + 0 + [\bar{r}\,\dot{\bar{r}}\,\dddot{\bar{r}}]$$
$$= \boxed{\left[\bar{r}\,\frac{d\bar{r}}{dt}\,\frac{d^3\bar{r}}{dt^3}\right]} \quad \underline{\text{Ans}}.$$

Ex: If $\bar{w}$ is a constant vector, $\bar{r}, \bar{s}$ are vector fns of a scalar variable $t$ and if $\frac{d\bar{r}}{dt} = \bar{w}\times\bar{r}$, $\frac{d\bar{s}}{dt} = \bar{w}\times\bar{s}$, then show that $\frac{d}{dt}(\bar{r}\times\bar{s}) = \bar{w}\times(\bar{r}\times\bar{s})$

$$\frac{d}{dt}(\bar{r}\times\bar{s}) = \frac{d\bar{r}}{dt}\times\bar{s} + \bar{r}\times\frac{d\bar{s}}{dt} = (\bar{w}\times\bar{r})\times\bar{s} + \bar{r}\times(\bar{w}\times\bar{s})$$
$$= (\bar{w}\cdot\bar{s})\bar{r} - \cancel{(\bar{r}\cdot\bar{s})\bar{w}} + \cancel{(\bar{r}\cdot\bar{s})\bar{w}} - (\bar{r}\cdot\bar{w})\bar{s}$$
$$= \bar{w}\times(\bar{r}\times\bar{s}) \quad \underline{\text{proved}}.$$

Ex. Evaluate: $\frac{d^2}{dt^2}\{\bar{r}\cdot\dot{\bar{r}}\times\ddot{\bar{r}}\} = \frac{d}{dt}\{\dot{\bar{r}}\cdot\dot{\bar{r}}\times\ddot{\bar{r}} + \bar{r}\cdot\ddot{\bar{r}}\times\ddot{\bar{r}} + \bar{r}\cdot\dot{\bar{r}}\times\dddot{\bar{r}}\}$

$$= \frac{d}{dt}\{0+0+\bar{r}\cdot\dot{\bar{r}}\times\dddot{\bar{r}}\} = \dot{\bar{r}}\cdot\dot{\bar{r}}\times\dddot{\bar{r}} + \bar{r}\cdot\ddot{\bar{r}}\times\dddot{\bar{r}} + \bar{r}\cdot\dot{\bar{r}}\times\ddddot{\bar{r}}$$
$$= \bar{r}\cdot\ddot{\bar{r}}\times\dddot{\bar{r}} + \bar{r}\cdot\dot{\bar{r}}\times\ddddot{\bar{r}}.$$

Ex: Differentiate $\frac{\bar{r}+\bar{b}}{\bar{r}^2+\bar{b}^2}$ w.r.t. $t$

$$\frac{d}{dt}\left\{\frac{\bar{r}+\bar{b}}{\bar{r}^2+\bar{b}^2}\right\} = \frac{(\dot{\bar{r}}+\bar{0})(\bar{r}^2+\bar{b}^2) - (\bar{r}+\bar{b})\,2\bar{r}\dot{\bar{r}}}{(\bar{r}^2+\bar{b}^2)^2}, \quad \bar{b} \text{ is a constant vector.}$$

7. Given $\bar{r} = a\cos t\,\hat{i} + a\sin t\,\hat{j} + bt\,\hat{k}$, show that

(a) $(\bar{r}')^2 = a^2+b^2$ ; (b) $(\bar{r}'\times\bar{r}'')^2 = a^2(a^2+b^2)$

(c) $[\bar{r}'\bar{r}''\bar{r}'''] = a^2 b$.

$\bar{r}' = -a\sin t\,\hat{i} + a\cos t\,\hat{j} + b\hat{k} \Rightarrow (\bar{r}')^2 = a^2+b^2$

$\bar{r}'' = -a\cos t\,\hat{i} - a\sin t\,\hat{j}$

$\bar{r}''' = a\sin t\,\hat{i} - a\cos t\,\hat{j}$

9. Verify that $\bar{r} = \bar{a}e^{m_1 t} + \bar{b}e^{m_2 t}$ satisfies the differential eqn $\ddot{\bar{r}} + K\dot{\bar{r}} + l\bar{r} = \bar{0}$ ($K, l$ are scalars constants) where $m_1, m_2$ are two unequal roots of the eqn $m^2+Km+l=0$; $\bar{a}, \bar{b}$ are arbitrary constant vectors.

Also if $m_1 = m_2$, then $\bar{r} = (\bar{a}+t\bar{b})e^{m_1 t}$ will satisfy the above D.E.

