This indicates that the energy flow is in the direction of wave propagation. Since $\vec{E}$ is perpendicular to $\vec{k}$, from Eq. (15.6-12) we can write in terms of magnitude,

$$kE = \mu\omega H \quad \text{or} \quad \sqrt{\epsilon}E = \sqrt{\mu}H \quad \text{[Using Eq. (15.6-14)]}$$
$$\text{or} \quad \tfrac{1}{2}\epsilon E^2 = \tfrac{1}{2}\mu H^2. \tag{15.6-18}$$

This shows that *in case of electromagnetic waves in isotropic dielectric electromagnetic energy is equally shared between electric and magnetic fields.*

Total electromagnetic energy density is

$$u = \tfrac{1}{2}\epsilon E^2 + \tfrac{1}{2}\mu H^2 = \epsilon H^2. \tag{15.6-19}$$

So Eq. (15.6-17) can also be written as

$$\vec{s} = \frac{u}{\epsilon\mu\omega} \cdot k\hat{n} = uv\hat{n}, \tag{15.6-20}$$

where $v = 1/\sqrt{\epsilon\mu} = \omega/k$ is the speed of the wave.

Thus, Poynting's vector equals the energy density (u) times the velocity of the wave. This means that the energy associated with wave propagates with the same velocity with which the field vectors $\vec{E}$ and $\vec{H}$ propagate.

15.7 Plane Electromagnetic Waves in a Conducting Medium

Let us consider a linear, homogeneous and isotropic conducting medium characterised by constant permittivity ϵ, permeability μ and conductivity σ. To simplify the discussion we assume that the medium is charge free ($\rho = 0$) and external-current free such that the currents existing in the medium are induced only by the electromagnetic wave itself. Thus, we take $\vec{J} = \sigma\vec{E}$. Any initial charge distribution within the conductor dies out quickly. The charges move to the surface and make $\rho = 0$ inside.

For such a medium Maxwell's equations take the following form:

$$\vec{\nabla} \cdot \vec{E} = 0 \tag{15.7-1}$$

$$\vec{\nabla} \cdot \vec{H} = 0 \tag{15.7-2}$$

$$\vec{\nabla} \times \vec{E} = -\mu\frac{\partial \vec{H}}{\partial t} \tag{15.7-3}$$

$$\vec{\nabla} \times \vec{H} = \sigma\vec{E} + \epsilon\frac{\partial \vec{E}}{\partial t} \tag{15.7-4}$$

Taking curl of Eq. (15.7-3) and using Eq. (15.7-4) we get

$$\vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) - \nabla^2\vec{E} = -\mu\frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{H}\right) = -\sigma\mu\frac{\partial \vec{E}}{\partial t} - \epsilon\mu\frac{\partial^2 \vec{E}}{\partial t^2}.$$

Now using Eq. (15.7-1) we get

$$\nabla^2 \vec{E} - \sigma\mu\frac{\partial \vec{E}}{\partial t} - \epsilon\mu\frac{\partial^2 \vec{E}}{\partial t^2} = 0. \qquad (15.7\text{-}5)$$

Similarly, taking curl of Eq. (15.7-4) and using Eqs. (15.7-3) and (15.7-1) we can get

$$\nabla^2 \vec{H} - \sigma\mu\frac{\partial \vec{H}}{\partial t} - \epsilon\mu\frac{\partial^2 \vec{H}}{\partial t^2} = 0. \qquad (15.7\text{-}6)$$

Suppose we are interested in the plane wave solutions and assume that

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{j(\vec{k}\cdot\vec{r}-\omega t)} \quad \text{and} \quad \vec{H}(\vec{r},t) = \vec{H}_0 e^{j(\vec{k}\cdot\vec{r}-\omega t)}, \qquad (15.7\text{-}7)$$

where $\vec{E}_0$ and $\vec{H}_0$ are complex amplitudes which are constants in space and time, $\vec{k} = k\hat{n}$ is the propagation vector.

Substituting the solution (15.7-7) in Eq. (15.7-5) or (15.7-6) wet get

$$\left[-k^2 + j\omega\sigma\mu + \omega^2\epsilon\mu\right]\vec{E} = 0.$$

For nonzero solution,

$$k^2 = \omega^2\epsilon\mu + j\omega\sigma\mu. \qquad (15.7\text{-}8)$$

k is thus, complex in character here. Let $k = \alpha + j\beta$. Then

$$k^2 = \alpha^2 - \beta^2 + j2\alpha\beta. \qquad (15.7\text{-}9)$$

Comparing Eqs. (15.7-8) and (15.7-9) we get,

$$\alpha^2 - \beta^2 = \omega^2\epsilon\mu \quad \text{and} \quad 2\alpha\beta = \omega\sigma\mu.$$

On solving these two equations we get

$$\begin{aligned} \alpha &= \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1\right]^{1/2}, \\ \beta &= \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}-1\right]^{1/2}. \end{aligned} \qquad (15.7\text{-}10)$$

In terms of α and β the field vectors become

$$\begin{aligned} \vec{E}(\vec{r},t) &= \vec{E}_0 \cdot e^{-\beta\hat{n}\cdot\vec{r}} \cdot e^{j(\alpha\hat{n}\cdot\hat{r}-\omega t)} \\ \text{and} \quad \vec{H}(\vec{r},t) &= \vec{H}_0 \cdot e^{-\beta\hat{n}\cdot\vec{r}} \cdot e^{j(\alpha\hat{n}\cdot\hat{r}-\omega t)}. \end{aligned} \qquad (15.7\text{-}11)$$

These solutions show that the field amplitudes are spatially attenuated. The physical reason of the attenuation is that the wave sets up electric current in the medium, which causes dissipation of energy in the form of Joule heating. The quantity β, the imaginary part of wave number k, is a measure of attenuation and is called *attenuation constant*. It depends on the frequency ω and conductivity σ.

