

FUNCTION OF SEVERAL VARIABLE 3011C

The Neighbourhood of a point :-

The set of values x, y other than a, b that satisfy the conditions $|x-a| < \delta, |y-b| < \delta$.

where δ is an arbitrarily small positive number, is said to form a neighbourhood of the point (a, b) . Thus the neighbourhood is the square.

$$(a-\delta, a+\delta; b-\delta, b+\delta)$$

where x takes any value from $a-\delta$ to $a+\delta$ except a and y from $b-\delta$ to $b+\delta$ except b ,

this is not only the way of specifying a neighbourhood of a point. There can be many other though equivalent ways; for example the points in the circle $x^2 + y^2 = \delta^2$ may be taken as a neighbourhood of the point $(0, 0)$.

Limit point :-

A point $(\bar{x}, \bar{y})$ is called a limit point or a point of condensation of a set of points, S , if every neighbourhood of $(\bar{x}, \bar{y})$ contains an infinite number of points of S . The limit points itself may or may not be a point of the set. For example, the point of the set $(0, 0)$ is a limit point of the set $\{(\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$.

The Limit of a function :-

A function $f(x, y)$ is said to ~~tend to~~ ^{tend to} a limit l as a point (x, y) tends to the point (a, b) , if to every arbitrarily small positive number ϵ , there corresponds a positive number δ , such that

$$|f(x, y) - l| < \epsilon.$$

for every point (x, y) , (different from (a, b)) which satisfies

$$|x-a| < \delta, |y-b| < \delta.$$

Symbolically we then write

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = l.$$

'l' is the limit (the double limit or the simultaneous limit) of f when x, y tends to a, b simultaneously.

Simultaneous and iterated limits :- Repeated limits

If $f(x, y)$ is a function of two variables x and y , we can define several kinds of limit. If (x_0, y_0) is the limiting point of a set of values on two dimensional space, then we have the limits

$$\lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} f(x, y), \quad \lim_{x \rightarrow x_0} \lim_{y \rightarrow y_0} f(x, y), \quad \lim_{y \rightarrow y_0} \lim_{x \rightarrow x_0} f(x, y).$$

We speak of the first type of limit as a simultaneous limit and the last two types as iterated limit.

Note that an iterated limit is nothing but a limit of a limit and can be found as in the case of function of a single variable. However, in simultaneous limit, we have something quite different from the ordinary simple limit.

EX: 1 Show that the simultaneous limit

$$\lim_{(x, y) \rightarrow (0, 0)} \left(\frac{xy^3}{x^2 + y^6} \right) \text{ does not exist,}$$

$\Rightarrow$ First let (x, y) approach $(0, 0)$ through the line $y = x$ which is a line through the origin. To do this we first put $y = x$ in the function and then allow x to approach 0, we have

$$\lim_{x \rightarrow 0} \frac{x^4}{x^2 + x^6} = \lim_{x \rightarrow 0} \frac{x^2}{1 + x^4} = \frac{0}{1} = 0$$

Again, we let (x, y) approach $(0, 0)$ through the curve $x = y^3$, i.e. we put $x = y^3$ in the function and then allow y to approach zero, in this

$$\lim_{y \rightarrow 0} \frac{y^6}{y^6 + y^6} = \lim_{y \rightarrow 0} \frac{1}{2} = \frac{1}{2}.$$

Since the limits obtained by two methods of approach are different, the simultaneous limit does not exist.

Ex: 2 let

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2}, & \text{if } x^4 + y^2 \neq 0 \\ 0, & \text{if } x = y = 0 \end{cases}$$

show that $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ does not exist.

Soln

If we approach $(0, 0)$ along any line $y = mx$ then $f(x, y) = f(x, mx) = \frac{mx^3}{x^4 + m^2 x^2} = \frac{mx}{1 + m^2} \rightarrow 0$ as $x \rightarrow 0$.

