Infinite Series of constant terms:

If u_n be a single valued function of n for all positive integral values of n, then the series

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots\cdots$$

$= \sum_{n=1}^{\infty} u_n$ or $\sum u_n$ consisting of infinite number of terms is called an infinite series of constant terms.

Since, $S_1 = u_1$, $S_2 = u_1 + u_2$, $S_3 = u_1 + u_2 + u_3$ and $S_n = u_1 + u_2 + \cdots + u_n$

$\{S_n\}$ is a sequence of infinite number of terms.

In the series

$$S_{n+p} = u_1 + u_2 + \cdots\cdots + u_n + u_{n+1} + u_{n+2} + \cdots + u_{n+p}$$

the portion $u_{n+1} + u_{n+2} + \cdots\cdots + u_{n+p}$ is called a partial residue and is denoted by ${}_pR_n$. The series $u_{n+1} + u_{n+2} + \cdots + u_{n+p} + \cdots \infty$ is called the residue of the series and is denoted by R_n.

An infinite series is said to be convergent when the sum of the first n-terms of the series cannot be numerically greater than some fixed quantity however greater n may be.

An infinite series is said to be divergent when the sum of the first n-terms of the series can be made numerically greater than any finite quantity by taking n sufficiently large.

If an infinite series is neither convergent nor divergent, then it is called an oscillating or a periodic convergent series.

<u>Infinite Series:</u>

Let $\{u_n\}$ be a sequence. Then the sequence $\{S_n\}$ defined by $S_1 = u_1$, $S_2 = u_1 + u_2$, $S_3 = u_1 + u_2 + u_3$, $\cdots\cdots$ is represented by the symbol $u_1 + u_2 + u_3 + \cdots\cdots\cdots$, which is said to be an infinite series (or a series) generated by the sequence $\{u_n\}$. The series is denoted by $\sum_{n=1}^{\infty} u_n$ or $\sum u_n$. u_n is said to be the nth term of the series. The elements of the sequence $\{S_n\}$ are called the partial sums of the series $\sum u_n$ and the sequence $\{S_n\}$ is called the sequence of partial sums of the series $\sum u_n$.

If $\{u_n\}$ be a real sequence then $\sum u_n$ is a series of real numbers.

We shall be mainly concerned with the series of real numbers.

The infinite series $\sum u_n$ is said to be convergent or divergent according as the sequence $\{S_n\}$ is convergent or divergent.

In case of Convergence, if $\lim s_n = s$ then s is said to be the sum of the series $\sum u_n$.
If, however, $\lim s_n = \infty$ (or $-\infty$) the series $\sum u_n$ is said to diverge to ∞ (or $-\infty$)

Examples: 1. Let us consider the series $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} + \cdots$

Let the series be $\sum_{n=1}^{\infty} u_n$. Then $u_n = \frac{1}{n(n+1)}$

Let $s_n = u_1 + u_2 + \cdots + u_n$

Then
$$s_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$$
$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$= 1 - \frac{1}{n+1},$$

and $\lim s_n = 1$. Hence the series $\sum u_n$ is convergent and the sum of the series is 1.

2. Let us consider the series $1 + 2 + 3 + \cdots + n + \cdots$

Let $s_n = 1 + 2 + 3 + \cdots + n$. Then $s_n = \frac{n(n+1)}{2}$

and $\lim s_n = \infty$. Hence the series is divergent.

3. The series $1 - 1 + 1 - 1 + \cdots\cdots$ oscillates finitely, while the series $1 - 2 + 3 - 4 + \cdots\cdots$ oscillates infinitely.

An expression of the form
$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots\cdots, \quad (1)$$
in which every term is followed by another according to some definite law, is called a series. If the series contains a finite number of terms, then it is called a finite series. In case the number of terms is not finite, it is called an infinite series. The above series will be denoted by $\sum_{n=1}^{\infty} u_n$ or simply by $\sum u_n$ and the sum of its first n terms by s_n, which is sometimes called the nth partial sum of the series.

If $R_n = u_{n+1} + u_{n+2} + \cdots\cdots$, then R_n is called the remainder after n terms of the series, while $u_{n+1} + u_{n+2} + \cdots + u_{n+p}$ is called the p-th partial remainder and is denoted by ${}_pR_n$.

* The infinite series (1) is said to converge to s, if, for an arbitrarily small positive quantity ϵ, there corresponds a positive integer m depending on ϵ such that $|s_n - s| < \epsilon$ for $n \geq m$.

* As n tends to ∞, the n-th partial sum S_n may tend to a finite limit S, say. We then call the series convergent and S is called the sum to infinity. Thus

$$\lim_{n\to\infty} S_n = S$$

S is not a sum in the normal sense but the limit of a sum. Henceforth, we shall write $Lt\, S_n$ for $\underset{n\to\infty}{Lt} S_n$.

If, on the other hand, S_n tends to $(+\infty)$ or $(-\infty)$, as n tends to infinity, then the series is said to be divergent.

If S_n tends to no limit, whether finite or infinite, then the series is said to oscillate. It oscillates finitely or infinitely according as S_n oscillates between finite limits or between $(-\infty)$ and $(+\infty)$.

A divergent or oscillating series is also called non-convergent.

Absolute and Conditional Convergence:

1. **Absolutely Convergent Series:**

If a convergent series whose terms are not all positive, remains convergent when all its terms are made positive, then it is called an absolutely convergent series.

Examples:

I. The series

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots$$

is absolutely convergent; for, by making all its terms positive, the series becomes

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\cdots$$

which is known to be convergent.

II. The series

$$1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \cdots$$

is absolutely convergent as the series of positive terms,

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\cdots \text{ being a G.P. series}$$

is convergent.

Another def^n: If $\sum u_n$ be a series of both positive and negative terms, it is said to be absolutely convergent when the series $\sum |u_n|$ is convergent.

If $\sum u_n$ is convergent but $\sum |u_n|$ is not convergent then it is called a conditionally convergent series.

2. Conditionally Convergent Series:

If any Convergent Series whose terms are not all positive, becomes divergent when all its terms are made positive. Then it is called Conditionally Convergent or Semi-Convergent or accidently Convergent Series.

Examples: I. The Series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\cdots$$

is called a Conditionally Convergent Series, because it is, as we know, a Convergent Series, but if all its term be made positive, it becomes

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\cdots\cdots$$

which is a divergent Series.

II. The Series

$$1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\cdots\cdots\cdots$$

is Conditionally Convergent Series; for, the Series with positive terms

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\cdots\cdots\cdots$$

is a divergent Series.

Alternating Series: A mixed Series whose terms are alternately positive and negative is called an alternating Series.

If $u_n>0$ for all n, the Series $\sum_{1}^{\infty}(-1)^{n+1}u_n$ / $\sum_{n=1}^{\infty}(-1)^{n-1}u_n$ is called an alternating Series. [A Series in which the Sign of Consecutive terms are alternately positive and negative, is called an alternating Series.]

Leibnitz's Test: Let $\{u_n\}$ be a monotone non-increasing Sequence of positive terms Such That $\lim u_n=0$.

