Sequence of Real Numbers

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A sequence is a function from the set of natural numbers. Now, if the co-domain set is the set of real numbers, complex numbers, functions, sets then we call the sequence as sequence of real numbers, complex numbers, functions, sets respectively.
Here we only discuss about sequence of real numbers, that means a function whose domain is the set of natural numbers and co-domain is the set of real numbers.
Suppose for each positive integer n, we are given a real number x_n. Then, the list of numbers, $x_1, x_2, \cdots, x_n, \cdots$ is called a sequence, and this ordered list is usually written as $(x_1, x_2, \cdots)$ or (x_n) or $\{x_n\}$. More precisely, we define a sequence as follows:

Definition A sequence of real numbers is a function from the set $\mathbb{N}$ of natural numbers to the set $\mathbb{R}$ of real numbers. If $x : \mathbb{N} \to \mathbb{R}$ is a sequence, and if $x_n = x(n)$ for $n \in \mathbb{N}$, then we write the sequence x as (x_n) or $(x_1, x_2, \cdots)$ or $\{x_n\}$ or $\{x_1, x_2, \cdots\}$.

Also we can say a sequence in $\mathbb{R}$ assigns to each natural number $n = 1, 2, \cdots$ a uniquely determined real number.
Examples

1. $\{n\} = \{1, 2, 3, 4, \cdots\}$
2. $\{\frac{1}{n}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$
3. $\{(-1)^n\} = \{-1, 1, -1, 1, -1, \cdots\}$
4. $\{\frac{1}{2^n}\} = \{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \cdots\}$
5. $\{a\} = \{a, a, a, \cdots\} \Rightarrow$ Constant sequence.

Definition (Range of a sequence) The distinct points in the sequence is called the range of of a sequence.

Remark 0.1 *It is to be noted that a sequence $\{x_1, x_2, \cdots\}$ is different from the set $\{x_n : n \in \mathbb{N}\}$. For instance, a number may be repeated in a sequence $\{x_n\}$, but it need not be written repeatedly in the set $\{x_n : n \in \mathbb{N}\}$. As an example, $\{1, 1/2, 1, 1/3, ..., 1, 1/n, ...\}$ is a sequence $\{x_n\}$ with $x_{2n-1} = 1$ and $x_{2n} = 1/(n+1)$ for each $n \in \mathbb{N}$, where as the set $\{x_n : n \in \mathbb{N}\}$ is same as the set $\{1/n : n \in \mathbb{N}\}$.*

1 Convergent Sequence

Given a sequence $\{x_n\}$, one is interested to know: what happens to x_n as n becomes large.

Definition A sequence $\{x_n\}$ is said to be converge to $x \in \mathbb{R}$ if $\forall\ \epsilon > 0$, there exist a natural number $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0,\ |x_n - x| < \epsilon$$

$$\Leftrightarrow x-\epsilon < x_n < x+\epsilon \ \forall n \geq n_0 \Rightarrow x_n \in (x-\epsilon, x+\epsilon) \ \forall n \geq n_0.$$

$(x-\epsilon, x+\epsilon)$ is called the ϵ-neighbourhood of x. x is called the limit of the sequence and we denote it by $\lim\limits_{n\to\infty} x_n = x$.

Geometrical Interpretation

Geometrically, $|x_n - x| < \epsilon \ \forall n \geq n_0 \Leftrightarrow x_n \in (x-\epsilon, x+\epsilon) \ \forall n \geq n_0$ means all except a finite number of elements must lie inside that interval $(x-\epsilon, x+\epsilon)$. ϵ is just to say that you can take an interval of any size. Suppose you take a smaller interval, that is a smaller ϵ, then you may have to change n_0 depending on that ϵ. You may have to take bigger n_0. But whatever it is that number is always finite.
If $\{x_n\}$ converges to x, then we write $\lim_{n\to\infty} x_n = x$, x is called the limit of $\{x_n\}$.
Geometrically, also we can say, sequence $\{x_n\}$ converges to x means if we take the distance from each x_n to x, then it is reducing, means $\{x_n\}$ is converging to x. So, the sequence must be convergent if the distance between x_n and x keep on reducing and reduced to zero. Mod is taken because x may be less than or greater than x_n.