For nonconductors $\sigma = 0$ and then we have $\beta = 0$ meaning that there is no attenuation of the field vectors.

For goods conductors $\sigma/\omega\epsilon \gg 1$ and then we have

$$\alpha \simeq \beta \approx \sqrt{\frac{\omega\sigma\mu}{2}}. \tag{15.7-12}$$

The quantity $1/\beta$ measures the depth at which electromagnetic wave entering a conductor is attenuated to $1/e$ of its initial amplitude at the surface. It is known as *skin depth* or *penetration depth* into a conducting medium. Thus, skin depth δ for a good conductor is

$$\delta = \frac{1}{\beta} = \sqrt{\frac{2}{\omega\sigma\mu}}. \tag{15.7-13}$$

Obviously, skin depth δ decreases with increase in frequency and conductivity. For good conductors at high frequencies δ is very small. That is why in high frequency circuits current flows only through the surface of good conductors. This phenomenon is called *skin effect*. Due to this effect the ac resistance of a conductor is greater than its dc resistance. For this in high frequency circuits it is better to use a number of fine stranded wires instead of a thick wire. It increases surface area for a given area of cross-section and reduces resistance. At microwave frequencies δ for Ag is very small ($\sim 10^{-3}$ mm). As a result, in the microwave region the performance of a waveguide made of pure Ag and another waveguide made of Ag-coated brass would appear to be identical. This technique is used to reduce the material cost of good conductors.

The attenuation of electromagnetic waves in the conducting sea water creates problem in radiocommunication with submerged submarine.

Wavelength, propagation speed and the index of refraction

The real part of k, i.e., α determines the wavelength, the propagation speed of the wave and the index of refraction of the conductor, in the usual way. Thus,

$$\lambda_c = \frac{2\pi}{\alpha}, \quad v = \frac{\omega}{\alpha} \quad \text{and} \quad n = \frac{c\alpha}{\omega}$$

For a good conductor $\alpha = \sqrt{\frac{\omega\sigma\mu}{2}}$ and hence,

$$\lambda_c = 2\pi\sqrt{\frac{2}{\omega\sigma\mu}}, \quad v = \omega\sqrt{\frac{2}{\omega\sigma\mu}} = \sqrt{\frac{2\omega}{\sigma\mu}} \quad \text{and} \quad n = c\sqrt{\frac{\sigma\mu}{2\omega}}.$$

$\alpha^2 - \beta^2 = \omega^2 \epsilon \mu$. & $2\alpha\beta = \omega \sigma \mu$.

$\therefore (\alpha^2 + \beta^2)^2 = (\alpha^2 - \beta^2)^2 + 4\alpha^2\beta^2$

Note that the skin depth δ as given by (15.7-13) for a good conductor can be expressed in terms of the wavelength λ_c in the conductor as

$$\delta = \frac{\lambda_c}{2\pi}.$$

Relative directions of $\vec{E}$, $\vec{H}$ and $\vec{k}$

Substituting the solutions (15.7-7) in Eqs. (15.7-1) and (15.7-2) we get

$$\vec{k} \cdot \vec{E} = 0 \quad \text{and} \quad \vec{k} \cdot \vec{H} = 0. \tag{15.7-14}$$

These equations indicates that $\vec{E}$ and $\vec{H}$ are both perpendicular to the direction of propagation. So *electromagnetic waves in a conducting medium are transverse in nature.*

Again, the substitution of the solutions (15.7-7) in Eqs. (15.7-3) and (15.7-4) gives

$$j\vec{k} \times \vec{E} = -\mu\left(-j\omega\vec{H}\right) \quad \text{or} \quad \vec{k} \times \vec{E} = \mu\omega\vec{H} \tag{15.7-15}$$

$$\text{and} \quad j\vec{k} \times \vec{H} = \sigma\vec{E} - j\omega\epsilon\vec{E} \quad \text{or} \quad \vec{k} \times \vec{H} = -\left(\omega\epsilon + j\sigma\right)\vec{E}. \tag{15.7-16}$$

These two equations imply that $\vec{E}$ and $\vec{H}$ are mutually perpendicular and also they are perpendicular to the direction of propagation vector $\vec{k}$.