Then the alternating Series $u_1-u_2+u_3-u_4+\cdots\cdots$ i.e., $\sum_{1}^{\infty}(-1)^{n+1}u_n$ is Convergent / i.e. $\sum_{1}^{\infty}(-1)^{n-1}u_n$ is Convergent.

Proof. Let $\{S_n\}$ be the Sequence of partial Sums of the Series.

$$\therefore S_n=\sum_{n=1}^{\infty}(-1)^{n-1}u_n$$

$$\therefore S_{2n+1}-S_{2n-1}=-u_{2n}+u_{2n+1}\leq 0\ \forall n\in\mathbb{N}$$

$\therefore$ the Subsequence $\{S_{2n+1}\}$ is monotone non-increasing.

Again $S_{2n+2} - S_{2n} = u_{2n+1} - u_{2n+2} \geq 0 \ \forall n \in \mathbb{N}$

$\therefore$ the subsequence $\{S_{2n}\}$ is monotone non-decreasing.

$S_{2n+1} = u_1 - u_2 + u_3 - u_4 + \cdots \cdots + u_{2n+1}$

$= (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2n-1} - u_{2n}) + u_{2n+1} \geq 0,$

as the sequence $\{u_n\}$ is non-increasing with positive terms. $\forall n$

$\therefore$ $\{S_{2n+1}\}$ is bounded below by 0 and non-increasing.

Hence sequence $\{S_{2n+1}\}$ is convergent.

Again

$S_{2n} = u_1 - u_2 + u_3 - u_4 + \cdots \cdots - u_{2n}$

$= u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots \cdots - u_{2n} < u_1$

$\therefore$ $\{S_{2n}\}$ is bounded above by u_1 and non-decreasing.

So $\{S_{2n}\}$ is convergent.

Therefore both the sequences $\{S_{2n}\}$ and $\{S_{2n+1}\}$ are convergent. Because $\lim (S_{2n+1} - S_{2n}) = \lim u_{2n+1} = 0$ (given), both sequences $\{S_{2n+1}\}$ and $\{S_{2n}\}$ converges to the same limit.

$\therefore$ $\lim S_{2n+1} = \lim S_{2n}$

$\therefore$ $\{S_n\}$ is convergent and so $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ is convergent.

<u>Another defn</u>: A series $\sum u_n$ is said to be <u>absolutely convergent</u> if the series $\sum u_n$ obtained on taking every term of the given series with a positive sign is convergent i.e, if the series $\sum |u_n|$ is convergent.

A series which is convergent but is not absolutely convergent is called a <u>conditionally convergent</u> series.

<u>Leibnitz Test</u>: If the alternating series

$u_1 - u_2 + u_3 - u_4 + \cdots \cdots \quad (u_n > 0 \ \forall n)$

is such that

i) $u_{n+1} \leq u_n, \ \forall n$ and

ii) $\lim u_n = 0,$

then the series converges.

Proof: Let $S_n = u_1 - u_2 + u_3 - u_4 + \cdots \cdots + (-1)^{n-1} u_n$

Now for all n, $S_{2n+2} - S_{2n} = u_{2n+1} - u_{2n+2} \geq 0$

$\Rightarrow S_{2n+2} \geq S_{2n}$

$\Rightarrow \{S_{2n}\}$ is a monotonic increasing sequence

Again $S_{2n} = u_1 - u_2 + u_3 - u_4 + \dots + u_{2n-1} - u_{2n}$

$= u_1 - (u_2 - u_3) - (u_4 - u_5) - \dots - (u_{2n-2} - u_{2n-1}) - u_{2n}$

But since $u_{n+1} \le u_n$, for all n, therefore, each bracket on the right hand side is positive and hence

$$S_{2n} \le u_1 \; \forall n$$

Thus, the monotonic increasing sequence is bounded above and is consequently convergent.

Let $\lim\limits_{n \to \infty} S_{2n} = S$

We shall now show that the sequence $\{S_{2n+1}\}$ also converges to the same limit S.

Now $S_{2n+1} = S_{2n} + u_{2n+1}$

$\therefore \lim\limits_{n\to\infty} S_{2n+1} = \lim\limits_{n\to\infty} S_{2n} + \lim\limits_{n\to\infty} u_{2n+1}$

But by condition (ii), $\lim\limits_{n\to\infty} u_{2n+1} = 0$

$\therefore \lim\limits_{n\to\infty} S_{2n+1} = \lim\limits_{n\to\infty} S_{2n} = S$

Thus, the sequences $\{S_{2n}\}$ and $\{S_{2n+1}\}$ both converge to the same limit S. We shall now show that the sequence $\{S_n\}$ also converges to S.

Since the sequences $\{S_{2n}\}$ and $\{S_{2n+1}\}$ both converge to S, therefore, $\exists$ positive integers m_1, m_2 respectively such that

$$|S_{2n} - S| < \epsilon, \; \forall n \ge m_1 \quad \text{——— (1)}$$

and $$|S_{2n+1} - S| < \epsilon, \; \forall n \ge m_2 \quad \text{——— (2)}$$

Thus from (1) and (2), we have

$$|S_n - S| < \epsilon \; \forall n \ge \max(2m_1, 2m_2+1)$$

$\Rightarrow \{S_n\}$ converges to S

$\Rightarrow$ The series $\sum (-1)^{n-1} u_n$ converges.

<u>Theorem</u>: To show that an absolutely convergent series is convergent. i.e.

If the series $\sum |u_n|$ is convergent, then the original series $\sum u_n$ is also convergent.

<u>Proof</u>: Since $\sum |u_n|$ is convergent, by the principle of convergence of an infinite series.

We have, corresponding to any pre-assigned positive number ϵ however small $\exists$ a positive number m such that for $n \geq m$

$$|u_{n+1}| + |u_{n+2}| + |u_{n+3}| + \dots + |u_{n+p}| < \epsilon \quad \text{—— (1)}$$

Now considering the series $\sum u_n$ we have

$$|S_{n+p} - S_n| = |{}_pR_n|$$

$$= |u_{n+1} + u_{n+2} + \dots + u_{n+p}|$$

$$\leq |u_{n+1}| + |u_{n+2}| + \dots + |u_{n+p}|$$

$$< \epsilon \text{ by (1) for } n \geq m$$

Hence $\sum u_n$ is Convergent.

Theorem: The series $\sum_{n=1}^{\infty} u_n^2$ is convergent if $\sum_{n=1}^{\infty} u_n$ is absolutely convergent.

Proof. Given that $\sum u_n$ is absolutely convergent.

$\therefore \sum |u_n|$ is Convergent.

Let $S_n = u_1 + u_2 + \dots + u_n$

$\therefore S_{n+p} = u_1 + u_2 + \dots + u_n + u_{n+1} + u_{n+2} + \dots + u_{n+p}$

So $|S_{n+p} - S_n| = |u_{n+1} + u_{n+2} + \dots + u_{n+p}|$

$$\leq |u_{n+1}| + |u_{n+2}| + \dots + |u_{n+p}|$$

$$< \epsilon/2L \text{ for } n \geq m \quad \text{—— (1)}$$

Also $\because \sum |u_n|$ is Convergent

$$|S_n| < L \ \forall n \quad \text{—— (2)}$$

$$\text{and } |S_{n+p}| < L \text{ for } p \geq 0 \quad \text{—— (3)}$$

Now $|S_{n+p}^2 - S_n^2| = |(S_{n+p} + S_n)(S_{n+p} - S_n)|$

$$= |S_{n+p} + S_n|\,|S_{n+p} - S_n|$$

$$< (L + L) \cdot \epsilon/2L \text{ by (1), (2) \& (3)}$$

i.e. $< \epsilon$

$\therefore \sum |u_n|^2$ is Convergent.

