

Fourier Series

A few general rules

1. If $f(x)$ is an odd function of x

i.e. $f(-x) = -f(x)$, then $\int_{-\infty}^{\infty} f(x) dx = 0$

Thus $\int_{-\pi}^{\pi} \sin nx dx = 0$ $\because \sin nx$ is an odd function of x .

2. If $f(x)$ is an even function of x

i.e. $f(-x) = f(x)$, then $\int_{-\infty}^{\infty} f(x) dx = 2 \int_0^{\infty} f(x) dx$

Thus $\int_{-\pi}^{\pi} \cos nx dx = 2 \int_0^{\pi} \cos nx dx$ $\because \cos nx$ is an even function of x .

$$\therefore \int_{-\pi}^{\pi} \cos nx dx = 2 \int_0^{\pi} \cos nx dx$$

$$= 2 \left[\frac{\sin nx}{n} \right]_0^{\pi} = 0$$

$$3. \int_{-\pi}^{\pi} \cos mx \cos nx dx = 2 \int_0^{\pi} \cos mx \cos nx dx$$

$$= 2 \int_0^{\pi} \{ \cos(m+n)x + \cos(m-n)x \} dx$$

$$= \left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n} \right]_0^{\pi} = 0$$

$$4. \int_{-\pi}^{\pi} \sin mx \sin nx dx = 2 \int_0^{\pi} \sin mx \sin nx dx$$

$$= \int_0^{\pi} \{ \cos(m-n)x - \cos(m+n)x \} dx$$

$$= \left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right]_0^{\pi} = 0$$

$$5. \int_{-\pi}^{\pi} \sin^2 nx dx = 2 \int_0^{\pi} \sin^2 nx dx$$

$$= \int_0^{\pi} (1 - \cos 2nx) dx$$

$$= \left[x - \frac{\sin 2nx}{2n} \right]_0^{\pi} = \pi$$

$$6. \int_{-\pi}^{\pi} \cos^2 nx dx = 2 \int_0^{\pi} \cos^2 nx dx$$

$$= \int_0^{\pi} (1 + \cos 2nx) dx$$

$$= \left[x + \frac{\sin 2nx}{2n} \right]_0^{\pi} = \pi$$

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Fourier's Expansion:

Let us assume that a given function of x defined in the interval $(-\pi, \pi)$ can be expanded in the trigonometrical series of the form

$$f(x) = a_0 + (a_1 \cos x + a_2 \cos 2x + \dots + a_n \cos nx) + \dots$$

Proof of x .

$$\dots + (b_1 \sin x + b_2 \sin 2x + \dots + b_n \sin nx)$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

$1/2$

where $a_0, a_1, a_2, \dots$ and $b_1, b_2, b_3, \dots$ are all constants.

even x .

Let us further assume that $f(x)$ is bounded and integrable in this interval, if not bounded let the improper integral $\int_{-\pi}^{\pi} f(x) dx$ be absolutely convergent.

Then integrating term by term between the limits $-\pi$ to π , it can be shown that

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \quad \text{--- (2)}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \quad \text{--- (3)}$$

$$\text{and } b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \quad \text{--- (4)}$$

$\int dx$

$= 0$

Then the series (1) with the coefficients a_0, a_n, b_n determined by (2), (3) and (4) is called the Fourier series corresponding to the function $f(x)$ in the interval $(-\pi, \pi)$ and the coefficients a_0, a_n, b_n are called Fourier coefficients or Fourier constants.

Dirichlet's Conditions:

A function $f(x)$ is said to satisfy Dirichlet's Condition in (a, b) in which it is defined if it satisfies the following two conditions:

(1) $f(x)$ is bounded in (a, b) and the interval can be broken up into a finite number of open sub-intervals in each of which $f(x)$ is monotonic.

(2) $f(x)$ has a finite number of points of infinite discontinuities in (a, b) , but when arbitrary small neighbourhood of these points are excluded, $f(x)$ becomes bounded in the rest of the interval, and the rest of the interval can be broken up into a finite number of open sub-intervals in each of which $f(x)$ is monotonic. Further the infinite integral $\int_a^b f(x) dx$

* Statement of the results known as Dirichlet's Integrals:

If $f(x)$ be bounded, monotonic and integrable in the closed interval $[0, b]$ and a is a point where $0 < a < b$, then

$$\lim_{n \rightarrow \infty} \int_0^a f(x) \frac{\sin nx}{x} dx = \frac{1}{2} f(+0) \text{ for } 0 < a$$

$$\text{and } \lim_{n \rightarrow \infty} \int_a^b f(x) \frac{\sin nx}{x} dx = 0 \text{ for } 0 < a < b$$

where $f(+0)$ denotes the limits of $f(x)$ as $x \rightarrow 0$ through positive values.

* Theorem-1: If the Series $\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

where a_0, a_n and b_n are constants, is uniformly convergent in $(-\pi, \pi)$ and if $f(x)$ is its sum, then a_0, a_n and b_n for all values of n

Proof: We have $f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \dots (1)$

Since the Series for $f(x)$ given by (1) is uniformly convergent in $(-\pi, \pi)$, it is integrable term by term in $(-\pi, \pi)$.

$$\begin{aligned} \therefore \int_{-\pi}^{\pi} f(x) dx &= \frac{1}{2} a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx dx \\ &\quad + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx dx \\ &= \frac{1}{2} a_0 [x]_{-\pi}^{\pi} + 0 + 0. \text{ as } \int_{-\pi}^{\pi} \sin nx dx = 0 \\ &\quad \text{and } \int_{-\pi}^{\pi} \cos nx dx = 0 \\ &= \frac{1}{2} a_0 \cdot 2\pi \\ &= a_0 \pi \end{aligned}$$

$$\therefore a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

Again, Since the Series for $f(x)$ in (1) is uniformly convergent, the Series obtained by multiplying it by $\cos nx$ or $\sin nx$ is uniformly convergent and so can be integrated term by term in $(-\pi, \pi)$.