So any straight line approach gives

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = 0$$

But putting $y = mx^2$

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \lim_{x \rightarrow 0} f(x, mx^2) = \lim_{x \rightarrow 0} \frac{x^2 \cdot mx^2}{x^4 + m^2 x^4} = \lim_{x \rightarrow 0} \frac{m}{1 + m^2} = \frac{m}{1 + m^2}$$

$$\therefore \lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \frac{m}{1 + m^2} \neq 0 \text{ for } m \neq 0$$

which is different for the different m selected.

Hence $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ does not exist.

thus the function possesses no limit at the origin, but a straight line approach gives the limit zero.

Ex: 3 show that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2xy^4}{(x^2+y^4)} \text{ does not exist.}$$

Soln:

If we put $x = my^2$ and let $y \rightarrow 0$, we get

$$\lim_{y \rightarrow 0} \frac{2my^4}{(m^2+1)y^4} = \lim_{y \rightarrow 0} \frac{2m}{1+m^2} = \frac{2m}{1+m^2}$$

which is different for different values of m .
Hence the limit does not exist.

Notes:

The determination of a simultaneous limit is a difficult matter but a simple consideration very often, examples us to show that the limit does not exist. It is possible to determine the simultaneous limit by changing to polar.

Ex: 4 show that $\lim_{(x,y) \rightarrow (0,0)} \frac{xy(x^2-y^2)}{x^2+y^2} = 0$

Soln:

Put $x = r \cos \theta$
 $y = r \sin \theta$ $\therefore r^2 = x^2 + y^2$

Now, $\left| \frac{xy(x^2-y^2)}{x^2+y^2} \right|$

$$= \left| r^2 \sin \theta \cos \theta \frac{r^2 \cos^2 \theta - r^2 \sin^2 \theta}{r^2 \cos^2 \theta + r^2 \sin^2 \theta} \right|$$

$$= \left| \frac{r^2}{2} 2 \sin \theta \cos \theta (\cos^2 \theta - \sin^2 \theta) \right| = \left| \frac{r^2}{2} \sin 2\theta \cos 2\theta \right|$$

$$= \left| \frac{r^2}{4} \sin 4\theta \right| \leq \frac{r^2}{4}, \quad [\because |\sin \theta| \leq 1]$$

$$\therefore \left| \frac{xy(x^2-y^2)}{x^2+y^2} \right| \leq \frac{r^2}{4} < \epsilon, \quad \text{if } \frac{r^2}{4} < \epsilon/2,$$

$$\left| \frac{xy(x^2-y^2)}{x^2+y^2} \right| < \epsilon, \quad \text{if } |x| < \sqrt{2\epsilon}, \quad |y| < \sqrt{2\epsilon}.$$

$$\left| \frac{xy(x^2-y^2)}{x^2+y^2} \right| < \epsilon, \quad \text{if } |x| < \sigma, \quad |y| < \sigma,$$

where $\sigma = \sqrt{2\epsilon}$ (say)

that there exists $\delta > 0$ such that
 $|f(x,y) - 0| < \epsilon$, where $|x| < \delta$, $|y| < \delta$.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0.$$

✓ Ex: 5 Show that the simultaneous limit exist for the function $f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Soln:

$$\text{Put } x = r \cos \theta, y = r \sin \theta.$$

$$\text{then } f(x,y) = \frac{r \cos \theta \cdot r \sin \theta}{\sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta}} = \frac{r^2 \sin \theta \cos \theta}{r} = r \sin \theta \cos \theta = \frac{r}{2} \sin 2\theta.$$

$$\therefore \left| \frac{xy}{\sqrt{x^2+y^2}} \right| = \left| \frac{r}{2} \sin 2\theta \right| \leq \frac{r}{2} \cdot 1 = \frac{r}{2}$$

$$\therefore \left| \frac{xy}{\sqrt{x^2+y^2}} \right| \leq \frac{r}{2} < \epsilon$$

$$\Rightarrow \left| \frac{xy}{\sqrt{x^2+y^2}} \right| < \epsilon, \text{ if } \frac{\sqrt{x^2+y^2}}{2} < \epsilon.$$