But $u_n^2 = |u_n|^2 \Rightarrow \sum u_n^2 = \sum |u_n|^2$

Hence $\sum u_n^2$ is Convergent.

Theorem: The sum of an absolutely convergent series is independent of the order of its terms. i.e.

If the terms of an absolutely convergent series are rearranged, the series remains convergent and its sum is unaltered.

Proof: Let the series $\sum u_n$ be absolutely convergent and converges to a finite quantity S.

We shall now show that S is not altered by derangement of the terms and the deranged series is also convergent.

Let $S_n = \sum_{n=1}^{\infty} u_n$ be the given absolutely convergent series before derangement,

and $T_n = \sum_{n=1}^{\infty} v_n$ be the series t by deranging the terms of the given series $\sum u_n$.

Let S_n tends to S.

Since the series $\sum_{n=1}^{\infty} u_n$ is absolutely convergent,

$\sum |u_n|$ is convergent

$\therefore |u_{n+1}| + |u_{n+2}| + |u_{n+3}| + \cdots$ to $\infty < \epsilon/2$ for $n \geq m$ ——— (1)

Let us now suppose that T_p contains all the terms of S_n.

If now $r \geq p$, then $T_r - T_p$ consists of a number of terms from the series $\sum_{n=1}^{\infty} u_n$ and the order of each term on the R.H.S is $> p$.

$\therefore$ $T_r - S_p$ consists of a number of terms of $\sum u_n$ and order of each term on R.H.S. beeing $> p$

Hence if $r \geq p$, then by virtue of (1).

$$|T_r - S_n| < \epsilon/2 \text{ ——— (2)}$$

But $|S - S_n| = |u_{n+1} + u_{n+2} + \cdots \text{ to } \infty|$

$\leq |u_{n+1}| + |u_{n+2}| + \cdots$ to ∞

$< \epsilon/2$ if $n \geq m$ ——— (3)

$\therefore |S - T_r| = |S - S_n + S_n - T_r|$

$= |(S - S_n) - [T_r - S_n]| \leq |S - S_n| + |T_r - S_n|$

i.e. $< \epsilon$ $\quad < \epsilon/2 + \epsilon/2$ by (2) & (3)

This shows that the series is convergent and has S as its sum.

Theorem: If $\sum u_n$ and $\sum v_n$ be two infinite absolutely convergent series whose sums are u and v, then the series $\sum(u_n+v_n)$, $\sum(u_n-v_n)$ are also absolutely convergent and their sums are $(u+v)$ & $(u-v)$.

Proof: We have

$$|u_n+v_n| \le |u_n| + |v_n|$$

$$\text{or, } \sum|u_n+v_n| \le \sum|u_n| + \sum|v_n|$$

Now we find that both $\sum u_n$ and $\sum v_n$ are monotonic increasing sequences.

Also since $\sum u_n$ and $\sum v_n$ has the sum u & v respectively, both of them are bounded above by u and v.

$\therefore \sum|u_n+v_n| < k$ where k is any constant

Thus the sequence $\sum|u_n+v_n|$ is monotonic increasing and bounded above.

So $\sum|u_n+v_n|$ is convergent and converges to the sum $(u+v)$.

Hence $\sum(u_n+v_n)$ is absolutely convergent and converges to the sum $(u+v)$,

Similiarly $\sum(u_n-v_n)$ is also absolutely convergent and converges to the sum $(u-v)$.

Theorem: A change in the order of the terms may involve a change in the sum for any conditionally convergent series.

This is illustrated by way of examples as follows:

(a) Consider the series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\cdots+(-1)^{n-1}\frac{1}{n}+\cdots$$

Let $S_n = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\cdots+(-1)^{n-1}\frac{1}{n}\cdots$

$$\therefore S_{n+p} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\cdots+(-1)^{n-1}\frac{1}{n}+\cdots$$
$$\cdots+(-1)^{n+p-1}\cdot\frac{1}{n+p}$$

$\therefore\ S_{n+p} - S_n = \frac{1}{n+1} - \frac{1}{n+2} + \cdots\cdots + (-1)^{p-1}\frac{1}{n+p}$ —(1)

Now $\frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \cdots\cdots$

$= \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \left(\frac{1}{n+3} - \frac{1}{n+4}\right) + \cdots$

$= \frac{1}{(n+1)(n+2)} + \frac{1}{(n+3)(n+4)} + \cdots$ whose last term is positive whether p is odd or even.

$\therefore\ \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \cdots + (-1)^{p-1}\frac{1}{n+p} > 0$

So from (1), we have

$S_{n+p} - S_n = \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \cdots + (-1)^{p-1}\cdot\frac{1}{n+p}$

$= \frac{1}{n+1} - \frac{1}{(n+2)(n+3)} - \frac{1}{(n+4)(n+5)} - \cdots$

$< \frac{1}{n+1}$ $\because$ Last term is negative whether p is odd or even.

If now $\frac{1}{n+1} < \epsilon$

then $n+1 > \frac{1}{\epsilon}$ i.e., $n > \frac{1}{\epsilon} - 1$

Thus $|S_{n+p} - S_n| < \epsilon$ for $n > \frac{1}{\epsilon} - 1$ i.e., $\geq m$

i.e., for $n \geq m$ and for $p \geq 0$

Hence by the Cauchy's principle of Convergence, the given series is Convergent.

Again considering all the terms positive, the series becomes

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$

$\because\ \frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$ etc

We get $S_4 > 1 + \frac{1}{2} + \frac{1}{2}$

$S_8 > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$

$S_{16} > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$ and so on.

$\therefore\ S_{2^n} > 1 + \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \text{to } n \text{ terms}\right)$

i.e., $> 1 + \frac{n}{2}$

$\therefore\ S_{2^n} \to \infty$ as $n \to \infty$

So the Series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}+\cdots$ is divergent.

Hence the Series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+(-1)^{n-1}\frac{1}{n}+\cdots$$

is Conditionally Convergent.

Now let $S=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$

we shall now show that by rearranging terms, its sum may change.

We have

$$S_{3n}=\left(1-\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{6}-\frac{1}{8}\right)+\left(\frac{1}{5}-\frac{1}{10}-\frac{1}{12}\right)+\cdots$$

$$=\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\cdots$$

$$=\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\cdots$$

$$=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots\right)$$

$$=\frac{1}{2}S$$

$\because\ Lt\ S_{3n}=Lt\ S_{3n+1}=Lt\ S_{3n+2}$ as $n\to\infty$

The Sum of the series when rearranged is $\frac{1}{2}S$.

(b) Now we shall show that in another rearrangement the sum of the above series $=\frac{3}{2}S$.