$\therefore$ Multiplying (1) by $\cos nx$ and integrating term by term, we get

$$\int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{2} a_0 \int_{-\pi}^{\pi} \cos nx dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos^2 nx dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos nx dx$$

$$= 0 + 2n\pi + 0 \quad \text{as} \quad \int_{-\pi}^{\pi} \cos^2 nx \, dx = \frac{\pi}{2} \quad \text{and} \quad \int_{-\pi}^{\pi} \sin nx \cos nx \, dx = 0$$

$$= 2n\pi.$$

$$\therefore a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

Also multiplying (1) by $\sin nx$ and integrating term by term, we get

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \sin nx \, dx &= \frac{1}{2} a_0 \int_{-\pi}^{\pi} \sin x \, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \sin x \cos nx \, dx \\ &\quad + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin^2 nx \, dx \\ &= 0 + 0 + b_n \pi \quad \text{as} \quad \int_{-\pi}^{\pi} \sin nx \, dx = 0, \quad \int_{-\pi}^{\pi} \sin x \cos nx \, dx = 0 \\ &\quad \text{and} \quad \int_{-\pi}^{\pi} \sin^2 nx \, dx = \pi \end{aligned}$$

$$\therefore b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

Theorem-2: If $f(x)$ satisfies Dirichlet's conditions, i.e. if $f(x)$ be bounded and integrable in $(-\pi, \pi)$, then it is possible to divide $(-\pi, \pi)$ into a finite number of sub-intervals in each of which $f(x)$ is monotonic.

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \frac{1}{2} [f(c-0) + f(c+0)] \quad \text{for } -\pi < c < \pi$$

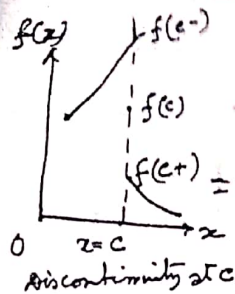
$$\text{and} = \frac{1}{2} [f(\pi-0) + f(-\pi+0)] \quad \text{for } c = \pm \pi$$

Solution: We here assume the equality

$$\int_{-\pi}^{\pi} f(x) \, dx = \int_{-\pi}^{\pi} f(a+x) \, dx \quad \text{--- (1)}$$

where a is any number,

$$\begin{aligned} \text{Now } \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) &\quad \text{for any } c \in (-\pi, \pi) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx + \sum_{n=1}^{\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \{ \cos nx \cos nc + \sin nx \sin nc \} \, dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx + \sum_{n=1}^{\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos n(x-c) \, dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \left\{ 1 + 2 \sum_{n=1}^{\infty} \cos n(x-c) \right\} \, dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+c) \left\{ 1 + 2 \sum_{n=1}^{\infty} \cos n(x+c-c) \right\} \, dx \quad \text{by (1)} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+c) \left\{ 1 + 2 \sum_{n=1}^{\infty} \cos nx \right\} \, dx \end{aligned}$$



$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+c) \frac{\sin\left(\left(m+\frac{1}{2}\right)x\right)}{\sin \frac{x}{2}} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^0 f(x+c) \frac{\sin\left(\left(m+\frac{1}{2}\right)x\right)}{\sin \frac{x}{2}} dx + \frac{1}{2\pi} \int_0^{\pi} f(x+c) \frac{\sin\left(\left(m+\frac{1}{2}\right)x\right)}{\sin \frac{x}{2}} dx$$

Putting $x = -2y$ in the first part and $x = 2z$ in the second part.

$$= -\frac{1}{\pi} \int_0^{\pi/2} f(-2y+c) \frac{\sin(2m+1)y}{\sin y} dy + \frac{1}{\pi} \int_0^{\pi/2} f(2z+c) \frac{\sin(2m+1)z}{\sin z} dz$$

$$= \frac{1}{\pi} \int_0^{\pi/2} f(-2y+c) \frac{\sin(2m+1)y}{\sin y} dy + \frac{1}{\pi} \int_0^{\pi/2} f(2z+c) \frac{\sin(2m+1)z}{\sin z} dz$$

$$= \frac{1}{\pi} [f(c-0) + f(c+0)] \int_0^{\pi/2} \frac{\sin x}{x} dx, \text{ as } m \rightarrow \infty$$

$$= \frac{1}{\pi} [f(c-0) + f(c+0)] \cdot \frac{\pi}{2} \text{ as } \int_0^{\pi/2} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

$$= \frac{1}{2} [f(c-0) + f(c+0)] \text{ for } -\pi < c < \pi.$$

At $c = \pi$, we have

$$\frac{1}{2} [f(c-0) + f(c+0)] = \frac{1}{2} [f(\pi-0) + f(\pi+0)]$$

$$= \frac{1}{2} [f(\pi-0) + f(\pi+0-2\pi)] \because f(x)$$

$$= \frac{1}{2} [f(\pi-0) + f(-\pi+0)] = f(x-2\pi)$$

Also at $c = -\pi$, we get

$$\frac{1}{2} [f(c-0) + f(c+0)] = \frac{1}{2} [f(-\pi-0) + f(-\pi+0)]$$

$$= \frac{1}{2} [f(-\pi-0+2\pi) + f(-\pi+0)] \because f(x) = f(x+2\pi)$$

$$= \frac{1}{2} [f(\pi-0) + f(-\pi+0)]$$

* Theorem-3: if a_r and b_r are the Fourier coefficients of $f(x)$ defined in $(-\pi \leq x \leq \pi)$ in the series

$$\frac{1}{2} a_0 + \sum_{r=1}^{\infty} (a_r \cos rx + b_r \sin rx)$$

then, $\frac{1}{2} a_0 + \sum_{r=1}^{\infty} (a_r \cos rx + b_r \sin rx)$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+t) \frac{\sin\left(\left(m+\frac{1}{2}\right)t\right)}{\sin \frac{t}{2}} dt$$

Solution: We have

$$\frac{1}{2} a_0 + \sum_{r=1}^{\infty} (a_r \cos rx + b_r \sin rx)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt + \sum_{r=1}^{\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos rt \cos rx dx + \sum_{r=1}^{\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin rt \sin rx dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt + \sum_{r=1}^{\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left\{ \cos rt \cos rx + \sin rt \sin rx \right\} dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt + \sum_{r=1}^{\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos r(t-x) dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \left\{ 1 + 2 \sum_{r=1}^{\infty} \cos r(t-x) \right\} dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \left[1 + 2 \left\{ \cos(t-x) + \cos 2(t-x) + \dots + \cos n(t-x) \right\} \right] dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \left[1 + 2 \cdot 2 \cos \frac{n-1}{2}(t-x) \cos \frac{n+1}{2}(t-x) \right] dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \left[1 + 4 \cdot \frac{\sin(n-1)(t-x)}{2 \sin \frac{n-1}{2}(t-x)} \cos \frac{n+1}{2}(t-x) \right] dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \left[1 + 2 \sin \frac{n-1}{2}(t-x) \cos \frac{n+1}{2}(t-x) \right] dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \left[1 + 2 \frac{\sin \frac{n(t-x)}{2} \cos \frac{n(t-x)}{2}}{\sin \frac{t-x}{2}} \right] dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \left[1 + \frac{\sin(2n+1)\left(\frac{t-x}{2}\right) - \sin\left(\frac{t-x}{2}\right)}{\sin\left(\frac{t-x}{2}\right)} \right] dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \left[\frac{\sin\left(\frac{t-x}{2}\right) + \sin(2n+1)\left(\frac{t-x}{2}\right) - \sin\left(\frac{t-x}{2}\right)}{\sin\left(\frac{t-x}{2}\right)} \right] dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \frac{\sin(2n+1)\left(\frac{t-x}{2}\right)}{\sin\left(\frac{t-x}{2}\right)} dt$$

$$= \frac{1}{2\pi} \int_{-\pi-x}^{\pi-x} f(x+u) \frac{\sin\left(n+\frac{1}{2}\right)u}{\sin \frac{u}{2}} du$$