Relative phase of $\vec{E}$ and $\vec{H}$

From Eq. (15.7-15) we have

$$\vec{H} = \frac{1}{\mu\omega}\left(\vec{k} \times \vec{E}\right) = \frac{k}{\mu\omega}\left(\hat{n} \times \vec{E}\right) = \frac{\alpha + j\beta}{\mu\omega}\left(\hat{n} \times \vec{E}\right). \tag{15.7-17}$$

This equation shows that *$\vec{E}$ and $\vec{H}$ are not in phase in a conductor.*

Writing $\alpha + j\beta = \sqrt{\alpha^2 + \beta^2} e^{j\phi}$; $\phi = \tan^{-1}(\beta/\alpha)$ and using (15.7-7), Eq. (15.7-17) may be rewritten as

$$\vec{H} = \frac{\sqrt{\alpha^2 + \beta^2}}{\mu\omega}\left(\hat{n} \times \vec{E}_0\right) \cdot e^{j\left(\vec{k}\cdot\vec{r} - \omega t - \phi\right)}, \tag{15.7-18}$$

where

$$\sqrt{\alpha^2 + \beta^2} = \omega\sqrt{\epsilon\mu}\left[1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{1/4}.$$

Thus, *$\vec{H}$ lags behind $\vec{E}$ in time by the phase angle*

$$\phi = \tan^{-1}\frac{\beta}{\alpha} = \frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\epsilon}\right).$$

For good conductors $\alpha \approx \beta$ and $\phi = 45\,°$. Therefore, *the phase difference between the $\vec{E}$ and $\vec{H}$ fields in a perfect conductor is* $45\,°$.

Relative magnitudes of $\vec{E}$ and $\vec{H}$ is

$$\frac{\left|\vec{H}\right|}{\left|\vec{E}\right|} = \frac{H_0}{E_0} = \frac{\sqrt{\alpha^2+\beta^2}}{\mu\omega} = \sqrt{\frac{\epsilon}{\mu}}\left[1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{1/4}. \tag{15.7-19}$$

For good conductors

$$\frac{\left|\vec{H}\right|}{\left|\vec{E}\right|} = \sqrt{\frac{\sigma}{\omega\mu}}.$$

Thus, in this case,

$$\left|\vec{H}\right| \gg \left|\vec{E}\right|,$$

which indicates that in *a good conducting medium the field energy is not equally shared between $\vec{E}$- and $\vec{H}$-fields but it is almost entirely magnetic in nature.*

Poynting's vector

The time average Poynting's vector is

$$\langle \vec{s} \rangle = \tfrac{1}{2} Re\left(\vec{E} \times \vec{H}^*\right).$$

Using Eq. (15.7-17),

$$\begin{aligned}\langle \vec{s} \rangle &= \tfrac{1}{2}\frac{\sqrt{\alpha^2+\beta^2}}{\mu\omega} \cdot Re\left[\vec{E} \times \left(\hat{n} \times \vec{E}^*\right)\right] e^{-j\phi} \\ &= \frac{\sqrt{\alpha^2+\beta^2}}{2\mu\omega} Re\left[\hat{n}\left(\vec{E}\cdot\vec{E}^*\right) - \vec{E}^*\left(\vec{E}\cdot\hat{n}\right)\right] e^{-j\phi}\end{aligned}$$

Now putting $\vec{E}\cdot\hat{n} = 0$ and using Eq. (15.7-11) we get

$$\langle \vec{s} \rangle = \frac{\sqrt{\alpha^2+\beta^2}}{2\mu\omega} \cdot E_0^2 e^{-2\beta\hat{n}\cdot\vec{r}} \cdot \cos\phi \cdot \hat{n}. \tag{15.7-20}$$

For good conductors $\alpha \simeq \beta \approx \sqrt{\dfrac{\omega\sigma\mu}{2}}$ and $\phi = 45\,°$. Then

$$\langle \vec{s} \rangle = \frac{1}{2} \cdot \sqrt{\frac{\sigma}{2\mu\omega}} E_0^2 e^{-2\beta\hat{n}\cdot\vec{r}}\hat{n}. \tag{15.7-21}$$

Thus, *energy flow is along the direction of propagation of the wave and is damped off exponentially.*

Average electric energy density is

$$\langle u_e \rangle = \tfrac{1}{2} Re \tfrac{1}{2}\left(\vec{E}\cdot\vec{D}^*\right) = \tfrac{1}{4}\epsilon Re\left(\vec{E}\cdot\vec{E}^*\right) = \tfrac{1}{4}\epsilon E_0^2 \cdot e^{-2\beta\hat{n}\cdot\vec{r}}.$$

Average magnetic energy density is

$$\langle u_m \rangle = \tfrac{1}{2} Re \tfrac{1}{2}\left(\vec{B}\cdot\vec{H}^*\right) = \tfrac{1}{4}\mu Re\left(\vec{H}\cdot\vec{H}^*\right) = \tfrac{1}{4}\mu H_0^2 e^{-2\beta\hat{n}\cdot\vec{r}}.$$

Thus,

$$\frac{\langle u_m \rangle}{\langle u_e \rangle} = \frac{\mu H_0^2}{\epsilon E_0^2} = \left[1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{1/2} \tag{15.7-22}$$

Obviously, for good conductors $\langle u_m \rangle \gg \langle u_e \rangle$, i.e., *the field energy inside is almost entirely magnetic in nature.*

Total time averaged energy density is

$$\langle u \rangle = \langle u_e \rangle + \langle u_m \rangle = \tfrac{1}{4} e^{-2\beta\hat{n}\cdot\vec{r}}\left[\epsilon E_0^2 + \mu H_0^2\right]$$

Using Eq. (15.7-19) we can write

$$\langle u \rangle = \tfrac{1}{4} e^{-2\beta\hat{n}\cdot\vec{r}} \cdot \epsilon E_0^2 \left[1 + \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}\right]. \tag{15.7-23}$$