Solution: Let $S=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$

$$=\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\cdots$$

$$=\sum_{r=1}^{n}\left[\frac{1}{2r-1}-\frac{1}{2r}\right]\quad\cdots\cdots(1)$$

Also $S=\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}\right)+\cdots$

$$=\sum_{r=1}^{n}\left[\frac{1}{4r-3}-\frac{1}{4r-2}+\frac{1}{4r-1}-\frac{1}{4r}\right]\quad\cdots(2)$$

Let the above series be rearranged as

$$\left(1+\frac{1}{3}-\frac{1}{2}\right)+\left(\frac{1}{5}+\frac{1}{7}-\frac{1}{4}\right)+\cdots$$

and let its nth term be denoted as S_{3n}

$$\therefore\ S_{3n}=\sum_{r=1}^{n}\left[\frac{1}{4r-3}+\frac{1}{4r-1}-\frac{1}{2r}\right]\quad\text{———}(3)$$

Subtracting ② from ③, we have

$$S_{3n} - S = \sum_{r=1}^{n}\left[\frac{1}{4r-2} - \frac{1}{4r}\right]$$

$$= \frac{1}{2}\sum_{r=1}^{n}\left[\frac{1}{2r-1} - \frac{1}{2r}\right]$$

$$= \frac{1}{2} S \text{ by } ①$$

$$\therefore S_{3n} = S + \frac{1}{2}S = S\left(1+\frac{1}{2}\right) = \frac{3}{2}S$$

Thus the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots$ when rearranged as in ③ its sum $= \frac{3}{2}S$.

<u>Theorem</u>: If the terms of the series $\sum (-1)^{n-1} u_n$ are alternately positive and negative and never increase in numerical value, then the series will converge, provided that the terms tend to zero as a limit.

<u>Proof</u>: Let the given series be

$$u_1 - u_2 + u_3 - u_4 + \cdots \quad \cdots \quad \cdots$$

where $u_1 > u_2 > u_3 > u_4 \cdots \quad \cdots \quad \cdots$

Now let $S_{2n} = \sum_{n=1}^{2n} (-1)^{n-1} u_n$

$$= (u_1 - u_2) + (u_3 - u_4) + \cdots + \cdots + (u_{2n-1} - u_{2n}) \quad \text{——— } ①$$

Also $S_{2n+1} = u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots - (u_{2n} - u_{2n+1})$ ——— ②

Since $u_1 > u_2 > u_3 > u_4 \cdots \quad \cdots$

From ①, it follows that

S_{2n} is positive and its value increases as n increases.

Also from ② it follows that

S_{2n+1} is positive but less than u_1 and its value diminishes as n increases.

$$\therefore S_{2n} = S_{2n+1} - u_{2n+1} < u_1$$

$$\text{and } S_{2n+1} = S_{2n} + u_{2n+1} > 0$$

$$\therefore \lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty}\left\{S_{2n} + u_{2n+1}\right\} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} u_{2n+1}$$

$$= \lim_{n\to\infty} S_{2n} \text{ as } \lim_{n\to\infty} u_{2n+1} = 0$$

Hence the given Series is Convergent and its Sum is lying between 0 and u_1.

Theorem: If $u_1, u_2, u_3, \ldots$ is a decreasing sequence
2018 of positive terms tending to zero as limit, then the Series $u_1 - \frac{1}{2}(u_1+u_2) + \frac{1}{3}(u_1+u_2+u_3) - \cdots + \frac{(-1)^{n-1}}{n}(u_1+u_2+\cdots+u_n) + \cdots$ is Convergent.

Proof: Let the given series be

$$v_1 - v_2 + v_3 - v_4 + \cdots \quad \cdots \quad \cdots = \sum (-1)^{n-1} v_n$$

$v_1 = u_1,\ v_2 = \frac{1}{2}(u_1+u_2) \cdots$

where $v_n = \dfrac{u_1+u_2+\cdots+u_n}{n}$

Now $\lim\limits_{n\to\infty} v_n = \lim\limits_{n\to\infty} \dfrac{u_1+u_2+\cdots+u_n}{n}$

$= 0$ by Cauchy's first theorem on limit

$\therefore \lim\limits_{n\to\infty} v_n = 0$

As $\{u_n\}$ strictly decreasing ...

$\therefore u_1 > u_2 > u_3 \cdots$

$2u_1 > u_1 + u_2$

$3(u_1+u_2) > 2(u_1+u_2+u_3)$

$4(u_1+u_2+u_3) > 3(u_1+u_2+u_3+u_4)$

and so on

$\therefore u_1 > \frac{1}{2}(u_1+u_2) > \frac{1}{3}(u_1+u_2+u_3) > \frac{1}{4}(u_1+u_2+u_3+u_4)$ and so on. Now, $v_{n+1} = \dfrac{u_1+u_2+\cdots+u_n+u_{n+1}}{n+1} = \dfrac{n v_n + u_{n+1}}{n+1}$

Thus $v_1 > v_2 > v_3 > v_4 \cdots$ $\quad \therefore v_{n+1} - v_n = \dfrac{u_{n+1} - v_n}{n+1} < 0$ *

So the given Series is an alternating Series with terms decreasing and ultimately tending to zero as a limit.

Hence the given Series is Convergent.

* as each of $u_1, u_2, \ldots u_n$ is greater than u_{n+1}.

$\therefore$ their A.M. $v_n > u_{n+1}$.

Also $\lim v_n = 0$

So, by Leibnitz's test the Series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} v_n$ i.e., the given Series is Convergent.

All

* Prove that the series $\frac{3}{1\cdot2} - \frac{5}{2\cdot3} + \frac{7}{3\cdot4} - \cdots \cdots$ converges conditionally.

Solution: The series is
$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n+1}{n(n+1)} = \sum_{n=1}^{\infty} (-1)^{n-1} u_n$$

where $u_n = \frac{2n+1}{n(n+1)}$

So, $\lim u_n = 0$

Now $u_{n+1} - u_n = \frac{2n+3}{(n+1)(n+2)} - \frac{2n+1}{n(n+1)}$

$$= \frac{2n^2+3n-(2n+1)(n+2)}{n(n+1)(n+2)}$$

$$= \frac{2n^2+3n-2n^2-5n-2}{n(n+1)(n+2)} = -\frac{2n+2}{n(n+1)(n+2)}$$

$$= -\frac{2}{n(n+2)} < 0$$

i.e. $u_{n+1} < u_n$

$\therefore$ $\{u_n\}$ is strictly decreasing sequence

Also $\lim u_n = 0$

Hence, by Leibnitz's test the series is convergent.

For testing absolute convergence we are to test the convergence of $\left|\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n+1}{n(n+1)}\right| = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)}$

Let the series be $\sum u_n$

$$\therefore u_n = \frac{2n+1}{n(n+1)}$$

Consider the divergent series $\sum v_n$ where $v_n = \frac{1}{n}$

$\therefore \lim \frac{u_n}{v_n} = \lim \frac{2n^2+n}{n^2+n} = \lim \frac{2+\frac{1}{n}}{1+\frac{1}{n}} = 2$, a non-zero finite number.