Put $t-x = u$
So that $dt = du$

$$= \frac{1}{2\pi} \int_{-\pi-x}^{-\pi} f(x+u) \frac{\sin\left(n+\frac{1}{2}\right)u}{\sin \frac{u}{2}} du + \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+u) \frac{\sin\left(n+\frac{1}{2}\right)u}{\sin \frac{u}{2}} du$$

$$= \frac{1}{2\pi} \int_{-\pi-x}^{-\pi} f(x+u) \frac{\sin\left(n+\frac{1}{2}\right)u}{\sin \frac{u}{2}} du + \frac{1}{2\pi} \int_{-\pi}^{\pi-x} f(x+u) \frac{\sin\left(n+\frac{1}{2}\right)u}{\sin \frac{u}{2}} du$$

$$- \frac{1}{2\pi} \int_{\pi-x}^{\pi} f(x+u) \frac{\sin\left(n+\frac{1}{2}\right)u}{\sin \frac{u}{2}} du - 0$$

$$\begin{aligned}
 \text{Now } & \frac{1}{2\pi} \int_{-\pi-x}^{\pi-x} f(x+u) \frac{\sin(n+\frac{1}{2})u}{\sin u/2} du \quad \text{Put } u = u' - 2\pi \\
 & = \frac{1}{2\pi} \int_{\pi-x}^{\pi} f(x+u'-2\pi) \frac{\sin(n+\frac{1}{2})(u'-2\pi)}{\sin(\frac{u'-2\pi}{2})} du' \quad \text{So that } du = du' \\
 & = \frac{1}{2\pi} \int_{\pi-x}^{\pi} f(x+u'-2\pi) \frac{\sin(2n+1)\pi - (n+\frac{1}{2})u'}{\sin(\pi - \frac{u'}{2})} du' \\
 & = \frac{1}{2\pi} \int_{\pi-x}^{\pi} f(x+u') \frac{\sin(n+\frac{1}{2})u'}{\sin u'/2} du' \quad \because f(x) = f(x-2\pi) \\
 & = \frac{1}{2\pi} \int_{\pi-x}^{\pi} f(x+u) \frac{\sin(n+\frac{1}{2})u}{\sin u/2} du
 \end{aligned}$$

Hence from (1), we have

$$\begin{aligned}
 \frac{1}{2}a_0 + \sum_{r=1}^{\infty} (a_r \cos rx + b_r \sin rx) \\
 & = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+u) \frac{\sin(n+\frac{1}{2})u}{\sin u/2} du \\
 & = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+t) \frac{\sin(n+\frac{1}{2})t}{\sin t/2} dt
 \end{aligned}$$

Problem: Find the Fourier Series of the following function defined by

$$f(x) = \begin{cases} -x, & \text{for } -\pi \leq x < 0 \\ x, & \text{for } 0 < x < \pi \end{cases}$$

Hence deduce that $\pi^2/8 = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

obtain F.S. for the function defined by

$$f(x) = \begin{cases} -\pi-x, & \text{for } -\pi \leq x < -\pi/2 \\ x, & \text{for } -\pi/2 \leq x < \pi/2 \\ \pi-x, & \text{for } \pi/2 \leq x < \pi \end{cases}$$

and the graph of the given function.

Solution: Let the F.S. corresponding to the given function $f(x)$ be $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ where

$$\begin{aligned}
 a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \left[\int_{-\pi}^0 -x dx + \int_0^{\pi} x dx \right] \\
 &= \frac{1}{\pi} \left[\int_0^{\pi} x dx + \int_0^{\pi} x dx \right]
 \end{aligned}$$

$$= \frac{1}{\pi} \left[2 \int_0^{\pi} x dx \right]$$

$$= \frac{2}{\pi} \int_0^{\pi} x dx = \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi} = \frac{2}{\pi} \cdot \frac{\pi^2}{2} = \pi$$

$$\text{Now } a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 -x \cos nx dx + \int_0^{\pi} x \cos nx dx \right]$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} x \cos nx dx + \int_0^{\pi} x \cos nx dx \right]$$

$$= \frac{2}{\pi} \int_0^{\pi} x \cos nx dx = \frac{2}{\pi} \left[\left(\frac{x \sin nx}{n} \right)_0^{\pi} - \int_0^{\pi} \frac{\sin nx}{n} dx \right]$$

$$= \frac{2}{\pi n} \left[\frac{\cos nx}{n} \right]_0^{\pi} = -\frac{2}{\pi n^2} \left[(-1)^n - 1 \right]$$

$$\text{u. } \frac{2}{\pi} \cdot \frac{1}{n^2} [(-1)^n - 1]$$

where n is even or odd

$$= -\frac{4}{\pi n^2} \text{ when } n \text{ is odd}$$

$$= 0 \text{ when } n \text{ is even}$$

$$\text{and } b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 -x \sin nx dx + \int_0^{\pi} x \sin nx dx \right]$$

$$= \frac{1}{\pi} \left[- \int_0^{\pi} x \sin nx dx + \int_0^{\pi} x \sin nx dx \right] \text{ as } f(-x) = f(x)$$

u. taking $f(x)$ as an odd function

$$= 0$$

Therefore, the F.S. for $f(x)$ is given by

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2} \{(-1)^n - 1\} \cos nx$$

$$= \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos nx$$

$$= \frac{\pi}{2} + \frac{2}{\pi} \left[-\frac{2}{1^2} \cos x - \frac{2}{3^2} \cos 3x - \frac{2}{5^2} \cos 5x - \dots \right]$$