In term of α, the real part of wave number as given by (15.7-10), it can be expressed as

$$\langle u \rangle = \frac{\alpha^2}{2\mu\omega^2} E_0^2 e^{-2\beta\hat{n}\cdot\vec{r}}. \tag{15.7-24}$$

15.8 Electromagnetic Waves in Ionised Gases

Let us consider a dilute medium of ionised gases such as the ionosphere. The motion of positive ions may be ignored as they are massive as compared to the electrons. Also, the damping of the motion of free electrons due to collisions is negligible here. So the nonrelativistic ($v \ll c$) equation of motion of an electron of mass m and charge e under the action of the incident electromagnetic field will be

$$m\frac{d\vec{v}}{dt} = e\vec{E}, \tag{15.8-1}$$

where $\vec{v}$ is the instantaneous velocity of the electron. Here we have neglected the force due to magnetic field which is only v/c times the force due to electric field.

Assuming $\vec{E} = \vec{E}(\vec{r})\, e^{-j\omega t}$, we get on integration,

$$\vec{v} = \frac{e\vec{E}(\vec{r})}{-jm\omega} = \frac{je\vec{E}(\vec{r})}{m\omega}.$$

If n_0 is the number of free electrons per unit volume then the complex current density,

$$\vec{J} = n_0 e\vec{v} = \frac{jn_0e^2\vec{E}(\vec{r})}{m\omega}. \tag{2}$$

Comparing this equation with $\vec{J} = \sigma\vec{E}$, we find that the complex conductivity of the medium is given by

$$\sigma = \frac{jn_0e^2}{m\omega}. \tag{3}$$

In case of dilute ionised gaseous medium we can take $\rho = 0$, $\epsilon \simeq \epsilon_0$ and $\mu \simeq \mu_0$. So here Maxwell's equations take the following form.

$$\vec{\nabla}\cdot\vec{E} = 0, \quad \vec{\nabla}\cdot\vec{H} = 0, \quad \vec{\nabla}\times\vec{E} = -\mu_0\frac{\partial\vec{H}}{\partial t} \quad \text{and} \quad \vec{\nabla}\times\vec{H} = \sigma\vec{E} + \epsilon_0\frac{\partial\vec{E}}{\partial t}. \tag{4}$$

Following the procedure as adopted in Section 15.7, we can arrive at the wave equation,

$$\nabla^2\vec{E} - \sigma\mu_0\frac{\partial\vec{E}}{\partial t} - \epsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2} = 0. \tag{5}$$

A similar equation for $\vec{H}$ can also be obtained.

Assuming a plane wave solution, $\vec{E}(\vec{r},t) = \vec{E}_0 e^{j(\vec{k}\cdot\vec{r}-\omega t)}$ and substituting it in Eq. (15.8-5) we get

$$-k^2 + j\omega\sigma\mu_0 + \omega^2\epsilon_0\mu_0 = 0 \quad \text{or} \quad k^2 = \epsilon_0\mu_0\omega^2\left[1 + \frac{j\sigma}{\epsilon_0\omega}\right]. \tag{6}$$

Using the relation (15.8-3) and putting $1/\sqrt{\epsilon_0\mu_0} = c$ we get

$$k^2 = \frac{\omega^2}{c^2}\left[1 - \frac{\omega_p^2}{\omega^2}\right], \tag{7}$$

where $\omega_p = \sqrt{n_0e^2/\epsilon_0 m}$ is known as *electron plasma frequency.*

As ω/k is the speed of the electromagnetic wave through the medium, the refractive index of the medium is given by

$$n = \frac{c}{(\omega/k)} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \tag{8}$$

For $\omega > \omega_p$, k is real and waves can propagate freely. For example, the typical electron number density of metals is $n_0 \approx 10^{28}\,\text{m}^{-3}$ and the corresponding $\omega_p \sim 10^{16}\text{s}^{-1}$. For ultraviolet light $\omega > 10^{16}\,\text{s}^{-1}$. For this ultraviolet light can generally propagate in metals. For $\omega < \omega_p$, k is purely imaginary, say jk_1. In this case,

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{-k_1(\hat{n}\cdot\vec{r})}\cdot e^{-j\omega t}$$
$$\text{and} \quad \vec{H}(\vec{r},t) = \vec{H}_0 e^{-k_1(\hat{n}\cdot\vec{r})}\cdot e^{-j\omega t}. \tag{9}$$

Thus, *for $\omega < \omega_p$, the field vectors fall off exponentially as they penetrate into the medium and cannot pass through it.*

In case of ionosphere electron concentration increases and hence, refractive index decreases as we go deeper into it. This may cause total internal reflection of the radio waves from the ionosphere, depending on its frequency and angle of incidence.

Phase velocity of electromagnetic waves through the ionised gas is, from (15.8-8),

$$\frac{\omega}{k} = \frac{c}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}}. \tag{15.8-10}$$

Obviously, $\omega/k > c$ for $\omega > \omega_p$. This does not violate the principle of relativity because it is the group velocity not the phase velocity with which signals move from one place to other and this group velocity is always less than c, the speed of light in free space.