Hence $\sum u_n$ and $\sum v_n$ converge or diverge together by comparison test. As $\sum v_n$ is divergent, therefore $\sum u_n$ is divergent.

Hence, the given series is conditionally convergent.

* Show that the series $\sum_{n=1}^{\infty} (-1)^n \sin\frac{1}{n}$ can not converge absolutely.

Solution: For testing absolute convergence we are to test the convergence of the series $\sum_{n=1}^{\infty} \sin\frac{1}{n}$.

Let the series be $\sum u_n$, then $u_n = \sin\frac{1}{n}$.

Consider the harmonic Series $\sum v_n$, $v_n = \frac{1}{n}$

Now,
$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{1/n} = 1, \text{ a non-zero}$$
finite number.

Hence, $\sum u_n$ and $\sum v_n$ Converge or diverge together by ratio test in limit form. As $\sum v_n$ is divergent, therefore, $\sum u_n$ is also divergent.

Hence, the given Series can not Converge absolutely.

* Show that the Series $\sum_{n=1}^{\infty} (-1)^{n-1} n^{-1/2}$ is Convergent. Examine its absolute Convergence.

Solution: Let the Series be $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$.

Then $u_n = n^{-\frac{1}{2}}$
$$\therefore u_{n+1} - u_n = \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}} < 0$$

i.e. $u_{n+1} < u_n$

So $\{u_n\}$ is strictly decreasing Sequence.
$$\therefore \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

Hence, by Leibnitz's test the Series is Convergent.

To test the absolute Convergence, Consider the Series
$$\sum_{n=1}^{\infty} \left|(-1)^{n-1} n^{-\frac{1}{2}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

The Series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series with $p = \frac{1}{2} < 1$ and hence divergent.

$\therefore$ the Series is not absolutely Convergent.

* Show that the Series $\frac{3}{2} + \frac{4}{3} + \frac{5}{4} + \frac{6}{5} + \cdots \cdots$ is not Convergent but the Series
$$\left(\frac{3}{2} - \frac{4}{3}\right) + \left(\frac{5}{4} - \frac{6}{5}\right) + \cdots \cdots \text{ is Convergent.}$$

Solution: Let the first Series be $\sum u_n$, then $u_n = \frac{n+2}{n+1}$
$$\therefore \lim u_n = \lim \frac{n+2}{n+1} = \lim \frac{1+\frac{2}{n}}{1+\frac{1}{n}} = 1 \neq 0$$

$\therefore$ $\sum u_n$ is not Convergent.

If the Second Series be $\sum v_n$, then $v_n = \frac{2n+1}{2n} - \frac{2n+2}{2n+1}$
$$= \frac{4n^2+4n+1-4n^2-4n}{2n(2n+1)}$$
$$= \frac{1}{2n(2n+1)} = \frac{1}{2n} - \frac{1}{2n+1}$$

If $\{S_n\}$ be the sequence of partial sums of $\sum v_n$, then $S_n = v_1 + v_2 + \cdots \cdots + v_n$

$$= \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{6} - \frac{1}{7}\right) + \cdots \cdots + \left(\frac{1}{2n} - \frac{1}{2n+1}\right)$$

$$= \frac{1}{2} - \left(\frac{1}{3} - \frac{1}{4}\right) - \left(\frac{1}{5} - \frac{1}{6}\right) - \cdots \cdots - \left(\frac{1}{2n-1} - \frac{1}{2n}\right) - \frac{1}{2n+1}$$

$$< \frac{1}{2} \quad \forall n \in \mathbb{N}$$

$\therefore \{S_n\}$ is bounded above.

Also $S_{n+1} - S_n = \dfrac{1}{2n+2} - \dfrac{1}{2n+3} > 0 \ \forall n \in \mathbb{N}$

$\therefore \{S_n\}$ is increasing and bounded above and so convergent.

$\therefore \sum v_n$ is convergent.

<u>Theorem</u>: Prove that an absolutely convergent series can be expressed as the difference of two convergent series of positive real numbers.

<u>Proof</u>. Let $\sum u_n$ be a mixed series and $\sum u_n$ be absolutely convergent.

Let $a_n = \dfrac{|u_n| + u_n}{2}$ and $b_n = \dfrac{|u_n| - u_n}{2}$

$\therefore u_n = a_n - b_n$

$\therefore \sum u_n = \sum a_n - \sum b_n$

Now, $\sum a_n$ and $\sum b_n$ are series of non-negative terms.

As $\sum u_n$ is absolutely convergent, therefore $\sum u_n$ and $\sum |u_n|$ are both convergent.

$\therefore \sum a_n$ and $\sum b_n$ are both convergent. A series of non-negative terms is identical to a series with positive terms neglecting the zero terms.

Hence, any absolutely convergent series can be expressed as the difference of two convergent series of positive real numbers.

* Show that the series $\sum \frac{(-1)^{n-1}}{n^5}(3n-2)$ is absolutely convergent.

Solution: The series may be written as
$$\sum \frac{(-1)^{n-1}}{n^4}\left(3-\frac{2}{n}\right) = \sum a_n b_n$$
where $a_n = \frac{(-1)^{n-1}}{n^4}$ and $b_n = 3-\frac{2}{n}$.

$\therefore \sum |a_n| = \sum \frac{1}{n^4}$ which is a p-series with $p=4>1$ and hence converges.

Hence, $\sum a_n$ converges absolutely.

Now $|b_n| = \left|3-\frac{2}{n}\right| < 3 \; \forall n \in \mathbb{N}$

$\therefore$ the sequence $\{b_n\}$ is bounded above.

$\therefore$ the series $\sum \frac{(-1)^{n-1}}{n^5}(3n-2)$ is absolutely convergent.

Comparison test for two series of positive terms:

(a) If $\sum u_n$ and $\sum v_n$ be two series of positive terms and if $\sum u_n$ is convergent, then $\sum v_n$ is also convergent provided $v_n \le u_n$ or $v_n \le k u_n \; \forall n$, k being a constant.

(b) If u_n and v_n are the n-th terms of two series of positive terms, then

(i) if $\sum u_n$ is convergent and $\frac{v_{n+1}}{v_n} \le \frac{u_{n+1}}{u_n}$, for all n or for $n \ge m$, then $\sum v_n$ is also convergent.

(ii) If $\sum u_n$ is divergent and $\frac{v_{n+1}}{v_n} \ge \frac{u_{n+1}}{u_n}$, for all n or for $n \ge m$, then $\sum v_n$ is also divergent.

Proof: (a) Since $\sum u_n$ is convergent, then
$$\lim_{n\to\infty} u_n = l \text{ (a fixed number)}$$

Now given $v_n \le u_n \; \forall n$
$$\therefore \sum v_n \le \sum u_n$$
$$\lim_{n\to\infty} \sum v_n \le \lim_{n\to\infty} \sum u_n \text{ i.e., } \le l$$
$\therefore \sum v_n$ is convergent.

Similarly if $v_n \le k u_n$ for all n, then
$$\lim_{n\to\infty} \sum v_n \le kl \text{ (a fixed number)}$$
$\therefore \sum v_n$ is convergent.