$$= \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right]$$

At $x = \pi$, sum of the F.S. is given by

$$S(\pi) = \frac{1}{2} [f(-\pi+0) + f(\pi-0)]$$

$$= \frac{1}{2} [\pi + \pi] = \pi$$

$$\therefore \pi = \frac{\pi}{2} + \frac{4}{\pi} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

$$\text{u. } \frac{\pi}{2} = \frac{4}{\pi} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

$$\text{u. } \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

2. Let the F.S. for the given function $f(x)$ is

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \text{ where}$$

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\ &= \frac{1}{\pi} \left[\int_{-\pi}^{-\pi/2} (-\pi-x) dx + \int_{-\pi/2}^{\pi/2} x dx + \int_{\pi/2}^{\pi} (\pi-x) dx \right] \\ &= \frac{1}{\pi} \left[\int_{-\pi}^{-\pi/2} -\pi dx + \int_{-\pi/2}^{\pi/2} x dx + 0 + \int_{\pi/2}^{\pi} \pi dx - \int_{\pi/2}^{\pi} x dx \right] \\ &= \frac{1}{\pi} \left[-\int_{\pi/2}^{\pi} \pi dx + \int_{\pi/2}^{\pi} \pi dx \right] \quad \left[\because \int_{-\pi/2}^{\pi/2} x dx = 0 \right] \end{aligned}$$

$$= 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$\begin{aligned} &= \frac{1}{\pi} \left[\int_{-\pi}^{-\pi/2} -(\pi+x) \cos nx dx + \int_{-\pi/2}^{\pi/2} x \cos nx dx + \int_{\pi/2}^{\pi} (\pi-x) \cos nx dx \right] \quad (*) \\ &= \frac{1}{\pi} \left[\int_{-\pi}^{-\pi/2} -\pi \cos nx dx - \int_{-\pi/2}^{\pi/2} x \cos nx dx + 0 + \int_{\pi/2}^{\pi} \pi \cos nx dx - \int_{\pi/2}^{\pi} x \cos nx dx \right] \\ &= 0 \end{aligned}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

$$\begin{aligned} &= \frac{1}{\pi} \left[\int_{-\pi}^{-\pi/2} -(\pi+x) \sin nx dx + \int_{-\pi/2}^{\pi/2} x \sin nx dx + \int_{\pi/2}^{\pi} (\pi-x) \sin nx dx \right] \\ &= \frac{1}{\pi} \left[\int_{\pi/2}^{\pi} \pi \sin nx dx - \int_{\pi/2}^{\pi} x \sin nx dx + 2 \int_{-\pi/2}^{\pi/2} x \sin nx dx + \int_{-\pi/2}^{\pi/2} \pi \sin nx dx - \int_{-\pi/2}^{\pi/2} x \sin nx dx \right] \\ &= \frac{2}{\pi} \left[\int_{\pi/2}^{\pi} \pi \sin nx dx - \int_{\pi/2}^{\pi} x \sin nx dx + \int_{-\pi/2}^{\pi/2} x \sin nx dx \right] \quad 0 \end{aligned}$$

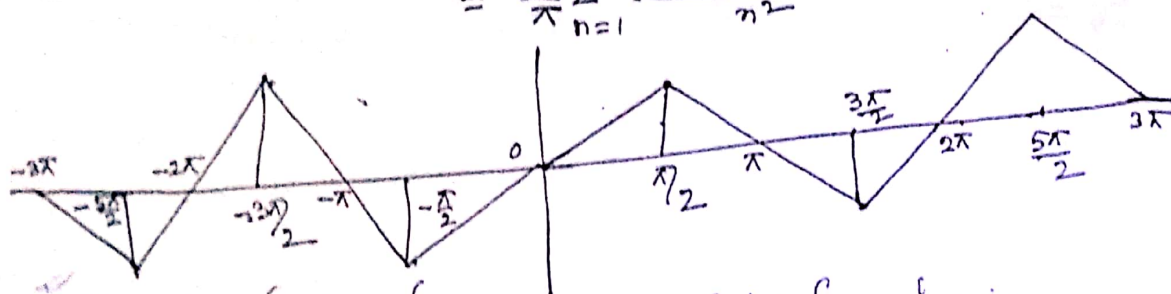
$$= \frac{2}{\pi} \left[\frac{\pi \cos nx}{n} + \frac{\pi \cos \frac{n\pi}{2}}{n} + \frac{\pi \cos nx}{n} - \frac{\pi \cos \frac{n\pi}{2}}{n} + \frac{\sin \frac{n\pi}{2}}{n^2} - \frac{\pi \cos \frac{n\pi}{2}}{2n} + \frac{\sin \frac{n\pi}{2}}{n^2} \right]$$

$$= \frac{2}{\pi} \left[\frac{2 \sin \frac{n\pi}{2}}{n^2} \right] = \frac{4}{\pi n^2} \sin \frac{n\pi}{2}$$

Hence the F.S. for the given function $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{\pi n^2} \sin \frac{n\pi}{2} \sin nx$$

$$= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2} \sin nx}{n^2}$$



* Fourier Series for Even and Odd function :-

Even Function: If f is an even function, i.e. $f(-x) = f(x)$, $\forall x$, then $f \cos nx$ is an even and $f \sin nx$ is an odd function and therefore

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f \cos nx \, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 f \cos nx \, dx + \int_0^{\pi} f \cos nx \, dx \right]$$

$$= \frac{2}{\pi} \int_0^{\pi} f \cos nx \, dx \quad \text{--- (1)}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f \sin nx \, dx = 0$$

So, the Fourier Series of an even function consists of terms of cosines only and the Coefficients a_n may be computed from (1).

Also, since for an even function,

$$f(0+) = f(0-) = f(0)$$

and $f(-\pi+) = f(\pi-)$
the sum of the series is $f(0)$ at 0 or $\pm$ (even multiple of π) and is $f(\pi-)$ at $\pm\pi$ or $\pm$ (odd multiple of π)

Odd function: If f is an odd function, i.e. $f(-x) = -f(x) \forall x$, then $f \cos nx$ is an odd and $f \sin nx$ is an even function and therefore

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f \cos nx \, dx = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} f \sin nx \, dx \quad \text{--- (2)}$$

So the Fourier Series of an odd function consists of sine terms only and the coefficients b_n may be calculated from ②. Also for an odd function

$$f(0-) = 0 = f(0+)$$

$$\text{and } f(-\pi+0) = -f(\pi-0)$$

The sum of the series is 0 at $x=0$ (or any multiple of π)
Problem: Find the Fourier Series generated by the periodic function $f(x)$ of period 2π . Also compute the value of sum of: $2\pi, -3\pi$

Solution: The function is monotonic and continuous on $[-\pi, \pi]$. Moreover it is an even function and therefore the Fourier series will consist of cosine terms only.