15.9 Polarization of Plane Electromagnetic Waves

We have seen that in a plane electromagnetic wave the $\vec{E}$-field vibrates in a plane perpendicular to the direction of propagation vector $\vec{k}$. In order to completely specify the wave we are to specify moment-by-moment direction of $\vec{E}$. Such a description corresponds to the polarization of the wave. The state of polarization of an electromagnetic wave is customarily given by the orientation of its $\vec{E}$-vector. If the $\vec{E}$-field of a plane wave is aligned along a fixed direction in space although its magnitude and sign may vary in time. It is said to be *linearly polarized.*

The plane containing the $\vec{E}$-vector and the direction of propagation $\left(\vec{k}\right)$ is called the *plane of vibration.* Suppose a linearly polarized electromagnetic wave is propagating along z-direction. The $\vec{E}$-vector of the wave may be in any direction in the xy-plane. So it can be considered as the superposition of two independent orthogonal plane polarized components having same amplitude and phase. Symbolically, a linearly polarized wave may be represented by

$$\begin{aligned} \vec{E}(z,t) &= \vec{E}_x(z,t) + \vec{E}_y(z,t) \\ &= \hat{\imath} E_{0x} e^{j(kz-\omega t)} + \hat{\jmath} E_{0y} e^{j(kz-\omega t)} \\ &= \left(\hat{\imath} E_{0x} + \hat{\jmath} E_{0y}\right) e^{j(kz-\omega t)}. \end{aligned} \tag{15.9-1}$$

The $\vec{E}$-vector has the amplitude $\sqrt{E_{0x}^2 + E_{0y}^2}$ and its direction makes an angle $\theta = \tan^{-1}(E_{0y}/E_{0x})$ with x-direction. θ is called *polarization angle.*

Other types of polarizations can be constructed by the superposition of two plane polarized waves, which are not in phase. Let us consider two plane polarized waves both moving along z-direction, one wave polarized along x-direction and the other along y-direction. Let us represent two such waves by

$$\vec{E}_x = \hat{\imath} E_{0x} e^{j(kz-\omega t+\phi_x)} \quad \text{and} \quad \vec{E}_y = \hat{\jmath} E_{0y} e^{j(kz-\omega t+\phi_y)}, \tag{15.9-2}$$

where ϕ_x and ϕ_y are some phase angles. Corresponding real fields are obtained by taking real parts. Thus,

$$\vec{E}_x = \hat{\imath} E_{0x} \cos(kz - \omega t + \phi_x) \quad \text{and} \quad \vec{E}_y = \hat{\imath} E_{0y} \cos(kz - \omega t + \phi_y)$$

Considering the special case when $\phi_x = \phi$ and $\phi_y = \phi \pm \pi/2$ we can write for the resultant wave by superposition as

$$\begin{aligned} \vec{E} = \vec{E}_x + \vec{E}_y &= \hat{\imath} E_{0x} \cos(kz - \omega t + \phi) \mp \hat{\jmath} E_{0y} \sin(kz - \omega t + \phi) \\ &= \hat{\imath} E_x \mp \hat{\jmath} E_y, \end{aligned} \tag{15.9-3}$$

where

$$\frac{E_x^2}{E_{0x}^2} + \frac{E_y^2}{E_{0y}^2} = 1.$$

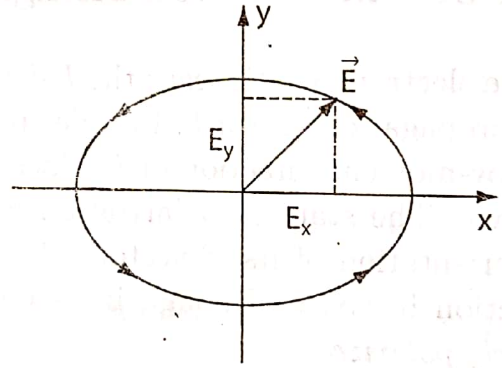

Fig 15.9-1: Left elliptically polarized wave.

This is equation of an ellipse for components of $\vec{E}$, with major and minor axes E_{0x} and E_{0y}, respectively. Thus, at any fixed plane z = constant the tip of $\vec{E}$-vector describes an ellipse. The sense of rotation—clockwise or anticlockwise—depends on the phase difference and can be seen by taking successive increasing values of ωt. Thus, the resultant of the components

$$\begin{aligned} E_x &= E_{0x} \cos(kz - \omega t + \phi) \\ \text{and} \quad E_y &= E_{0y} \sin(kz - \omega t + \phi) \end{aligned} \tag{15.9-4}$$

represents a vector $\vec{E}$, which rotates clockwise at an angular frequency ω, as seen by an observer towards whom the wave is moving. Such a wave is *right-elliptically polarized.* On the other hand, the resultant of the components

$$E_x = E_{0x} \cos(kz - \omega t + \phi) \quad \text{and} \quad E_y = -E_{0y} \sin(kz - \omega t + \phi) \tag{15.9-5}$$

is *left-elliptically polarized* (Fig 15.9-1).

If we let $E_{0x} = E_{0y} = E_0$ and keep the same phase conditions as above then $\vec{E} = \hat{\imath}E_x \mp \hat{\jmath}E_y$ with $E_x^2 + E_y^2 = E_0^2$.