(b).(i): $\frac{v_{n+1}}{v_n} \le \frac{u_{n+1}}{u_n}$ for all n

or for $n \geq m$ $\therefore \frac{v_{n+1}}{u_{n+1}} \le \frac{v_n}{u_n}$

Changing n to $n-1, n-2, \ldots m$ we get

$$\frac{v_{n+1}}{u_{n+1}} \le \frac{v_n}{u_n} \le \frac{v_{n-1}}{u_{n-1}} \le \ldots \le \frac{v_m}{u_m} = k \text{ say}$$

$\therefore v_n \le k u_n$ for all n or for $n \geq m$

Hence by (a) Σv_n is convergent.

(ii) $\frac{v_{n+1}}{v_n} \geq \frac{u_{n+1}}{u_n}$ for all n or for $n \geq m$

$$\therefore \frac{v_{n+1}}{u_{n+1}} \geq \frac{v_n}{u_n}$$

Changing n to $n-1, n-2, \ldots m$, we get

$$\frac{v_{n+1}}{u_{n+1}} \geq \frac{v_n}{u_n} \geq \frac{v_{n-1}}{u_{n-1}} \geq \ldots \geq \frac{v_m}{u_m} = k \text{ (say)}$$

$\therefore \frac{v_n}{u_n} \geq k$ for all n or for $n \geq m$

or $v_n \geq k u_n$

Since Σu_n is divergent, then Σv_n is greater than or equal to a divergent series.

Hence Σv_n is divergent.

Limit form: If Σu_n and Σv_n are two positive term series such that $\lim_{n\to\infty} \left(\frac{u_n}{v_n}\right) = l$, where l is a non-zero finite number, then the two series converge or diverge together.

Proof: Evidently $l > 0$.

Let ϵ be a positive number such that $l - \epsilon > 0$

Since $\lim_{n\to\infty} \frac{u_n}{v_n} = l$, therefore there exists positive integer m such that

$$\left|\frac{u_n}{v_n} - l\right| < \epsilon \ \forall \ n \geq m$$

$\Rightarrow l-\epsilon < \frac{u_n}{v_n} < l+\epsilon, \forall n \geq m$

$\Rightarrow (l-\epsilon)v_n < u_n < (l+\epsilon)v_n, \forall n \geq m$ ——— (1)

Now if Σv_n is Convergent, then from (1)

$u_n < (l+\epsilon)v_n, \forall n \geq m$

So that by Test-I, Σu_n is Convergent.

Again if Σv_n is divergent, then from (1)

$u_n > (l-\epsilon)v_n, \forall n \geq m$

So that by Test-I, Σu_n is divergent.

Similiarly, we may show that Σv_n Converges or diverges with Σu_n.

Hence, the two Series behave alike.

<u>Problem</u>: Show that the series Σu_n, given by $u_n = \frac{1}{n}\sin\frac{1}{n}$ is Convergent.

<u>Solution</u>: Consider the Series Σv_n, when $v_n = \frac{1}{n^2}$.

This is a p-Series with $p = 2 > 1$ and hence Convergent.

Now, $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = \lim_{h\to 0+} \frac{\sin h}{h}$, when $h = \frac{1}{n}$

$= 1$, a non-zero finite number.

$\therefore \Sigma \frac{1}{n}\sin\frac{1}{n}$ is Convergent.

<u>Problem</u>: Test the Convergence of the Series Σu_n where $u_n = \sqrt{n^4-1} - \sqrt{n^4+1}$

<u>Solution</u>: Here $u_n = \frac{(\sqrt{n^4-1} - \sqrt{n^4+1})(\sqrt{n^4-1} + \sqrt{n^4+1})}{\sqrt{n^4-1} + \sqrt{n^4+1}}$

$$= -\frac{2}{\sqrt{n^4-1} + \sqrt{n^4+1}}$$

The denominator of u_n is of dimension 2 in n.

$\therefore$ Choose the Series $\sum_{n=1}^{\infty} v_n$ given by $v_n = \frac{1}{n^2}$ for Comparison. Σv_n is a p-Series with $p = 2 > 1$ and hence Convergent.

Now $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \lim\limits_{n\to\infty} \frac{-2n^2}{\sqrt{n^4-1}+\sqrt{n^4+1}}$

$$= \lim_{n\to\infty} \frac{-2}{\sqrt{1-\frac{1}{n^4}}+\sqrt{1+\frac{1}{n^4}}}$$

$= -2$, a finite non-zero number.

$\therefore \sum u_n$ is convergent by comparison test.

Problem: Show that the series $\sum \sin\frac{1}{n}$ is divergent.

Solution: Let $u_n = \sin\frac{1}{n}$. Consider $\sum v_n$, where $v_n = \frac{1}{n}$ for comparison.

Now $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \lim\limits_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = \lim\limits_{h\to 0+} \frac{\sin h}{h}$ where $h=\frac{1}{n}$

$= 1$, a non-zero finite number.

$\therefore$ So, $\sum u_n$ and $\sum v_n$ have the same convergence status. As $\sum v_n$ is a harmonic series, it is divergent.

Hence the given series also diverges.

Problem: Test the convergence of $\sum\limits_{n=1}^{\infty} \left\{(n^5+1)^{\frac{1}{5}} - n\right\}$.

Solution: Here

$$u_n = (n^5+1)^{1/5} - n$$

$$= n\left\{\left(1+\frac{1}{n^5}\right)^{\frac{1}{5}} - 1\right\}$$

$$= n\left\{1 + \frac{1}{5}\cdot\frac{1}{n^5} + \frac{\frac{1}{5}\left(\frac{1}{5}-1\right)}{2!}\left(\frac{1}{n^5}\right)^2 + \cdots - 1\right\}$$

by Binomial expansion

$$= \frac{1}{5}\cdot\frac{1}{n^4} - \frac{2}{25}\,\frac{1}{n^9} + \cdots$$

For comparison, choose the series $\sum v_n$, where $v_n = \frac{1}{n^4}$. This is a p-series with $p = 4 > 1$ and hence convergent.

Now $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \lim\limits_{n\to\infty} n^4\left(\frac{1}{5}\cdot\frac{1}{n^4} - \frac{2}{25}\cdot\frac{1}{n^9} + \cdots\right)$

$= \frac{1}{5}$ which is a non-zero finite no.

Hence $\sum u_n$ and $\sum v_n$ have their same convergence status. As $\sum v_n$ is convergent.

$\therefore \sum u_n$ is also convergent by the comparison test.

D'Alembert's Ratio Test

If $\sum u_n$ is a positive term series, such that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$, then the series (i) converges, if $l<1$,

(ii) diverges, if $l>1$, and

(iii) the test fails, if $l=1$.

Proof: Case-I $0<l<1$

Let us select a positive number ϵ, such that $l+\epsilon<1$.