$$\Rightarrow \left| \frac{xy}{\sqrt{x^2+y^2}} \right| < \epsilon, \text{ if } \sqrt{\frac{x^2}{4} + \frac{y^2}{4}} < \epsilon.$$

$$\Rightarrow \left| \frac{xy}{\sqrt{x^2+y^2}} \right| < \epsilon, \text{ if } \frac{x^2}{4} < \epsilon^2, \frac{y^2}{4} < \epsilon^2$$

$$\Rightarrow \left| \frac{xy}{\sqrt{x^2+y^2}} \right| < \epsilon, \text{ if } |x| < \sqrt{2}\epsilon, |y| < \sqrt{2}\epsilon.$$

thus for $\epsilon > 0$, $\exists \delta > 0$ such that

$$\left| \frac{xy}{\sqrt{x^2+y^2}} \right| < \epsilon, \text{ if } |x| < \delta, |y| < \delta, \text{ where } \delta = \sqrt{2}\epsilon. \text{ (Says)}$$

$$\therefore \lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$$

Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{\sqrt{1+x^2y^2} - 1}{x^2+y^2} = 0$

Soln:

$\therefore x, y$ are small, as $(x,y) \rightarrow (0,0)$

$$\begin{aligned} \therefore \frac{\sqrt{1+x^2y^2} - 1}{x^2+y^2} &= \frac{(1+x^2y^2)^{1/2} - 1}{x^2+y^2} \approx \frac{(1+\frac{1}{2}x^2y^2) - 1}{x^2+y^2} \\ &= \frac{\frac{1}{2}x^2y^2}{x^2+y^2} = \frac{x^2y^2}{2(x^2+y^2)} \end{aligned}$$

Now, Put $x = r \cos \theta$,
 $y = r \sin \theta$.

$$\begin{aligned} \therefore \left| \frac{x^2y^2}{2(x^2+y^2)} \right| &= \left| \frac{r^4 \cos^2 \theta \sin^2 \theta}{2r^2 (\sin^2 \theta + \cos^2 \theta)} \right| = \left| \frac{r^2}{2} \cos^2 \theta \sin^2 \theta \right| \\ &\leq \frac{r^2}{2} |\cos^2 \theta| |\sin^2 \theta| \leq \frac{r^2}{2} \cdot \begin{cases} |\cos \theta| \leq 1 \\ |\sin \theta| \leq 1 \end{cases} \\ \therefore \left| \frac{x^2y^2}{2(x^2+y^2)} \right| &\leq \frac{r^2}{2} = \frac{x^2+y^2}{2} = \frac{x^2}{2} + \frac{y^2}{2} \end{aligned}$$

$$\left| \frac{x^2y^2}{2(x^2+y^2)} \right| \leq \frac{x^2}{2} + \frac{y^2}{2}$$

$$\Rightarrow \left| \frac{x^2y^2}{2(x^2+y^2)} \right| < \epsilon, \text{ if } \frac{x^2}{2} < \epsilon/2, \frac{y^2}{2} < \epsilon/2$$

$$\Rightarrow \left| \frac{x^2y^2}{2(x^2+y^2)} \right| < \epsilon, \text{ if } |x| < \sqrt{2\epsilon}, |y| < \sqrt{2\epsilon}$$

$$\Rightarrow \left| \frac{x^2y^2}{2(x^2+y^2)} \right| < \epsilon, \text{ if } |x| < \delta, |y| < \delta$$

where $\delta = \sqrt{2\epsilon}$ (say)

thus for any $\epsilon > 0$, $\exists \delta > 0$ such that

$$\left| \frac{\sqrt{1+x^2y^2} - 1}{x^2+y^2} \right| < \epsilon, \text{ when } |x| < \delta, |y| < \delta.$$

$$\Rightarrow \left| \frac{\sqrt{1+x^2y^2} - 1}{x^2+y^2} - 0 \right| < \epsilon, \text{ when } |x| < \delta, |y| < \delta.$$

$$\therefore \lim_{(x,y) \rightarrow (0,0)} \frac{\sqrt{1+x^2y^2} - 1}{x^2+y^2} = 0$$