Some Examples:

1. Show that the sequence $\{x_n\}$, $x_n = 1/n, \ n \in \mathbb{N}$ is convergent.
   As in this questions you have to show the sequence is convergent but the limit is not given, that means it is not asked to check whether the sequence is converging to some given point or not. As the limit is not given, at first you have to guess that limit then by applying definition you have to prove that your guess is right. Later by applying many results you will be able to find the limit easily. Till now as we know only the definition, so let's see how to apply the definition.
   So, here the sequence is $\{x_1, x_2, x_3 \cdots\} = \{1, 1/2, 1/3, \cdots\}$. As we see here as $n \to \infty$ $x_n \to 0$.
   $|x_n - x| = |1/n - 0| = 1/n$. To show $\forall \epsilon > 0$, there exist $n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow 1/n < \epsilon \Leftrightarrow n > 1/\epsilon$. Now if you take a n_0, which is bigger than $1/\epsilon$, that will prove the result. So we take $n_0 = [1/\epsilon] + 1$. Then $n > 1/\epsilon > [1/\epsilon] + 1 = n_0$.
   Now does such n_0 in $\mathbb{N}$ exit? Answer is yes. We know that $\mathbb{N}$ is not bounded above. So, whatever ϵ is given $1/\epsilon$ is a positive real number. We can always find some natural number bigger than that, because $\mathbb{N}$ is not bounded above.

2. Show that $\{\frac{n^2+1}{n^2}\}$ converges to 1.

3. Show that $\lim_{n\to\infty} x_n = 2$, where $x_n = \frac{2n+1}{n+1}$.

4. $\{x_n\}$, $x_n = 2 \ \forall n$ is convergent, in particular show that every constant sequence is convergent.

5. For a given $k \in \mathbb{N}$, let $a_n = \frac{1}{n^{1/k}}, \ \forall n \in \mathbb{N}$.

6. Show that the sequence $\{\frac{\cos n\pi}{\sqrt{n}}\}$ is convergent.
   Sol^n: As $|\frac{\cos n\pi}{\sqrt{n}}| < \frac{1}{\sqrt{n}}$, given any $\epsilon > 0$, if we choose a natural number n_0 such that $\frac{1}{n_0} < \epsilon^2$, then for every $n > n_0$,
   $$|\frac{\cos n\pi}{\sqrt{n}}| \leq \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{n_0}} < \epsilon.$$
   Hence $\lim_{n\to\infty} \frac{\cos n\pi}{\sqrt{n}} = 0$.

7. Show that $\lim_{n\to\infty} \frac{5}{3n+1} = 0$.

8. Show that $\lim_{n\to\infty} \frac{n^{2/3}\sin(n!)}{n+1} = 0$.

9. Show that $\lim_{n\to\infty} \frac{n}{n+1} - \frac{n+1}{n} = 0$.

Some Important Limits:

1. $x_n = r^n$, $|r| < 1$. Show that $\lim_{n\to\infty} r^n = 0$.

Case I: If $r = 0$, then the sequence is $\{0, 0, \cdots\} \Rightarrow$ constant sequence converges to zero.

Case II: $r \neq 0$, $|r| < 1 \Rightarrow \frac{1}{|r|} > 1$.
Let $\frac{1}{|r|} = a + 1$, where $a > 0$. Now $|r^n - 0| = |r^n| = |r|^n = \frac{1}{(a+1)^n}$.
We have, $(1+a)^n = 1 + n1 + \frac{n(n-1)}{2!}a^2 + \cdots \Rightarrow (1+a)^n > na$. So, $|r|^n < \epsilon \Rightarrow \frac{1}{(1+a)^n} < \frac{1}{na} < \epsilon \Rightarrow \frac{1}{na} < \epsilon \Rightarrow n > \frac{1}{a\epsilon}$. So, for $\forall \epsilon > 0$, $|r^n - 0| < \epsilon$ holds if $n > \frac{1}{a\epsilon}$.
Let, $n_0 = [\frac{1}{a\epsilon}] + 1$. So, $\forall \epsilon > 0$, $|r^n - 0| < \epsilon \ \forall n \geq n_0$.