Let $l+\epsilon = \alpha < 1, \alpha \neq 0$

Since $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$, therefore there exists a positive number m such that

$$\left|\frac{u_{n+1}}{u_n} - l\right| < \epsilon, \forall n \geq m$$

$$\Rightarrow l-\epsilon < \frac{u_{n+1}}{u_n} < l+\epsilon, \forall n \geq m$$

$$\Rightarrow \frac{u_{n+1}}{u_n} < l+\epsilon = \alpha, \forall n \geq m$$

For $n \geq m$, $\frac{u_n}{u_m} = \frac{u_{m+1}}{u_m} \cdot \frac{u_{m+2}}{u_{m+1}} \cdots \cdots \frac{u_n}{u_{n-1}} < \alpha^{n-m}$

$$\Rightarrow u_n < \left(\frac{u_m}{\alpha^m}\right)\alpha^n, \forall n \geq m, \alpha<1$$

Since m is a fixed integer, therefore $\left(\frac{u_m}{\alpha^m}\right)$ is a fixed number, say K.

Thus, $\forall n \geq m$, we have

$$u_n < K\alpha^n$$

But since $\sum \alpha^n$ is a convergent geometric series (common ratio, $\alpha<1$), therefore by comparison test $\sum u_n$ converges.

Case-II : $l>1$.

Let us select a positive number ϵ, such that $l-\epsilon>1$. Let $l-\epsilon = \beta > 1$.

Since $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$, therefore there exists a positive number m_1 such that

$$\left|\frac{u_{n+1}}{u_n} - l\right| < \epsilon \quad \forall n \geq m_1$$

$$\Rightarrow \ l-\epsilon < \frac{u_{n+1}}{u_n} < l+\epsilon, \ \forall \ n \geq m_1$$

$$\Rightarrow \ \frac{u_{n+1}}{u_n} > l-\epsilon = \beta, \ \forall \ n \geq m_1$$

Now for $n \geq m_1$,

$$\frac{u_n}{u_{m_1}} = \frac{u_{m_1+1}}{u_{m_1}} \cdot \frac{u_{m_1+2}}{u_{m_1+1}} \cdot \dots \cdot \frac{u_n}{u_{n-1}} > \beta^{n-m_1}$$

$$\Rightarrow \ u_n > \frac{u_{m_1}}{\beta^{m_1}} \cdot \beta^n, \ \forall \ n \geq m_1$$

Since m_1 is a fixed number, therefore $\frac{u_{m_1}}{\beta^{m_1}}$ is a fixed finite number, say K_1.

Thus, for $n \geq m_1$, we have

$$u_n > K_1 \beta^n$$

But $\Sigma \beta^n$ is a divergent geometric Series (Common ratio, $\beta > 1$), therefore by Comparison test, Σu_n diverges.

__Note__: The test fails for $l = 1$ in the Sense that it fails to give any definite information.

For example, Consider the two Series $\Sigma(1/n)$ & $\Sigma(1/n^2)$

$\Sigma(1/n)$ diverges when $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)$

$= 1$

and $\Sigma\left(\frac{1}{n^2}\right)$ Converges when $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2$

$= 1.$

__Logarithmic Ratio Test__:

If Σu_n is a positive term Series such that

$$\lim_{n\to\infty}\left(n \log \frac{u_n}{u_{n+1}}\right) = l,$$

then the Series Converges for $l > 1$, and diverges for $l < 1$.

__Proof__: First, $l > 1$.

Let us Select $\epsilon > 0$, Such that $l - \epsilon > 1$

Let $l - \epsilon = \alpha > 1$.

Since $\lim_{n\to\infty}\left(n \log \frac{u_n}{u_{n+1}}\right) = l$, therefore there exists a positive number m such that

$$l - \epsilon < n \log \frac{u_n}{u_{n+1}} < l + \epsilon, \ \forall \ n \geq m$$

$$\Rightarrow \ n \log \frac{u_n}{u_{n+1}} > l - \epsilon = \alpha$$

$$\Rightarrow \ \log \frac{u_n}{u_{n+1}} > \alpha/n$$

$$\Rightarrow \frac{u_n}{u_{n+1}} > e^{\alpha/n}, \forall n \geq m$$

Now since $\{(1+\frac{1}{n})^n\}$ is a monotonic increasing sequence converging to e, therefore

$$\left(1+\frac{1}{n}\right)^n \leq e, \forall n$$

So that we get

$$\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)^\alpha = \frac{(n+1)^\alpha}{n^\alpha} = \frac{v_n}{v_{n+1}}, \forall n \geq m$$

where $v_n = \frac{1}{n^\alpha}$

But since for $\alpha > 1$, Σv_n converges, therefore by comparison test, Σu_n also converges.

Secondly, $l < 1$.

Let us select $\epsilon > 0$, such that $l + \epsilon < 1$.

Let $l + \epsilon = \beta < 1$

Since $\lim_{n\to\infty}\left(n \log \frac{u_n}{u_{n+1}}\right) = l$, therefore there exists a positive integer m_1 such that

$$n \log \frac{u_n}{u_{n+1}} < l + \epsilon = \beta$$

$$\Rightarrow \log \frac{u_n}{u_{n+1}} < \beta/n, \forall n \geq m_1$$

$$\Rightarrow \frac{u_n}{u_{n+1}} < e^{\beta/n}, \forall n \geq m_1$$

Now since $\{(1+\frac{1}{n-1})^n\}$ is a monotonic decreasing sequence converging to e, therefore

$$\left(1+\frac{1}{n-1}\right)^n > e, \forall n \geq 2$$

So that we get

$$\frac{u_n}{u_{n+1}} < \left(1+\frac{1}{n-1}\right)^\beta = \frac{n^\beta}{(n-1)^\beta} = \frac{v_n}{v_{n+1}}, \forall n$$

where $v_n = \frac{1}{(n-1)^\beta}$

But since for $\beta < 1$, Σv_n diverges, therefore by comparison test, Σu_n also diverges.

<u>Problem:</u> Show that for any fixed value of x, the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ is convergent.

<u>Solution:</u> Let $u_n = \frac{\sin nx}{n^2}$, so that $|u_n| = \frac{|\sin nx|}{n^2}$

Now $\left|\frac{\sin nx}{n^2}\right| \leq \frac{1}{n^2}, \forall n$ and $\Sigma \frac{1}{n^2}$ converges as it is a p-series with $p = 2 > 1$ and hence convergent.

Hence, by comparison test, the series $\sum_{n=1}^{\infty}\left|\frac{\sin nx}{n^2}\right|$ converges. Since every absolutely convergent series is convergent, therefore $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ is convergent.

2018 Cauchy's root test: If Σu_n is a positive term series, such that $\lim_{n\to\infty} (u_n)^{1/n} = l$, then the series

(i) Converges, if $l < 1$,

(ii) diverges, if $l > 1$, and

(iii) the test fails to give any definite information, if $l = 1$.

Proof:

Case-I: $l < 1$.

Let us select a positive number ϵ, such that $l + \epsilon < 1$.

Let $l + \epsilon = \alpha < 1$

Since $\lim_{n\to\infty} (u_n)^{1/n} = l$, therefore there exists a positive integer m such that

$$|(u_n)^{1/n} - l| < \epsilon, \forall n \geq m$$

$$\Rightarrow l - \epsilon < (u_n)^{1/n} < l + \epsilon, \forall n \geq m$$

$$\Rightarrow (l-\epsilon)^n < u_n < (l+\epsilon)^n = \alpha^n, \forall n \geq m$$

$$\Rightarrow u_n < \alpha^n, \forall n \geq m.$$

But since $\Sigma \alpha^n$ is a convergent geometric series (common ratio $\alpha < 1$), therefore, by comparison test, the series Σu_n converges.