— 0 —

Exo 7 Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist, but

$\lim_{(x,y) \rightarrow (0,0)} xy \left(\frac{x^2 - y^2}{x^2 + y^2} \right)$ exists and its value is zero, [2011/88]

Soln: 0

Exo 8 Let $f(x,y) = \frac{x^2 y^2}{x^2 y^2 + (x-y)^2}$, $(x,y) \neq (0,0)$

Show that the repeated limits exist but the simultaneous limit does not exist as $(x,y) \rightarrow (0,0)$ [2011/91]

Soln: 0

$$\text{Now, } \lim_{x \rightarrow 0} \lim_{y \rightarrow 0} \frac{x^2 y^2}{x^2 y^2 + (x-y)^2}$$

$$= \lim_{x \rightarrow 0} \frac{x^2 \times 0}{0 + x^2} = 0$$

$$\text{Again, } \lim_{y \rightarrow 0} \lim_{x \rightarrow 0} \frac{x^2 y^2}{x^2 y^2 + (x-y)^2} = \lim_{y \rightarrow 0} \frac{0}{0 + y^2} = 0$$

Hence the repeated limits exist and they are equal and its value is zero.

Simultaneous limit 0 -

$$\text{Put } (x-y) = mn^2$$

$$\text{i.e. } y = -mn^2 + x = x - mn^2$$

$$\text{Now, } f(x,y) = f(x, x - mn^2)$$

$$= \frac{x^2 (x - mn^2)^2}{x^2 (x - mn^2)^2 + (mn^2)^2}$$

$$= \frac{1 - mn}{(1 - mn) + m^2} \rightarrow \frac{1}{1 + m^2} \text{ as } x \rightarrow 0$$

$$\therefore f(x,y) \rightarrow \frac{1}{1 + m^2} \text{ as } x \rightarrow 0$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{x^2 y^2 + (x-y)^2} = \frac{1}{1 + m^2}$$

which different for different values of m . Thus the simultaneous limit does not exist.

Hence for the given function the repeated limits exist but the simultaneous limit does not exist.

Soln: 0

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{1 + x^2 + y^2} = 0$$

Now,

$$|f(x,y) - 0| = \left| \frac{x^2 - y^2}{1 + x^2 + y^2} - 0 \right|$$

$$= \left| \frac{x^2 - y^2}{1 + x^2 + y^2} \right| \leq \frac{|x^2 - y^2|}{|1 + x^2 + y^2|}$$

$$\left[\because x^2 - y^2 \leq x^2 + y^2 \text{ and } |1 + x^2 + y^2| \geq 1 \Rightarrow \frac{1}{1 + x^2 + y^2} \leq 1 \right]$$

$$\Rightarrow |f(x,y) - 0| \leq |x^2 + y^2| \leq |x| + |y|$$

$$\Rightarrow |f(x,y) - 0| \leq \epsilon, \text{ if } |x| < \epsilon/2, |y| < \epsilon/2.$$

thus for $\epsilon > 0$, $\exists \delta > 0$ such that
 $|f(x,y) - 0| < \epsilon$, whenever even $|x| < \delta, |y| < \delta$
 where $\sqrt{\epsilon/2} = \delta$ (say)

this implies that -

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{1 + x^2 + y^2} = 0$$

(Proved)

Ex 1/0 Show that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = 0$$

Soln:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x \sin(x^2 + y^2)}{(x^2 + y^2)}$$

$$= \lim_{(x,y) \rightarrow (0,0)} x \cdot \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{(x^2 + y^2)}$$

$$= 0 \cdot 1 = 0.$$

Ex 1/1 Show that

$$\lim_{(x,y) \rightarrow (2,1)} \frac{\sin^{-1}(xy-2)}{\tan^{-1}(3xy-6)} = 1/3$$