This shows that the resultant wave is *circularly polarized.* The magnitude of $\vec{E}$ remains constant, although its direction rotates. The components

$$E_x = E_0 \cos(kz - \omega t + \phi) \quad \text{and} \quad E_y = E_0 \sin(kz - \omega t + \phi) \tag{15.9-6}$$

represent a wave, which is *right-circularly polarized.* On the other hand, the components

$$E_x = E_0 \cos(kz - \omega t + \phi) \quad \text{and} \quad E_y = -E_0 \sin(kz - \omega t + \phi) \tag{15.9-7}$$

represent a *left-circularly polarized wave.*

It is interesting to note that a circularly polarized wave may be regarded as the superposition of two orthogonal linearly polarized waves having a phase difference of $\pi/2$. Also, a linearly polarized wave can be regarded as the superposition of two oppositely polarized circular waves of equal amplitude. For example, if we add the right-circular wave of Eq. (15.9-6) to the left-circular wave of Eq. (15.9-7), we get for the resultant wave

$$E_x = 2E_0 \cos(kz - \omega t + \phi) \quad \text{and} \quad E_y = 0$$

and the resultant $\vec{E}$-vector is

$$\vec{E} = \hat{\imath}E_x + \hat{\jmath}E_y = \hat{\imath}2E_0 \cos(kz - \omega t + \phi).$$

Obviously, it is a linearly polarized wave.

15.10 Wave equation of Potentials with Sources—Gauge Transformations

So far we have considered EM waves without inquiring how these waves are produced. We did not consider any external charge and current sources. But it is found that *accelerated charges can produce EM radiations.* There are a number of procedures for the calculation of radiation from accelerated charges. However, the most fruitful one is the potential formulation. For a prescribed time dependent charge and current distributions one finds the scalar and vector potentials from which the fields are then obtained.

Since $\vec{\nabla} \cdot \vec{B} = 0$ we can always write $\vec{B}$ as the curl of some vector function $\vec{A}$, i.e.,

$$\vec{B} = \vec{\nabla} \times \vec{A}. \tag{15.10-1}$$

where $\vec{A}$ is called the *vector potential.* Now from Maxwell's Eq. (15.3-3),

$$\vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0.$$

So $\vec{E} + \frac{\partial \vec{A}}{\partial t}$ can be written as the gradient of some scalar function ϕ, i.e.,

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\phi \quad \text{or} \quad \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}, \tag{15.10-2}$$

where ϕ is known as *scalar potential.*

Now, substituting $\vec{H} = \vec{B}/\mu$, $\vec{E} = \vec{D}/\epsilon$ and Eqs. (15.10-1) and (15.10-2) into Maxwell's Eq. (15.3-4) we get

$$\frac{1}{\mu}\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \vec{J} - \epsilon\frac{\partial}{\partial t}\left(\vec{\nabla}\phi + \frac{\partial \vec{A}}{\partial t}\right)$$

$$\text{or} \quad \nabla^2 \vec{A} - \mu\epsilon\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu\vec{J} + \vec{\nabla}\left[\vec{\nabla}\cdot\vec{A} + \mu\epsilon\frac{\partial \phi}{\partial t}\right]. \tag{15.10-3}$$

Using Eq. (15.10-2) in the Maxwell's equation, $\vec{\nabla}\cdot\vec{D} = \rho$, we get

$$-\epsilon\left[\vec{\nabla}\cdot\vec{\nabla}\phi + \vec{\nabla}\cdot\frac{\partial \vec{A}}{\partial t}\right] = \rho \quad \text{or} \quad \nabla^2\phi + \frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{A}\right) = -\frac{\rho}{\epsilon}. \tag{15.10-4}$$

We now note that $\vec{A}$ defined by Eq. (15.10-1) is not unique because $\vec{B}$ remains unchanged if we replace $\vec{A}$ by

$$\vec{A}' = \vec{A} + \vec{\nabla}\psi, \tag{15.10-5a}$$

where ψ is a scalar function. To keep $\vec{E}$ unchanged as well, ϕ is to be simultaneously changed to

$$\phi' = \phi - \frac{\partial \psi}{\partial t}. \tag{15.10-5b}$$

The transformation (15.10-5) is called a *gauge transformation* and the invariance of the fields under such transformation is called *gauge invariance.* The freedom implied by (15.10-5) means that we can always choose a set of potentials $\vec{A}$, ϕ such that

$$\vec{\nabla}\cdot\vec{A} + \mu\epsilon\frac{\partial \phi}{\partial t} = 0. \tag{15.10-6}$$

This choice is referred to as the *Lorentz gauge.* Substituting this Lorentz gauge in Eqs. (15.10-3) and (15.10-4) we get uncoupled wave equations for each of $\vec{A}$ and ϕ.

$$\nabla^2 \vec{A} - \mu\epsilon\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu\vec{J} \tag{15.10-7}$$

$$\nabla^2 \phi - \mu\epsilon\frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon}. \tag{15.10-8}$$

It may be pointed out here that even the potentials, satisfying Lorentz condition, are not unique. If $\vec{A}$ and ϕ satisfy the Lorentz condition (15.10-6), $\vec{A}'$ and ϕ' will also satisfy the same condition namely

$$\vec{\nabla} \cdot \vec{A}' + \mu\epsilon \frac{\partial \phi'}{\partial t} = 0,$$

if the gauge function ψ satisfies the homogeneous scalar wave equation

$$\nabla^2 \psi - \mu\epsilon \frac{\partial^2 \psi}{\partial t} = 0.$$

All potentials in this restricted class are said to belong to the Lorentz gauge.