$\dot{\bar{r}} = \bar{a}m_1 e^{m_1 t} + \bar{b}m_2 e^{m_2 t}$

$\ddot{\bar{r}} = \bar{a}m_1^2 e^{m_1 t} + \bar{b}m_2^2 e^{m_2 t}$

$\therefore \ddot{\bar{r}} + K\dot{\bar{r}} + l\bar{r} = \bar{a}(m_1^2+Km_1+l)e^{m_1 t} + \bar{b}(m_2^2+Km_2+l)e^{m_2 t}$

Now, $= \bar{a}.0e^{m_1 t} + \bar{b}.0e^{m_2 t} = \bar{0}$ Satisfied

8. If $\bar{r}$ be the displacement vector at any time t from the origin of a moving point, where $\bar{r} = \bar{a}\cos\omega t + \bar{b}\sin\omega t$, $\omega$ is a scalar constant.

(i) Find $\bar{V}$ and show that $\bar{r}\times\bar{V}$ = a constant vector $\vec{r} = \omega(\bar{a}\times\bar{b})$

(ii) Show that the acel$^n$ is directed towards the origin and $\propto$ to the displacement. $\left|\frac{d^2\bar{r}}{dt^2}\right| = |-\omega^2\bar{r}| \propto \bar{r}$

(iii) What is the physical interpretation of this motion?

The motion is that of a particle moving on the circumference of a circle with constant angular speed $\omega$. The force directed towards the centre of the circle is the centrepetal force.



Scanned with CamScanner

# MECHANICS :

1) Kinematics → study of the motion of particles along curve

2) Dynamics → study of the forces on moving objects.

**Q.** Show that the acceleration $\bar{a}$ of a particle which travels along a space curve with velocity $\bar{V}$ is given by $\bar{a} = \frac{dv}{dt}\hat{T} + \frac{v^2}{\rho}\hat{N}$. $\bar{T}$ : unit tangent vector, $\bar{N}$ : " principal normal, $\rho$ : radius of curvature

Solution : $\bar{V} = v\bar{T}$

$$\bar{a} = \frac{d\bar{V}}{dt} = \frac{dv}{dt}\bar{T} + v\frac{d\bar{T}}{dt} = \frac{dv}{dt}\bar{T} + v\frac{d\bar{T}}{ds}\frac{ds}{dt}$$

or, $\bar{a} = \frac{dv}{dt}\bar{T} + v^2 K\bar{N}$ $\left[ v = \frac{ds}{dt},\ \frac{d\bar{T}}{ds} = K\bar{N} \right]$

$\bar{a} = \frac{dv}{dt}\bar{T} + \frac{v^2}{\rho}\bar{N}$ $[\rho = 1/K]$.

$\frac{v^2}{\rho}$ → called the centripetal accln.

**Q** If $\bar{r}$ be the p.v. of a particle of mass $m$ relative to point O and $\bar{F}$ is the external force on the particle, then $\bar{r}\times\bar{F} = \bar{M}$ is the torque or moment of $\bar{F}$ about O. Show that $\bar{M} = \frac{d\bar{H}}{dt}$, where $\bar{H} = \bar{r}\times m\bar{V}$, $\bar{V}$ is the vel. of particle.

Soln: $\bar{M} = \bar{r}\times\bar{F} = \bar{r}\times\frac{d}{dt}(m\bar{V})$, by Newton's Law.

Now $\frac{d}{dt}(\bar{r}\times m\bar{V}) = \bar{r}\times\frac{d}{dt}(m\bar{V}) + \frac{d\bar{r}}{dt}\times m\bar{V}$

$= \bar{r}\times\frac{d}{dt}(m\bar{V}) + \underbrace{\bar{V}\times m\bar{V}}_{=0}$

or, $\bar{M} = \frac{d}{dt}(\bar{r}\times m\bar{V}) = \frac{d\bar{H}}{dt}$, $\bar{H}$ is called the angular momentum

i.e., Torque is equal to the time rate of change of angular momentum.

**Q** A particle moves along the curve $\bar{r} = ((t^3-4t), (t^2+4t), (8t^2-3t^3))$, find the magnitude of the tangential & normal components of its accln when $t=2$.

$\bar{a} = \frac{dv}{dt}\bar{T} + \frac{v^2}{\rho}\bar{N}$, $\frac{dv}{dt} = 16$, $\frac{v^2}{\rho} = 2\sqrt{73}$.