Soln:

$$\lim_{(x,y) \rightarrow (2,1)} \frac{\sin^{-1}(xy-2)}{\tan^{-1}(3xy-6)}, \quad [\text{Put } xy-2 = z]$$

$$= \lim_{z \rightarrow 0} \frac{\sin^{-1} z}{\tan^{-1} z}, \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{z \rightarrow 0} \frac{1}{\sqrt{1-z^2}} \quad \left[\text{Using by L'Hospital rule} \right]$$

$$= \lim_{z \rightarrow 0} \frac{1+9z^2}{3\sqrt{1-z^2}} = \frac{1+0}{3} = 1/3.$$

$$\therefore \lim_{(m,y) \rightarrow (2,1)} \frac{\sin^2(m y - 2)}{\tan^2(3 m y - 6)} = 1/3 \quad (\text{proved})$$

← 0 →

Ex! 12 Show that the limit, when $(m,y) \rightarrow (0,0)$ does not exist for the function

$$f(m,y) = \frac{m^2 y^2}{m^2 y^2 + (m^2 - y^2)^2}$$

Soln

Put $m = m n^3 = y$

then $f(n, m - m n^3)$

$$= \frac{n^2 (m - m n^3)^2}{n^2 (m - m n^3)^2 + (n^2 - (m - m n^3)^2)^2}$$

$$= \frac{n^4 (1 - m n^2)^2}{n^4 (1 - m n^2)^2 + n^4 \{1 - (1 - m n^2)^2\}^2} \rightarrow 1 \text{ as } n \rightarrow 0.$$

Again, we put $y = m n$.

then

$$f(m,y) = f(n, m n) = \frac{n^2 m^2 n^2}{n^2 m^2 n^2 + (n^2 - m^2 n^2)^2}$$

$$= \frac{m^2}{m^2 + (1 - m^2)^2} \text{ as } n \rightarrow 0$$

which is different for different values of m .
Thus the simultaneous limit does not exist.

$$\therefore \lim_{(m,y) \rightarrow (0,0)} \frac{m^2 y^2}{m^2 y^2 + (m^2 - y^2)^2} \text{ does not exist.}$$

← 0 →

✓ Ex! 13 Show that the limit ~~exists~~ at the origin but the repeated limits does not exist,

$$\text{where } f(m,y) = \begin{cases} x \sin(1/y) + y \sin(1/x), & m \neq 0 \\ 0, & m = 0 \end{cases}$$

Soln Now the limits $\lim_{y \rightarrow 0} f(x, y)$ and $\lim_{x \rightarrow 0} f(x, y)$ do not exist, and hence $\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x, y)$ does not exist.

Again,

$$|x \sin 1/y + y \sin 1/x| \leq |x| + |y| < \epsilon.$$

$$\text{if } |x| < \epsilon/2, |y| < \epsilon/2$$

$$\therefore |(x \sin 1/y + y \sin 1/x) - 0| < \epsilon; \text{ if } |x| < \epsilon/2, |y| < \epsilon/2.$$

thus for $\epsilon > 0$, $\exists \delta > 0$ such that

$$|x \sin 1/y + y \sin 1/x - 0| < \epsilon, \text{ whenever } |x| < \delta, |y| < \delta \text{ when } \delta \leq \epsilon/2$$

$$\Rightarrow \lim_{(x, y) \rightarrow (0, 0)} (x \sin 1/y + y \sin 1/x) = 0$$

Ex 14 Show that the repeated limits exist at the origin and are equal, but the simultaneous limit does not exist, where

$$f(x, y) = \begin{cases} 1, & \text{if } xy \neq 0 \\ 0, & \text{if } xy = 0 \end{cases}$$

Soln

$$\text{Now, } \lim_{y \rightarrow 0} f(x, y) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\therefore \lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x, y) = 1$$

$$\text{Similarly } \lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x, y) = 1$$

hence the repeated limits exist and are two equal

Again, since there are points arbitrarily near $(0, 0)$ at which f is equal to 0 and points arbitrarily near $(0, 0)$ at which f is equal to 1. therefore there is an $\epsilon > 0$, such that

$$|f(x, y) - f(0, 0)| = |f(x, y)| = 1 \neq \epsilon.$$

for all points in any neighbourhood of $(0, 0)$.

Hence $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ does not exist.