2. $\lim_{n\to\infty} a^{1/n} = 1$ if $a > 0$.
Sol^n : If $a = 1$ the sequence $\{1, 1, 1, \cdots\}$ converges to 1.
For $a > 1$,

$$\Rightarrow a^{1/n} > 1$$

$$\Rightarrow a^{1/n} = 1 + x_n, \ \ \textit{where } x_n > 0.$$

Then showing $a^{1/n} \to 1$ is equivalent to showing $x_n \to 0$ as $n \to \infty$. Now,

$$a = (1 + x_n)^n = 1 + nx_n + \frac{n(n-1)}{2!}x_n^2 + \cdots$$

$$\Rightarrow a > 1 + nx_n \Rightarrow x_n < \frac{a-1}{n}.$$

So, $|x_n - 0| < \epsilon \Rightarrow |x_n| < \frac{a-1}{n} < \epsilon \Rightarrow n > \frac{a-1}{\epsilon}$.
Therefore for $n \geq n_0 (= [(a-1)/\epsilon] + 1)$, $|x_n| < \epsilon$.
Let $0 < a < 1$, then $b = 1/a \Rightarrow b > 1$. So $\lim_{n\to\infty} a^{1/n} = \lim_{n\to\infty} \frac{1}{b^{1/n}} = 1$

3. $\lim_{n\to\infty} n^{1/n} = 1$.

4. $\lim_{n\to\infty} \frac{1}{n^p} = 0$, $p > 0$.

5. $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$, $p > 0$, $\alpha > 0$

2 Divergent Sequence

Now, suppose we want to show that a sequence is not convergent. This is usually more difficult thing to do.
Let us first see how one can attempt to show it by using the definition. We shall see easier methods a little later.
By the definition saying that a sequence does not converge means what?
It means no such real number exist that means whatever x you take, for that x such a thing is not going to happen. For convergence we say for every ϵ something happens, if that is false means there exists some ϵ for which it does not happen. That means "for every $\epsilon > 0$ there exist n_0 such that $\forall n \geq n_0$ $|x_n - x| < \epsilon$" is false, that means whatever n_0 you take, you can always find some $n \geq n_0$ such that $|x_n - x| > \epsilon$.
To show that a sequence is not convergent using only the definition:
To show $\{x_n\}$ is not convergent. There are two things. Suppose you wanted to show that a sequence $\{x_n\}$ does not converge a given real number x. Then you need not take for every x in $\mathbb{R}$. But now just want to say that $\{x_n\}$ is not convergent. That means we need to show $\forall x \in \mathbb{R}$, there exist $\epsilon > 0$ such

that $|x_n - x| > \epsilon_0$. If you want to show a sequence is not convergent by using definition, this is what you need to show. We will see easier methods later.

1. **Suppose I want to show that $\{1, -1, 1, -1, \cdots\}$ is not convergent.**
   By the definition, geometrically it means that all the terms of the sequence go closer to x. Roughly speaking it means that, $\forall \epsilon > 0$, there exist n_0 such that $\forall n \geq n_0$, $|x_n - x| < \epsilon$, means after n_0 all terms are close to x. Now, if all terms are close to x, the question is, can there exist some x such that 1 is close to x and -1 is also close to x? Now that is obviously can't happen. suppose we take $\epsilon_0 = 1/2$. Thus if the sequence $\{x_n\}$ converges to x, then all terms of that sequence must lie in the interval $(x - 1/2, x + 1/2)$. But whatever n_0 we take, you can always find some n such that $x_n = 1$. Similarly we can always find some n for which $x_n = -1$ and both of them have to lie on that interval. So, 1, -1 must belong to the interval $(x - 1/2, x + 1/2)$. Now can this happen? The length of the interval is 1 and 1 and -1 both of them belong to an interval, then obviously it's length has to be at least 2. **So, this sequence does not converge.**
   It is very difficult to prove this kind of thing for every non-convergent sequence. So, proving that a sequence is convergent or not using definition along, can be quite difficult task and that is why we shall see easier methods to show both of the sequence convergent and divergent.
   Using another way we can show that the sequence is not convergent. Suppose we consider $\{x_n\}$ converges to -1, then for every ϵ, in particular we take $\epsilon = 1/2$, there must should exist a n_0 such that
   $$|x_n + 1| < 1/2 \ \forall n \geq n_0$$
   $$\Rightarrow -1 - 1/2 < x_n < -1 + 1/2 \ i.e., \ \ -3/2 < x_n < -1/2 \ \forall n \geq n_0,$$
   which is not true for every odd n, $n > n_0$. Hence $x_n \to -1$ is not true. Similarly we can show that $x_n \to x$, is not possible for any $x \in \mathbb{R}$.