Many other gauge transformations are found in the literature. Common purpose of all these transformations is to simplify Eqs. (15.10-3) and (15.10-4). Suitability of a gauge depends on the problem at hand. Let us consider another useful gauge-the so-called *Coulomb gauge*. In this gauge one takes

$$\vec{\nabla} \cdot \vec{A} = 0. \qquad (15.10\text{-}9)$$

With this choice Eqs. (15.10-3) and (15.10-4) become

$$\nabla^2 \vec{A} - \mu\epsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J} + \mu\epsilon \vec{\nabla} \frac{\partial \phi}{\partial t} \qquad (15.10\text{-}10)$$

$$\nabla^2 \phi = -\frac{\rho}{\epsilon}. \qquad (15.10\text{-}11)$$

Eq. (15.10-11) can be easily solved to find ϕ as in electrostatics

$$\phi(\vec{r}, t) = \frac{1}{4\pi\epsilon} \int \frac{\rho(\vec{r}', t)}{|\vec{r} - \vec{r}'|} dV.$$

But here it is difficult to find $\vec{A}$ by solving (15.10-10). The Coulomb gauge is often utilized in cases where there are no charge or current distributions, i.e., $\rho = 0$, $\vec{J} = 0$. In this case, one takes $\phi = 0$ and $\vec{A}$ satisfies the equation

$$\nabla^2 \vec{A} - \mu\epsilon \frac{\partial^2 \vec{A}}{\partial t^2} = 0.$$

In this case the fields are derivable from a single potential

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad \text{and} \quad \vec{E} = -\frac{\partial \vec{A}}{\partial t}.$$

Problem 13. *The current flowing through a long solenoid of radius a is varied in such a manner that the axial magnetic field inside increases with time according to the law* $\vec{B} = \hat{r}B_0t^2$, *where* B_0 *is a constant. Find the displacement current density as a function of the distance r from the axis of the solenoid* ($r < a$).

Solution : From Maxwell's equation

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\hat{r}2B_0t.$$

Integrating over a cross-section of radius r we get

$$\int \vec{\nabla} \times \vec{E} \cdot d\vec{S} = -\int 2B_0t\hat{r} \cdot \hat{r}\, dS \quad \text{or} \quad \oint \vec{E} \cdot d\vec{l} = -2B_0t \cdot \pi r^2$$

$$\text{or,} \quad E \cdot 2\pi r = -2B_0t\pi r^2$$

$$\text{or,} \quad E = -B_0tr.$$

$\therefore$ The magnitude of displacement current density is

$$\left|\vec{J}_d\right| = \epsilon_0 \left|\frac{\partial \vec{E}}{\partial t}\right| = \epsilon_0 B_0 r.$$

Problem 14.

(*i*) *Show that for a good conductor skin depth* $\delta = \lambda_c/2\pi$, *where* λ_c *is the wavelength of electromagnetic waves in the conductor.* **[C.U. 2003]**

(*ii*) *Show that for an electromagnetic wave incident on a good conductor the electric vector reduces to about* 1% *at a depth of* 0.73 λ_c.

(*iii*) *Find the wavelength and the propagation speed in copper for radio waves at* 1 MHz. *For copper assume* $\mu = \mu_0$, $\epsilon = \epsilon_0$ *and conductivity* $\sigma = 5.8 \times 10^7 (\Omega\text{-m})^{-1}$.

Solution :

(*i*) Suppose the wave is incident normally on the surface of the conductor along z-axis. Then the electric field inside the conductor is given by Eq. (15.7-11) as

$$\vec{E}(z,t) = \vec{E}_0 e^{-\beta z} \cdot e^{j(\alpha z - \omega t)}.$$

For a good conductor

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}}$$

and skin depth

$$\delta = \frac{1}{\beta} = \sqrt{\frac{2}{\omega\mu\sigma}}.$$

The wavelength of the wave in the conductor is

$$\lambda_c = \frac{2\pi}{\alpha} = \frac{2\pi}{\beta} = 2\pi\delta.$$

$\therefore$ Skin depth $\delta = \lambda_c/2\pi$.

(*ii*) Let us write

$$E(z) = E_0 e^{-\beta z} = E_0 e^{-z/\delta} = E_0 e^{-2\pi z/\lambda_c}.$$

Now putting

$$\frac{E(z)}{E_0} = \frac{1}{100}$$

we get

$$z = \frac{\lambda_c}{2\pi} \ln \frac{E_0}{E(z)} = \frac{\lambda_c}{2\pi} \ln 100 = 0.73\,\lambda_c.$$

(*iii*) The required wavelength

$$\begin{aligned}\lambda_c &= \frac{2\pi}{\alpha} = 2\pi\sqrt{\frac{2}{\omega\mu\sigma}} \\ &= 2\pi\sqrt{\frac{2}{2\pi \times 10^6 \times 4\pi \times 10^{-7} \times 5.8 \times 10^7}}\ \text{m} \\ &= 4.15 \times 10^{-4}\ \text{m}\end{aligned}$$

and the propagation speed

$$v = \frac{\omega}{k_c} = \frac{\omega}{2\pi} \times \lambda_c = \frac{2\pi \times 10^6}{2\pi} \times 4.15 \times 10^{-4}\ \text{ms}^{-1} = 415\ \text{ms}^{-1}.$$

Problem 15. *The electric field associated with an electromagnetic wave is*

$$\vec{E} = \hat{x}E_0 \cos(kz - \omega t) + \hat{y}E_0 \sin(kz - \omega t),$$

where E_0 *is a constant. Find the corresponding magnetic field* $\vec{H}$ *and the Poynting's vector* $\vec{s}$.