Ex: 15 Show that $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ and $\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x,y)$ exist, but $\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x,y)$ does not,

$$\text{where } f(x,y) = \begin{cases} y + x \sin \frac{1}{y}, & y \neq 0 \\ 0, & y = 0 \end{cases}$$

$$\Rightarrow \text{Now, } |f(x,y) - 0| = |y + x \sin \frac{1}{y} - 0| \leq |y| + |x| < \epsilon \quad \text{if } |x| < \epsilon/2, |y| < \epsilon/2$$

$$\therefore |f(x,y) - 0| < \epsilon, \quad \text{if } |x| < \epsilon/2, |y| < \epsilon/2$$

thus, there exist $\delta > 0$, for $\epsilon > 0$,

$$|f(x,y) - 0| < \epsilon, \quad \text{whenever } |x| < \delta/2, |y| < \delta/2$$

$\Rightarrow \lim_{(x,y) \rightarrow (0,0)} f(x,y)$ exists and its value is zero.

Again,

$$\begin{aligned} \lim_{x \rightarrow 0} f(x,y) &= \lim_{x \rightarrow 0} (y + x \sin \frac{1}{y}) \\ &= y + 0 \cdot \sin \frac{1}{y} = y \end{aligned}$$

$$\therefore \lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x,y) = \lim_{y \rightarrow 0} y = 0$$

$$\text{but } \lim_{y \rightarrow 0} (y + x \sin \frac{1}{y})$$

$$= \lim_{y \rightarrow 0} y + x \cdot \lim_{y \rightarrow 0} \sin \frac{1}{y}$$

$$= 0 + x \cdot \lim_{y \rightarrow 0} \sin \frac{1}{y}$$

$$= x \cdot \lim_{y \rightarrow 0} \sin \frac{1}{y} = \text{does not exist,}$$

$\therefore \lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x,y)$ does not exist.

Continuity :-

A function f is said to be continuous at a point (a,b) of its domain of definition

$$\text{if, } \lim_{(x,y) \rightarrow (a,b)} f(x,y) = f(a,b)$$

Definition:
A function f is said to be continuous at a point (a, b) of its domain of definition if for $\epsilon > 0$, there exists a neighbourhood N of (a, b) such that
 $|f(x, y) - f(a, b)| < \epsilon, \forall (x, y) \in N$

Note:

The definition of continuity of a function f at a point (a, b) requires that besides (a, b) f is defined in a certain neighbourhood of (a, b) and moreover the limit of f when $(x, y) \rightarrow (a, b)$ exists and equals to the value $f(a, b)$.

Note:

A function is said to be continuous in region if it is continuous at every point of the same.

✓ Ex: 16 Show that the function
 $f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & x \neq 0, y \neq 0 \\ 0, & x = y = 0 \end{cases}$

is continuous at the origin in (x, y) position.

Soln:

Let $x = r \cos \theta$,
 $y = r \sin \theta$.

$$\therefore \left| \frac{xy}{\sqrt{x^2+y^2}} \right| = \left| \frac{r^2 \cos \theta \sin \theta}{r} \right| = |r \cos \theta \sin \theta|$$

$$\leq r |\cos \theta| |\sin \theta|$$

$$\leq r$$

or, $\left| \frac{xy}{\sqrt{x^2+y^2}} \right| \leq r, \forall \theta$ since $|\cos \theta| \leq 1$
 $\& |\sin \theta| \leq 1$.

$$\Rightarrow \left| \frac{xy}{\sqrt{x^2+y^2}} \right| \leq \sqrt{x^2+y^2} < \epsilon \quad \text{if } \begin{matrix} x^2 < \epsilon^2/2 \\ \text{or } y^2 < \epsilon^2/2 \end{matrix}$$

$$\Rightarrow \left| \frac{xy}{\sqrt{x^2+y^2}} \right| < \epsilon, \quad \text{if } |x| < \epsilon/\sqrt{2} \text{ or } |y| < \epsilon/\sqrt{2}$$

thus $\left| \frac{xy}{\sqrt{x^2+y^2}} \right| < \epsilon$, when $|x| < \epsilon/\sqrt{2}$, $|y| < \epsilon/\sqrt{2}$.

$$\Rightarrow \lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$$

$$\Rightarrow \lim_{(x,y) \rightarrow (0,0)} f(x, y) = f(0, 0)$$

Here f is cont. at $(0, 0)$.