The function may be restated as

$$f(x) = \begin{cases} -x, & \text{for } -\pi \leq x < 0 \\ x, & \text{for } 0 \leq x \leq \pi \end{cases}$$

$$\text{Therefore } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 f(x) dx + \int_0^{\pi} f(x) dx \right]$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 -x dx + \int_0^{\pi} x dx \right]$$

$$= \frac{2}{\pi} \int_0^{\pi} x dx = \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi} = \frac{2}{\pi} \cdot \frac{\pi^2}{2} = \pi$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \cos nx dx$$

$$= \frac{2}{\pi n^2} (\cos n\pi - 1)$$

$$= \begin{cases} 0 & \text{for } n \text{ even} \\ -\frac{4}{\pi n^2} & \text{for } n \text{ odd} \end{cases}$$

we, thus, obtain the series

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right]$$

The series converges at all points and its sum is equal to the given function.

Sum of the series at 2π is the same as at 0.

$$\text{At } 0, \quad 0 = \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

$$\Rightarrow \frac{4}{\pi} \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right) = \frac{\pi}{2}$$

$$\text{At } -3\pi \text{ (same as at } \pi) \Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

$$\pi = \frac{\pi}{2} + \frac{4}{\pi} \left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

$$\Rightarrow 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8} \text{ same as above}$$

Problem: Show that for $0 < x < \pi$,

$$(i) \quad x(\pi-x) = \frac{\pi^2}{6} - 4 \sum_{n=1}^{\infty} \frac{\cos 2nx}{(2n)^2}$$

$$(ii) \quad x(\pi-x) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^3}$$

Reduce from (i) & (ii) respectively that

$$(iii) \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}, \quad (iv) \quad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$

Solution: We define $f(0) = 0 = f(\pi)$. So f is continuous on $[0, \pi]$.

In the half-range series $0 < x < \pi$, cosine series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \text{ where}$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} x(\pi-x) dx = \frac{2}{\pi} \left[\frac{\pi^3}{2} - \frac{\pi^3}{3} \right]$$

$$= \frac{2}{\pi} \cdot \frac{\pi^3}{6} = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x(\pi-x) \cos nx dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi} \pi x \cos nx dx - \int_0^{\pi} x^2 \cos nx dx \right]$$

$$= \frac{2}{\pi} \left[\pi \left\{ \frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right\}_0^{\pi} - \left\{ \frac{x^2 \sin nx}{n} \right\}_0^{\pi} - \int_0^{\pi} 2x \frac{\sin nx}{n} dx \right]$$

$$= \frac{2}{\pi^2} [(-1)^n - 1] + \frac{2}{\pi} \left\{ -\frac{2x \cos nx}{n^2} + \frac{2 \sin nx}{n^3} \right\}_0^{\pi}$$

$$= \frac{2}{\pi^2} [(-1)^n - 1] = \frac{4}{\pi^2} (-1)^n$$

$$= -\frac{4}{\pi^2}, \text{ when } n \text{ is even}$$

$$= 0, \text{ when } n \text{ is odd}$$

$$\text{Hence } x(\pi-x) = \frac{\pi^2}{6} - 4 \sum_{n=1}^{\infty} \frac{\cos 2nx}{(2n)^2}$$

(ii) In the half-range $0 < x < \pi$, sine series is of the form

$$\sum_{n=1}^{\infty} b_n \sin nx, \text{ where}$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x(\pi-x) \sin nx dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi} \pi x \sin nx dx - \int_0^{\pi} x^2 \sin nx dx \right]$$

$$= \frac{2}{\pi} \left[\pi \left\{ -\frac{x \cos nx}{n} + \frac{\sin nx}{n^2} \right\}_0^{\pi} - \int_0^{\pi} x^2 \sin nx dx \right]$$

$$= -\frac{2\pi(-1)^n}{n} - \frac{2}{\pi} \left[\left\{ \frac{x^2 \cos nx}{n} \right\}_0^{\pi} + \int_0^{\pi} \frac{2x \cos nx}{n} dx \right]$$

$$= \frac{2\pi(-1)^{n+1}}{n} + \frac{2}{\pi} (-1)^n - \frac{2}{\pi} \int_0^{\pi} \frac{2x \cos nx}{n} dx$$

$$= \frac{2\pi(-1)^{n+1}}{n} + \frac{2\pi(-1)^n}{n} - \frac{2}{\pi} \left\{ \frac{2x \sin nx}{n^2} + \frac{2 \cos nx}{n^3} \right\}_0^\pi$$

$$= -\frac{2\pi(-1)^n}{n} + \frac{2\pi(-1)^n}{n} - \frac{4}{\pi n^3} [(-1)^n - 1]$$

$$= -\frac{4}{\pi n^3} [(-1)^n - 1]$$

$$= \frac{8}{\pi n^3}, \text{ when } n \text{ is odd}$$

$$= 0, \text{ when } n \text{ is even}$$

Thus $x(\pi-x) = \frac{8}{\pi} \sum_{m=1}^{\infty} \frac{\sin(2m-1)x}{(2m-1)^3}$

(iii) In $0 < x < \pi$,

$$\int_0^\pi [f(x)]^2 dx = \frac{\pi}{2} \left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 \right]$$

$$\Rightarrow \int_0^\pi \{x(\pi-x)\}^2 dx = \frac{\pi}{2} \left[\frac{\pi^4}{18} + 16 \sum_{m=1}^{\infty} \frac{1}{(2m)^6} \right]$$

$$\Rightarrow \int_0^\pi x^2(\pi^2 - 2\pi x + x^2) dx = \frac{\pi^5}{36} + 8\pi \sum_{m=1}^{\infty} \frac{1}{(2m)^6}$$

$$\Rightarrow \left[\frac{x^3}{3} - 2\pi \frac{x^2}{2} + \frac{x^3}{3} \right]_0^\pi = \frac{\pi^5}{36} + \frac{8\pi}{2^6} \sum_{m=1}^{\infty} \frac{1}{m^6}$$

$$\Rightarrow \pi^5 \left[\frac{1}{3} - \frac{1}{2} + \frac{1}{3} \right] = \frac{\pi^5}{36} + \frac{\pi}{2} \sum_{m=1}^{\infty} \frac{1}{m^6}$$

$$\Rightarrow \frac{\pi^5}{30} = \frac{\pi^5}{36} + \frac{\pi}{2} \sum_{m=1}^{\infty} \frac{1}{m^6}$$