Scanned with CamScanner

Ex: If $\bar{\omega}$ is a constant vector, $\bar{r}$ and $\bar{s}$ are vector functions of $t$ and if $\frac{d\bar{r}}{dt} = \bar{\omega}\times\bar{r}$, $\frac{d\bar{s}}{dt} = \bar{\omega}\times\bar{s}$, then show that

$$\frac{d}{dt}(\bar{r}\times\bar{s}) = \bar{\omega}\times(\bar{r}\times\bar{s})$$

Soln: $\frac{d}{dt}(\bar{r}\times\bar{s}) = \frac{d\bar{r}}{dt}\times\bar{s} + \bar{r}\times\frac{d\bar{s}}{dt} = (\bar{\omega}\times\bar{r})\times\bar{s} + \bar{r}\times(\bar{\omega}\times\bar{s})$

$$= \bar{s}\times(\bar{r}\times\bar{\omega}) + \bar{r}\times(\bar{\omega}\times\bar{s})$$

$$= (\bar{s}.\bar{\omega})\bar{r} - \cancel{(\bar{s}.\bar{r})}\bar{\omega} + \cancel{(\bar{r}.\bar{s})}\bar{\omega} - (\bar{r}.\bar{\omega})\bar{s}$$

$$= (\bar{\omega}.\bar{s})\bar{r} - (\bar{\omega}.\bar{r})\bar{s} = \bar{\omega}\times(\bar{r}\times\bar{s}) \text{ proved.}$$

Ex: At any instant $t$ the displacement vector of a moving point is $\bar{r} = \bar{a}\cos\omega t + \bar{b}\sin\omega t$, where $\bar{a}, \bar{b}$ are constant vectors and $\omega$ is scalar constant.

(a) Find the vel. $\bar{V}$ & S.T. $\bar{r}\times\bar{V} =$ a constant vector,

(b) S.T. the acceleration is directed towards the origin and is proportional to the displacement. What is the physical interpretation of this motion?

Soln: (a) $\bar{r} = \bar{a}\cos\omega t + \bar{b}\sin\omega t$

$\therefore \bar{V} = \frac{d\bar{r}}{dt} = -\omega\bar{a}\sin\omega t + \bar{b}\omega\cos\omega t$

$\bar{r}\times\bar{V} = (\bar{a}\cos\omega t + \bar{b}\sin\omega t)\times\omega(-\bar{a}\sin\omega t + \bar{b}\cos\omega t)$

$= \omega(\bar{a}\times\bar{b}\cos^2\omega t + \bar{a}\times\bar{b}\sin^2\omega t) = \omega\,\bar{a}\times\bar{b} =$ a constant vector.

(b) $\frac{d^2\bar{r}}{dt^2} = -\omega^2(\bar{a}\cos\omega t + \bar{b}\sin\omega t) = -\omega^2\bar{r}$ (Centripetal accln)

$\left|\frac{d^2\bar{r}}{dt^2}\right| \propto |\bar{r}|$

The motion is that of a particle moving on the circumference of a circle with constant angular speed $\omega$. The force directed towards the centre of the circle, is the centripetal force.

Ex: A particle moves along the curve $\bar{r} = (t^3-4t)\hat{i} + (t^2+4t)\hat{j} + (8t^2-3t^3)\hat{k}$, $t$ is the time. Find the magnitudes of the tangential and normal components of its accln when $t = 2$.

Soln:- $\bar{V} = \frac{d\bar{r}}{dt} = (3t^2-4)\hat{i} + (2t+4)\hat{j} + (16t-9t^2)\hat{k} = 8\hat{i} + 8\hat{j} - 4\hat{k} = 4(2\hat{i}+2\hat{j}-\hat{k})$

$\bar{a} = \frac{d^2\bar{r}}{dt^2} = 6t\hat{i} + 2\hat{j} + (16-18t)\hat{k} = 12\hat{i} + 2\hat{j} - 20\hat{k}$

$v = \frac{ds}{dt} = \left|\frac{d\bar{r}}{dt}\right| = \sqrt{(3t^2-4)^2 + (2t+4)^2 + (16t-9t^2)^2} = \sqrt{64+64+16} = \sqrt{144} = 12$

$\bar{T} = \frac{d\bar{r}}{ds} = \frac{d\bar{r}}{dt}\Big/\frac{ds}{dt} = \frac{8\hat{i}+8\hat{j}-4\hat{k}}{12} = \frac{2\hat{i}+2\hat{j}-\hat{k}}{3}$.

$\therefore$ Tangential component of $\bar{a} = \bar{a}.\bar{T} = (12\hat{i}+2\hat{j}-20\hat{k}).\frac{(2\hat{i}+2\hat{j}-\hat{k})}{3}$

$= \frac{1}{3}.(24+4+20) = \underline{16}.$ Ans.