✓ Ex 12 show that the function $f(x, y)$ defined as

$$f(x, y) = \begin{cases} \frac{2xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is continuous with respect to each of the variables x and y separately but is not continuous at the point $(0, 0)$.

Soln We consider y be constant.

$$\text{then } \lim_{x \rightarrow 0} f(x, y) = \lim_{x \rightarrow 0} \frac{0}{x^2 + y^2} = 0$$

$$\therefore \lim_{x \rightarrow 0} f(x, y) = f(0, y)$$

Hence $f(x, y)$ is continuous when y is constant.

Again, let x be constant.

$$\text{then } \lim_{y \rightarrow 0} f(x, y) = 0 \text{ and } f(x, 0) = 0$$

$$\therefore \lim_{y \rightarrow 0} f(x, y) = f(x, 0)$$

Thus $f(x, y)$ is ^{continuous} ~~constant~~ when x is constant.

Hence $f(x, y)$ is continuous separately w.r.t. x and y .

2nd part :-

$$\text{Put } x = r \cos \theta, \\ y = r \sin \theta.$$

$$\therefore \left| \frac{2xy}{x^2 + y^2} - 0 \right| = \left| \frac{2xy}{x^2 + y^2} \right| = \left| \frac{2r^2 \cos \theta \sin \theta}{r^2} \right| \\ \leq 2 |\cos \theta| |\sin \theta| \leq 2 \quad \forall \theta.$$

$\therefore \lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ does not exist.

Hence $f(x, y)$ is not continuous at $(0, 0)$.

Ex: 22 - It is possible to define $f(0,0)$ suitably so that the function f defined by

$$f(x,y) = \frac{xy(x^2+y^2)}{x^4+y^4}, \quad (x,y) \neq (0,0)$$

is continuous at $(0,0)$.

✓ Ex: 23 Discuss the continuity of the function

$$f(x,y) = \frac{x^2-y^2}{x^2+y^2}, \quad x \neq 0, y \neq 0$$

and $f(0,0) = 0$, at the point $(0,0)$.

Soln: let $x = r \cos \theta$
 $y = r \sin \theta$

$$\therefore \left| \frac{x^2-y^2}{x^2+y^2} \right| = \left| \frac{r^2 \cos 2\theta}{r^2} \right| = |\cos 2\theta| \leq 1 \quad \forall \theta$$

$\Rightarrow \lim_{(x,y) \rightarrow (0,0)} f(x,y)$ does not exist.

hence $f(x,y)$ is not continuous at $(0,0)$

Ex: 24 Show that the function $f(x,y)$ of the two variable x and y defined by

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$.

[Let's go]

Soln:

The function $f(x,y)$ will be continuous if for $\epsilon > 0$, $\exists \delta > 0$, such that

$$|f(x,y) - f(0,0)| < \epsilon \quad \text{whenever } |x| < \delta$$

$$\text{Now, } |f(x,y) - f(0,0)| = \left| \frac{x^2 y}{x^4+y^4} \right| = \left| \frac{x^2 y}{x^4+y^4} \right|$$

$$\leq \frac{|x^2 y|}{x^4+y^4} \quad \left[\begin{array}{l} |x^2 y| < |x^4 y| \\ x^2 < x^2 y^2 \\ |y| < \sqrt{x^2+y^2} \end{array} \right]$$

$$|f(x,y) - f(0,0)| < (x^2 + y^2)^{3/2} < \epsilon. \quad \text{if } x^2 \leq \frac{\epsilon^{2/3}}{2}, \quad y^2 \leq \frac{\epsilon^{2/3}}{2}$$

$$\text{thus for } \epsilon > 0, \exists \delta > 0 \text{ such that} \quad \text{if } |x| < \frac{\epsilon^{1/3}}{\sqrt{2}}, \quad |y| < \frac{\epsilon^{1/3}}{\sqrt{2}}.$$

whenever $|x| < \delta, |y| < \delta$
 Hence $f(x,y)$ is continuous at $(0,0)$ when $\delta = \frac{\epsilon^{1/3}}{\sqrt{2}}$.