$$\Rightarrow \frac{\pi^5}{30} - \frac{\pi^5}{36} = \frac{\pi}{2} \sum_{m=1}^{\infty} \frac{1}{m^6}$$

$$\Rightarrow \frac{\pi^5}{180} = \frac{\pi}{2} \sum_{m=1}^{\infty} \frac{1}{m^6}$$

$$\Rightarrow \frac{\pi^4}{90} = \sum_{m=1}^{\infty} \frac{1}{m^6}$$

That is, $\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^4}{90}$

(iv) $\int_0^\pi [f(x)]^2 dx = \frac{\pi}{2} \sum_{n=1}^{\infty} b_n^2$

$$\Rightarrow \frac{\pi^5}{30} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{64}{\pi^2 (2m-1)^6} \text{ for } n=2m-1$$

$$\Rightarrow \frac{\pi^4}{15} = \frac{64}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^6} = \frac{64}{\pi^2} \left[\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \dots \right]$$

$$\Rightarrow \frac{\pi^6}{960} = \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \dots$$

Now let $S = \sum_{n=1}^{\infty} \frac{1}{n^6} = \left(\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \dots \right) + \left(\frac{1}{2^6} + \frac{1}{4^6} + \frac{1}{6^6} + \dots \right)$

Then $S = \frac{\pi^6}{960} + \left(\frac{1}{2^6} + \frac{1}{4^6} + \frac{1}{6^6} + \frac{1}{8^6} + \dots \right)$

$$u. S = \frac{\pi^6}{960} + \frac{1}{2^6} \left[\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \dots \right]$$

$$= \frac{\pi^6}{960} + \frac{1}{2^6} S$$

$$\Rightarrow S - \frac{S}{64} = \frac{\pi^6}{960}$$

$$\Rightarrow \frac{63S}{64} = \frac{\pi^6}{960} \Rightarrow S = \frac{64}{63} \cdot \frac{1}{960} \pi^6$$

$$\Rightarrow \sum \frac{1}{n^6} = \frac{1}{63} \cdot \frac{1}{15} \pi^6 = \frac{\pi^6}{945}$$

$$** \therefore \sum \frac{1}{n^6} = \frac{\pi^6}{945}$$

Theorem: If f be bounded and integrable on $[-\pi, \pi]$ and if a_n, b_n are its Fourier Coefficients, then $\sum_{n=1}^{\infty} (a_n^2 + b_n^2)$ is convergent.

Proof. We have $\int_{-\pi}^{\pi} \left\{ f - \sum_{n=1}^m (a_n \cos nx + b_n \sin nx) \right\}^2 dx \geq 0$ as $\left\{ f - \sum_{n=1}^m (a_n \cos nx + b_n \sin nx) \right\}^2$ is non-negative.

$$\text{Therefore } \int_{-\pi}^{\pi} \left\{ f^2 + \left[\sum_{n=1}^m (a_n \cos nx + b_n \sin nx) \right]^2 - 2f \sum_{n=1}^m (a_n \cos nx + b_n \sin nx) \right\} dx \geq 0$$

$$\Rightarrow \int_{-\pi}^{\pi} f^2 dx + \sum_{n=1}^m a_n^2 \int_{-\pi}^{\pi} \cos^2 nx dx + \sum_{n=1}^m b_n^2 \int_{-\pi}^{\pi} \sin^2 nx dx + \sum_{n=1}^m 2a_n b_n \int_{-\pi}^{\pi} \sin 2nx dx - 2 \sum_{n=1}^m a_n \int_{-\pi}^{\pi} f \cos nx dx - 2 \sum_{n=1}^m b_n \int_{-\pi}^{\pi} f \sin nx dx \geq 0$$

$$\Rightarrow \int_{-\pi}^{\pi} f^2 dx + \sum_{n=1}^m a_n^2 \cdot \pi + \sum_{n=1}^m b_n^2 \cdot \pi + \sum_{n=1}^m 2a_n b_n \cdot 0 - 2 \sum_{n=1}^m a_n \cdot 0 - 2 \sum_{n=1}^m b_n \cdot 0 \geq 0$$

$$\Rightarrow \int_{-\pi}^{\pi} f^2 dx + \pi \sum_{n=1}^m a_n^2 + \pi \sum_{n=1}^m b_n^2 - 2\pi \sum_{n=1}^m a_n^2 - 2\pi \sum_{n=1}^m b_n^2 \geq 0 \text{ as } a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f \cos nx dx$$

$$\Rightarrow \int_{-\pi}^{\pi} f^2 dx - \pi \sum_{n=1}^m a_n^2 - \pi \sum_{n=1}^m b_n^2 \geq 0 \Rightarrow a_n \pi = \int_{-\pi}^{\pi} f \cos nx dx$$

$$\Rightarrow \int_{-\pi}^{\pi} f^2 dx - \pi \sum_{n=1}^m (a_n^2 + b_n^2) \geq 0 \Rightarrow b_n \pi = \int_{-\pi}^{\pi} f \sin nx dx$$

$$\Rightarrow \int_{-\pi}^{\pi} f^2 dx \geq \pi \sum_{n=1}^m (a_n^2 + b_n^2)$$

$$\Rightarrow \sum_{n=1}^m (a_n^2 + b_n^2) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f^2 dx$$

$$\Rightarrow \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f^2 dx \text{ taking } m \rightarrow \infty. \text{ Known as Bessel's inequality}$$

Again f is integrable $\Rightarrow f'$ is integrable.

$$\Rightarrow \int_{-A}^A f' dx \text{ exists on } [-A, A]$$

$$\Rightarrow \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \text{ is convergent.}$$

Theorem If f be non-decreasing, integrable and bounded with A in $[-A, A]$, then $\int_{-A}^A [f(x)]^2 dx = \pi \left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right]$

Proof: Let f satisfy all the Dirichlet's conditions in $[-A, A]$.