2. The sequence $\{x_n\}$, where $x_n = n$, $\forall n$ is not convergent. Suppose, $\lim\limits_{n\to\infty} n = x$. Then given $\epsilon = 1$, there exist $n_0 \in \mathbb{N}$ such that $|x_n - x| = |n - x| < 1 \ \forall n \geq n_0$, which is not true by Archimedian property. In some sense $\{x_n\}$ is not convergent as it outgrows every real number.

3. Show that $\{\frac{n^2}{n+1}\}$ is not convergent using definition.

4. Show that $\left\{(-1)^n \left(\frac{1}{2} - \frac{1}{n}\right)\right\}$ is not convergent using only definition.

3 Bounded Sequence

Definition A sequence $\{x_n\}$ is said to be bounded if there exist real numbers k_1 and k_2 such that $k_1 \leq x_n \leq k_2 \ \forall n$.
A sequence is said to be bounded below if there exist some real number k_1 such that $k_1 \leq x_n \ \forall n$.
A sequence is said to be bounded above if there exist some real number k_2 such that $x_n \leq k_2 \ \forall n$.
So, if a sequence is both bounded above and below then that is bounded sequence.

Also we can say a sequence is bounded if there exist some real number A such that $|x_n| \leq A \ \forall n$. This is equivalent to the first definition.
Suppose we take $A = \max(k_1, k_2) \Rightarrow A \geq k_1, \ A \geq k_2 \Rightarrow, -A \leq -k_1 \ A \geq k_2$. So,

$$-k_1 < k_1 \leq x_n \leq k_2 \Rightarrow -A \leq -k_1 < k_1 \leq x_n \leq k_2 \leq A \Rightarrow |x_n| \leq A.$$

Example 1. Consider the sequence $\left\{\frac{(-1)^n}{n^2}\right\}$. Since for every n, $|\frac{(-1)^n}{n^2}| = \frac{1}{n^2} < 1$, the sequence is bounded.

2. Consider the sequence $\left\{\frac{2^n}{n}\right\}$. It is easy to show, using induction that $2^n > n^2$ for $n \geq 5$. Hence the sequence is not bounded.

Theorem 3.1 *Every convergent sequence is bounded.*

Proof To show a sequence is bounded, means you need to show the sequence is bounded above and bounded below. That means you have to find two real numbers k_1, k_2 such that $k_1 \leq x_n \leq k_2\ \forall n$.
Let $\{x_n\}$ be a convergent sequence, converging to x. Then $\forall \epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that

$$|x_n - x| < \epsilon\ \forall n \geq n_0$$

$$\Rightarrow x - \epsilon < x_n < x + \epsilon\ \forall n \geq n_0. \tag{1}$$

By Equation 1 we have found upper and lower bounds for the sequence except the first $n_0 - 1$ terms. But to show the boundedness of the sequence we have to find both upper and bounds of all the terms of the sequence.
We consider $k_2 = \max\{x_1, x_2, \cdots, x_{n_0-1}, x + \epsilon\}$. Then

$$k_2 \geq x_n\ \forall n \leq n_0 - 1 \quad \textit{and}\ k_2 \geq x + \epsilon. \tag{2}$$

So by Equation 1 and 2 $x_n \leq k_2\ \forall n$.
Similarly for $k_1 = \min\{x_1, x_2, \cdots, x_{n_0-1}, x - \epsilon\}$, $k_1 \leq x_n\ \forall n$. So we get two real numbers k_1 and k_2 such that $k_1 \leq x_n \leq k_2\ \forall n$.
Hence the proof.

Remark 3.2 *A bounded sequence does not have to be convergent. See, for example* $\{1, -1, 1, -1, \cdots\}$*. This sequence is bounded, it's lower bound is -1 and upper bound is 1. But this sequence is not convergent.*

Remark 3.3 *A convergent sequence is bounded. Hence an unbounded sequence must be divergent. For example,* $\{n\}$ *is divergent as it is not bounded.*

Example State if the following sequences converge or diverge. If the sequence converges, find its limit. If it diverges, explain why.
1. $\{n^2 - 2n + 1\}$ **2.** $\{n^3\}$ **3.** $\{1 + \frac{(-1)^n}{n}\}$ **4.** $\{(-1)^n\}$
5. $\{\sqrt{1 - 1/n}\}$ **6.** $\{\frac{1}{2}, \frac{2}{3}\frac{3}{4}, \frac{4}{5}, \cdots\}$ **7.**$\{1 + \frac{(-1)^n}{2n-1}\}$ **8.**$\{(-3/2)^n\}$