Ex: 2.5 Let $f(x,y) = \begin{cases} \frac{2xy}{(x^2+y^2)^n}, & \text{if } (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Prove that (i) f is continuous at $(0,0)$ if $n = 1/2$
 (ii) f is discontinuous at $(0,0)$ if $n = 1$.

Soln: Now, $|f(x,y) - 0| = \left| \frac{2xy}{(x^2+y^2)^n} \right|$

Put $x = r \cos \theta$
 $y = r \sin \theta$

$$\therefore \left| \frac{2xy}{(x^2+y^2)^n} \right| = \left| \frac{2r^2 \cos \theta \sin \theta}{r^{2n}} \right| = \left| r^{2-2n} \sin 2\theta \right|$$

$$\leq r^{2-2n} [\sin 2\theta] \quad \text{[} \sin 2\theta \in [-1, 1] \text{]}$$

$$= (x^2+y^2)^{1-n}$$

$$= (x^2+y^2)^{-(n+1)}$$

$$= \begin{cases} 1, & \text{if } n = 1 \\ (x^2+y^2)^{-(n+1)}, & \text{if } n \neq 1 \end{cases}$$

$\therefore$ if $n = 1$, then $|f(x,y) - 0| < 1 < \epsilon$.

So that for $n = 1$, f is not continuous at $(0,0)$

But if $n = 1/2$, then
 $|f(x,y) - 0| \leq (x^2+y^2)^{1/2} < \epsilon, \quad \text{if } x^2 \leq \epsilon^2/2, \quad y^2 \leq \epsilon^2/2.$

$\therefore |f(x,y) - 0| < \epsilon, \quad \text{whenever } |x| < \delta, |y| < \delta; \quad \delta = \epsilon/\sqrt{2}.$

thus for $\epsilon > 0, \exists \delta > 0$, such that

$|f(x,y) - 0| < \epsilon, \quad \text{whenever } |x| < \delta, |y| < \delta.$

$\Rightarrow \lim_{(x,y) \rightarrow (0,0)} f(x,y) = f(0,0)$

Hence $f(x,y)$ is continuous at $(0,0)$ when $n = 1/2$

show that the following functions are not continuous at the origin.

$$(i) f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

$$(ii) f(x,y) = \begin{cases} \frac{x^2+y^2}{x-y} & , x \neq y \\ 0 & , x = y \end{cases}$$

Soln

① put $y = mx$

$$\text{then } f(x,y) = f(x, mx) = \frac{x^2 mx}{x^2 + m^2 x^2} = \frac{m}{1+m^2}$$

$$\therefore \lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{x \rightarrow 0} \frac{m}{1+m^2} = \frac{m}{1+m^2}$$

which is different for different values of m .

Hence the function is not cont. at $(0,0)$.

(ii) let (x,y) approach $(0,0)$ along the line curve $x-y = mn^2$.

$$\text{then } y = x - mn^2 = n(1-mn^2)$$

$$\begin{aligned} \therefore f(x,y) &= f(n, n-mn^2) \\ &= \frac{n^3 + (1-mn^2)^2 n^2}{mn^2} \\ &= \frac{1 + (1-mn^2)^2}{m} \Rightarrow \frac{2}{m} \text{ as } n \rightarrow 0. \end{aligned}$$

$$\text{thus } \lim_{(x,y) \rightarrow (0,0)} f(x,y) = \frac{2}{m}$$

which is different for different values of m .
Hence the simultaneous limit does not exist at $(0,0)$, thus the function $f(x,y)$ is not continuous at $(0,0)$.

Ex! 27 Show that the following functions are cont. at the origin

① $f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

② $f(x,y) = \begin{cases} \frac{x^2 y^3}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$