Also let f is continuous on $[-A, A]$. Then the Fourier Series $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ of f has the sum $f(x) \forall x \in [-A, A]$.

$$\text{So } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Now multiplying by f and integrating term by term in $[-A, A]$, we get

$$\int_{-A}^A [f(x)]^2 dx = \frac{a_0}{2} \int_{-A}^A f(x) dx + \sum_{n=1}^{\infty} a_n \int_{-A}^A f(x) \cos nx dx + \sum_{n=1}^{\infty} b_n \int_{-A}^A f(x) \sin nx dx$$

$$= \frac{a_0}{2} \cdot \frac{a_0 \pi}{2} + \sum_{n=1}^{\infty} a_n \cdot \pi a_n + \sum_{n=1}^{\infty} b_n \cdot \pi b_n$$

$$= \frac{a_0^2}{2} \cdot \pi + \pi \sum_{n=1}^{\infty} a_n^2 + \pi \sum_{n=1}^{\infty} b_n^2$$

which is known as Parseval's formula. [Since the series being F.S, a_0, a_n, b_n are Fourier Coefficients]

$$a_0 = \frac{1}{\pi} \int_{-A}^A f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-A}^A f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{-A}^A f(x) \sin nx dx$$

Note: ① In $0 < x < \pi$, (Half-range with Cosine series), the Parseval's formula stands as

$$\int_0^{\pi} [f(x)]^2 dx = \frac{\pi}{2} \left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 \right]$$

② In $0 < x < \pi$, (Half-range with Sine series), the Parseval's formula stands as

$$\int_0^{\pi} [f(x)]^2 dx = \frac{\pi}{2} \sum_{n=1}^{\infty} b_n^2$$

Problem: Expand $f(x) = x$ in Fourier series on interval $-\pi \leq x \leq \pi$

Solution: Observe that $f(x) = x$ is bounded and integrable on $-\pi \leq x \leq \pi$, since it is continuous there. Further $f'(x) = 1 > 0$ indicates that $f(x)$ is monotone increasing on the entire interval. Thus $f(x)$ satisfies Dirichlet's conditions on $[-\pi, \pi]$. Hence the Fourier series corresponding to $f(x) = x$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = 0$, $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = 0$

Since $x \cos nx$ and x are odd functions and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx dx = \frac{2}{\pi} \int_0^{\pi} x \sin nx dx$$

Since $x \sin nx$ is even.

Thus

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} x \sin nx dx \\ &= \frac{2}{\pi} \left[-\frac{x \cos nx}{n} \right]_0^{\pi} + \frac{2}{\pi} \int_0^{\pi} \frac{\cos nx}{n} dx \\ &= -\frac{2}{n} \cos n\pi + \frac{2}{\pi} \left[\frac{\sin nx}{n} \right]_0^{\pi} \\ &= \begin{cases} -\frac{2}{n}, & n \text{ is even} \\ \frac{2}{n}, & n \text{ is odd} \end{cases} \end{aligned}$$

Hence $f(x) = x$ generates the Fourier series in the form

$$\begin{aligned} &\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \\ &= \sum_{n=1}^{\infty} b_n \sin nx \\ &= b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + b_4 \sin 4x + \dots \\ &= 2 \left\{ \frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots \right\} \quad \text{--- (1)} \end{aligned}$$

Observations: See that $f(x) = x$ is also a continuous function on the interval. Hence eqn. (1) is equal to $f(x)$ at any point $-\pi < x < \pi$. That means $-\pi < x < \pi$,

$$x = 2 \left\{ \frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots \right\}$$

Thus when $x = \frac{\pi}{2}$, then

$$\frac{\pi}{2} = 2 \left\{ 1 - \frac{1}{3} + \frac{1}{5} - \dots \right\}$$

$$\text{whereby } \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \dots$$

Problem: Expand in Fourier Series $x+x^2$ on $-\pi < x < \pi$ and deduce that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

Solution: Let $f(x) = x+x^2$ on $-\pi < x < \pi$. We may define $f(x)$ at the end points $x = \pm\pi$ arbitrarily. But for convenience we take $f(x) = x+x^2$ on $-\pi \leq x \leq \pi$. Now $f(x)$ is bounded and integrable on $[-\pi, \pi]$ because of its continuity on the same closed interval. Further $f(x) = x+x^2$, so that $f'(x) = 1+2x$ if $f'(x) > 0$ for $x > -\frac{1}{2}$ and $f'(x) < 0$ for $x < -\frac{1}{2}$. Thus $f(x)$ is monotone decreasing on $-\pi \leq x < -\frac{1}{2}$ and monotone increasing on $-\frac{1}{2} < x \leq \pi$ whereby $f(x)$ is piecewise monotone on $-\pi \leq x \leq \pi$. Hence $f(x)$ satisfies Dirichlet's condition on $[-\pi, \pi]$.

$$\text{Now } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x+x^2) dx$$

$$= \frac{1}{\pi} \left[\left\{ \frac{x^2}{2} \right\}_{-\pi}^{\pi} + \left\{ \frac{x^3}{3} \right\}_{-\pi}^{\pi} \right]$$

$$= \frac{1}{\pi} \left[0 + \frac{\pi^3}{3} + \frac{\pi^3}{3} \right] = \frac{1}{\pi} \cdot \frac{2\pi^3}{3} = \frac{2}{3}\pi^2$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x+x^2) \cos nx dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos nx dx + \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx dx$$

$$= \frac{1}{\pi} \left[\left\{ \frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right\}_{-\pi}^{\pi} + \left\{ \frac{x^2 \sin nx}{n} \right\}_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{2x \sin nx}{n} dx \right]$$

$$= \frac{2}{\pi n} \left[- \int_{-\pi}^{\pi} x \sin nx dx \right]$$

$$= \frac{2}{\pi n} \left[\frac{x \cos nx}{n} - \frac{\sin nx}{n^2} \right]_{-\pi}^{\pi}$$

$$= \frac{2}{\pi n} \left[\frac{\pi}{n} \cos n\pi + \frac{\pi}{n} \cos n\pi \right]$$

$$= -\frac{4}{n^2} \text{ when } n \text{ is odd}$$

$$= \frac{4}{n^2} \text{ when } n \text{ is even}$$

$$\therefore a_n = \frac{4}{n^2} \cos n\pi = \begin{cases} \frac{4}{n^2}, & n \text{ even} \\ -\frac{4}{n^2}, & n \text{ odd} \end{cases}$$

$$\text{Similarly } b_n = \begin{cases} -\frac{2}{n}, & n \text{ even} \\ \frac{2}{n}, & n \text{ odd} \end{cases}$$

$$\text{Thus } f(x) = x+x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \frac{\pi^2}{3} - 4 \left\{ \frac{\cos 2x}{2} - \frac{\cos 4x}{2} + \frac{\cos 6x}{3^2} - \dots \right\} + 2 \left\{ \frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \right\}$$

And $x+x^2$ is a continuous function, hence $-\pi < x < \pi$,
 $x+x^2 = \frac{\pi^2}{3} - 4 \left\{ \frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \dots \right\} + 2 \left\{ \frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \right\}$