Solution: We know that

$$\vec{H} = \frac{\vec{k} \times \vec{E}}{\mu\omega}.$$

Here $\vec{k} = \hat{z}k$.

$$\vec{H} = \frac{k}{\ldots}\hat{z} \times [\hat{x}E_0 \cos(\ldots$$

Poynting's vector

$$\vec{s} = \vec{E} \times \vec{H} = \hat{z}\frac{kE_0^2}{\mu\omega}\left[\cos^2(kz - \omega t) + \sin^2(kz - \omega t)\right] = \hat{z}\frac{kE_0^2}{\mu\omega}.$$

Problem 16. *An electromagnetic wave is propagating through a nonconducting medi characterised by permittivity* $5\epsilon_0$ *and permeability* $2\mu_0$. *The magnetic field associated u the wave is*

$$\vec{H} = \hat{y}2\cos(3z - \omega t)\ A/m,$$

where z *is in metre. Find the value of* ω. *Also find the electric field associated with wave. What is the direction of propagation of the wave?*

Solution: Speed of the wave is given by

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}}$$

$$\text{or} \quad \omega = \frac{k}{\sqrt{\mu\epsilon}} = \frac{3}{\sqrt{2\mu_0 \cdot 5\epsilon_0}} = \frac{3}{\sqrt{10}}c$$

$$= \frac{3}{\sqrt{10}} \times 3 \times 10^8\,\text{rad/s} = 2.846 \times 10^8\,\text{rad/s}.$$

From the equation

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t},$$

we get

$$j\vec{k} \times \vec{H} = -j\omega\epsilon\vec{E}.$$

$$\therefore \quad \vec{E} = -\frac{\vec{k} \times \vec{H}}{\epsilon\omega} = \hat{x}\frac{k \cdot 2}{\epsilon\omega}\cos(3z - \omega t)$$

$$= \hat{x}\frac{3 \cdot 2 \cdot \cos(3z - \omega t)}{5 \times 8.854 \times 10^{-12} \times 2.846 \times 10^8}$$

$$= \hat{x}476.2\cos(3z - 2.846 \times 10^8 t)\ \text{V/m}$$

The constant phase planes are

$$\phi = 3z - \omega t = \text{constant}.$$

The direction of propagation is perpendicular to these planes, hence,

$$\frac{\vec{k}}{k} = \frac{\vec{\nabla}\phi}{\left|\vec{\nabla}\phi\right|} = \hat{z},$$

i.e., the wave propagates along $+z$-direction.

Problem 19. *Consider the propagation of EM waves through dilute ionised gases such as the ionosphere. Hence, show that the critical frequency below which wave propagation through it is not possible is given by $f_c \simeq 9\sqrt{n_0}$, where n_0 is the number of electrons per metre^3.*

Solution: The effective refractive index of the medium is given by (15.8-8),

$$n = \sqrt{1 - \left(\frac{\omega_p}{\omega}\right)^2},$$

where $\omega_p = \sqrt{\frac{n_0 e^2}{\epsilon_0 m}}$ is the electron plasma frequency.

For free wave propagation n should be real, i.e., $\omega \geq \omega_p$.

$\therefore$ The critical frequency

$$f_c = \frac{\omega_p}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{n_0 e^2}{\epsilon_0 m}} = \frac{1}{2\pi} \cdot \frac{1.6 \times 10^{-19}}{\sqrt{8.854 \times 10^{-12} \times 9.1 \times 10^{-31}}}\sqrt{n_0} \approx 9\sqrt{n_0}\, Hz.$$

Problem 20. *Calculate the retarded potentials of an infinite, straight wire in which a constant current I_0 is turned on abruptly at $t = 0$. Also, find the resulting electric and magnetic fields.*

Solution: Consider an infinite straight wire in which a constant current I_0 is started at $t = 0$. Thus, current $I(t) = 0$ for $t < 0$ and $I(t) = I_0$ for $t \geq 0$. Since the wire is electrically neutral charge density $\rho = 0$ everywhere in the wire and hence, scalar potential produced by the wire at the point of observation P is zero. Assuming the wire to be along z-axis (Fig 15.P-20), we can write for the vector potential at P as

$$\vec{A}(\vec{r}, t) = \hat{k}\frac{\mu_0}{4\pi}\int_{-\infty}^{+\infty} \frac{I(t - R/c)dz}{R}.$$

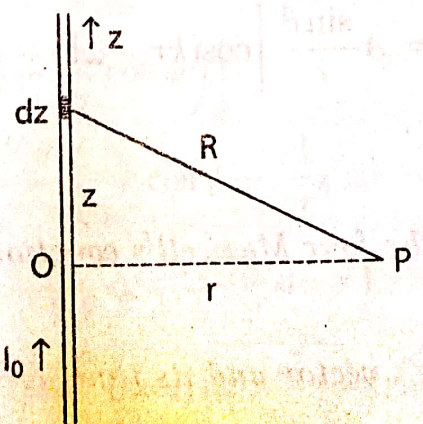

Fig 15.P-20