Next at $x = \pm \pi$, the sum of the series

$$= \frac{1}{2} [f(-\pi+0) + f(\pi-0)]$$

$$= \frac{1}{2} [-\pi + \pi^2 + \pi + \pi^2] = \pi^2$$

whereby

$$\pi^2 = \frac{\pi^2}{3} - 4 \left\{ -\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \dots \right\}$$

$$\therefore \pi^2 - \frac{\pi^2}{3} = 4 \left\{ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right\}$$

$$\therefore \frac{2\pi^2}{3} = 4 \left\{ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right\}$$

$$\therefore \frac{2\pi^2}{12} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

$$\therefore \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$

Problem: Find the Fourier series of the periodic function f with period 2π , where

$$f(x) = 0, -\pi < x < a; f(x) = 1, a \leq x \leq b; f(x) = 0, b < x < \pi$$

Find the sum of the series for $x = 4\pi + a$ and deduce that $\sum_{n=1}^{\infty} \frac{\sin n(b-a)}{n} = \frac{\pi - b + a}{2}$

Solution: Here $f(x)$ satisfies Dirichlet's Condition on $[a, b]$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_a^b f(x) dx = \frac{1}{\pi} \int_a^b 1 dx = \frac{b-a}{\pi}$$

$$a_n = \frac{1}{\pi} \int_a^b \cos nx dx = \frac{1}{\pi} \left[\frac{\sin nx}{n} \right]_a^b = \frac{1}{n\pi} [\sin bn - \sin an]$$

$$b_n = \frac{1}{\pi} \int_a^b \sin nx dx = \frac{1}{\pi} \left[-\frac{\cos nx}{n} \right]_a^b = \frac{1}{n\pi} [\cos an - \cos bn]$$

Since f satisfies Dirichlet's Conditions on convergence, we have

$$f(x) = \frac{b-a}{2\pi} + \frac{1}{\pi} \left\{ \sum_{n=1}^{\infty} \frac{1}{n} (\sin bn - \sin an) \cos nx + \sum_{n=1}^{\infty} \frac{1}{n} (\cos an - \cos bn) \sin nx \right\}$$

At $x = 4\pi + a$, i.e. at $x = a$ (period 2π), the series converges to $\frac{1}{2} [f(a-0) + f(a+0)] = \frac{1}{2} [0 + 1] = \frac{1}{2}$, a being an arbitrary discontinuity.

$$\therefore \frac{1}{2} = \frac{b-a}{2\pi} + \frac{1}{\pi} \left\{ \sum_{n=1}^{\infty} \frac{1}{n} (\sin bn - \sin an) \cos na + \sum_{n=1}^{\infty} \frac{1}{n} (\cos an - \cos bn) \sin na \right\}$$

$$\therefore \frac{1}{2} - \frac{b-a}{2\pi} = \frac{1}{\pi} \left\{ \sum_{n=1}^{\infty} \frac{\sin bn \cos an - \cos bn \sin an}{n} \right\}$$

$$\therefore \frac{\pi - b + a}{2\pi} = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin n(b-a)}{n} \quad \text{i.e.} \quad \frac{\pi - b + a}{2} = \sum_{n=1}^{\infty} \frac{\sin n(b-a)}{n}$$

Riemann-Lebesgue Lemma

If a function f is bounded and integrable on $[a, b]$,
 Show that $\lim_{n \rightarrow \infty} \int_a^b f(x) \cos nx \, dx = 0$ and $\lim_{n \rightarrow \infty} \int_a^b f(x) \sin nx \, dx = 0$

Solution: Let $I_n = \int_a^b f(x) \cos nx \, dx$

Let further, $\epsilon > 0$ be an arbitrary number.

Since f is bounded and integrable on $[a, b]$,
 there exists a partition $P = \{a = x_0, x_1, x_2, \dots, x_p = b\}$

such that the oscillatory sum

$$U(P, f) - L(P, f) = \sum_{i=1}^p (M_i - m_i) \Delta x_i < \frac{\epsilon}{2}$$

where M_i, m_i are bounds of f in Δx_i .

$$\begin{aligned} \text{Now } I_n &= \sum_{i=1}^p \int_{x_{i-1}}^{x_i} f(x) \cos nx \, dx \\ &= \sum_{i=1}^p f(x_{i-1}) \int_{x_{i-1}}^{x_i} \cos nx \, dx + \sum_{i=1}^p \int_{x_{i-1}}^{x_i} \{f(x) - f(x_{i-1})\} \cos nx \, dx \\ \therefore |I_n| &\leq \sum_{i=1}^p |f(x_{i-1})| \left| \int_{x_{i-1}}^{x_i} \cos nx \, dx \right| + \sum_{i=1}^p \int_{x_{i-1}}^{x_i} |f(x) - f(x_{i-1})| \cos nx \, dx \end{aligned}$$

But for all $x \in \Delta x_i$, we have

$$|f(x) - f(x_{i-1})| \leq M_i - m_i \quad (i = 1, 2, \dots, p)$$

So that

$$|f(x) - f(x_{i-1})| \cos nx \leq M_i - m_i$$

$$\begin{aligned} \sum_{i=1}^p \int_{x_{i-1}}^{x_i} |f(x) - f(x_{i-1})| \cos nx \, dx &\leq (M_i - m_i)(x_i - x_{i-1}) \\ &< \frac{\epsilon}{2} \end{aligned}$$

$$\begin{aligned} \text{Also } \left| \int_{x_{i-1}}^{x_i} \cos nx \, dx \right| &= \left| \frac{1}{n} \{ \sin nx_i - \sin nx_{i-1} \} \right| \\ &\leq \frac{1}{n} \{ |\sin nx_i| + |\sin nx_{i-1}| \} \\ &\leq \frac{2}{n} \end{aligned}$$

$$\therefore |I_n| \leq \frac{2}{n} \sum_{i=1}^p |f(x_{i-1})| + \frac{\epsilon}{2}$$

For a fixed P , $f(x_{i-1})$ is a fixed quantity. Also there exists a positive number m such that for all $n \geq m$,

$$\frac{2}{n} \sum_{i=1}^p |f(x_{i-1})| < \frac{\epsilon}{2}$$

$$\therefore |I_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \forall n \geq m$$

$$\Rightarrow \lim_{n \rightarrow \infty} \int_a^b f(x) \cos nx \, dx = 0$$

It may similarly be shown that

$$\lim_{n \rightarrow \infty} \int_a^b f(x) \sin nx \, dx = 0$$

one
dx

along
x