Case-II: $l > 1$.

Let us select a positive number ϵ, such that $l - \epsilon > 1$

Let $l - \epsilon = \beta > 1$

Since $\lim_{n\to\infty} (u_n)^{1/n} = l$, therefore there exists a positive integer m_1 such that

$$l - \epsilon < (u_n)^{1/n} < l + \epsilon, n \geq m_1$$

$$\Rightarrow (l-\epsilon)^n < u_n < (l+\epsilon)^n, n \geq m_1$$

$$\Rightarrow u_n > (l-\epsilon)^n = \beta^n, n \geq m_1$$

But since $\Sigma \beta^n$ is a divergent geometric series (common ratio $\beta > 1$), therefore, by comparison test, the series Σu_n diverges.

Note: The test fails to give any definite information for $l = 1$.

Consider the two Series $\Sigma\left(\frac{1}{n}\right)$ and $\Sigma\left(\frac{1}{n^2}\right)$.

$\Sigma\left(\frac{1}{n}\right)$ diverges when $\lim_{n\to\infty}\left(\frac{1}{n}\right)^{1/n} = 1$, and $\Sigma\left(\frac{1}{n^2}\right)$ Converges when $\lim_{n\to\infty}\left(\frac{1}{n^2}\right)^{1/n} = 1$.

Cauchy's Integral Test: If u is a non-negative monotonic decreasing integrable function such that $u_n = u(n)$ for all positive integral values of n, then the Series $\sum_{n=1}^{\infty} u_n$ and $\int_1^{\infty} u(x)\,dx$ Converge or diverge together.

Proof. As u is a monotonic decreasing, we have

$$u(n) \geq u(x) \geq u(n+1), \text{ whenever } n \leq x \leq n+1$$

Also, Since u is non-negative and integrable,

$$\int_n^{n+1} u(n)\,dx \geq \int_n^{n+1} u(x)\,dx \geq \int_n^{n+1} u(n+1)\,dx$$

$$\Rightarrow u(n) \geq \int_n^{n+1} u(x)\,dx \geq u(n+1)$$

$$\text{or } u_n \geq \int_n^{n+1} u(x)\,dx \geq u_{n+1} \quad \ldots \quad \ldots (1)$$

Let us write $S_n = u_1 + u_2 + \ldots + u_n$ and

$I_n = \int_1^n u(x)\,dx$, and putting $n = 1, 2, \ldots (n-1)$, and adding, we get $u_1 + u_2 + \cdots + u_{n-1} \geq \int_1^2 u(x)dx + \int_2^3 u(x)dx + \cdots + \int_{n-1}^{n} u(x)dx \geq u_2 + u_3 + \cdots + u_n$

$$\text{i.e. } S_n - u_n \geq I_n \geq S_n - u_1$$

$$\Rightarrow 0 < u_n \leq S_n - I_n \leq u_1 \quad \ldots \quad \ldots (2)$$

Let us Consider the Sequence $\{(S_n - I_n)\}$,

$$(S_n - I_n) - (S_{n-1} - I_{n-1})$$
$$= S_n - S_{n-1} - (I_n - I_{n-1})$$
$$= u_n - \int_{n-1}^{n} u(x)\,dx$$
$$\leq 0 \quad [\text{using } (1)]$$

Therefore, the Sequence $\{(S_n - I_n)\}$ is monotone decreasing, bounded by 0 and u_1.

Hence, the sequence Converges and has a limit such that

$$0 \leq \lim(S_n - I_n) \leq u_1 \quad \ldots \quad \ldots \quad \ldots (3)$$

Thus the series Σu_n converges or diverges with the integral $\int_1^\infty u(x)\,dx$, if convergent, the sum of the series differs from the integral by less than u_1; if divergent, the limit of $(S_n - I_n)$ still exists and lie between 0 and u_1.

Problem: Show that the series $\Sigma\left(\frac{1}{n^p}\right)$ converges if $p>1$, and diverges if $p \le 1$.

Solution: Let $u(x) = \frac{1}{x^p}$, so that for $x \ge 1$, the function u is a non-negative monotonic decreasing integrable function such that

$$u_n = u(n) = \frac{1}{n^p}, \forall n \in \mathbb{N}$$

By Integral test, $\sum_{n=1}^{\infty} u_n$ and $\int_1^\infty u(x)\,dx$ converge or diverge together.

Let us now test the convergence of the infinite integral.

$$\because \int_1^X u(x)\,dx = \int_1^X \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{1-p}\left(X^{1-p}-1\right), & \text{if } p \ne 1 \\ \log X, & \text{if } p=1 \end{cases}$$

$$\therefore \int_1^\infty u(x)\,dx = \lim_{X\to\infty} \int_1^X u(x)\,dx = \begin{cases} \frac{1}{p-1}, & \text{if } p>1 \\ \infty, & \text{if } 0<p\le 1 \end{cases}$$

Thus $\int_1^\infty u(x)\,dx$ converges if $p>1$, and diverges if $0<p\le 1$.

Hence the infinite series $\Sigma\left(\frac{1}{n^p}\right)$ converges if $p>1$, and diverges if $0<p\le 1$.

But, when $p<0$, the series $\Sigma\left(\frac{1}{n^p}\right)$ diverges for then the nth term n^{-p} does not tend to zero as $n\to\infty$.

Hence the series $\Sigma\left(\frac{1}{n^p}\right)$ converges when $p>1$, and diverges for $p \le 1$.

Problem: The series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$, $p>0$ converges for $p>1$ and diverges for $p\le 1$.

Solution: Let $u(x) = \frac{1}{x(\log x)^p}$, so that for $x \ge 2$, the function u is a non-negative monotonic decreasing integrable function, such that

$$u_n = u(n) = \frac{1}{n(\log n)^p}, \forall p>0, n\in\mathbb{N}$$

By Integral test, $\sum_{n=2}^{\infty} u_n$ and $\int_2^\infty u(x)\,dx$ converge or diverge together.

Let us now test the Convergence of the infinite integral.

$$\because \int_2^x u(x)\,dx = \int_2^x \frac{1}{x(\log x)^p}\,dx, \quad p>0$$

$$= \begin{cases} \dfrac{(\log x)^{1-p} - (\log 2)^{1-p}}{1-p}, & \text{if } p \neq 1 \\ \log\log x - \log\log 2, & \text{if } p = 1 \end{cases}$$

$$\therefore \int_2^\infty u(x)\,dx = \lim_{x\to\infty} \int_2^x u(x)\,dx$$

$$= \begin{cases} \dfrac{(\log 2)^{1-p}}{p-1}, & \text{if } p>1 \\ \infty, & \text{if } 0<p\leq 1 \end{cases}$$

Thus $\int_2^\infty u(x)\,dx$ Converges if $p>1$, and diverges if $0<p\leq 1$.

Hence the Series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$, $p>0$, Converges if $p>1$, and diverges if $p\leq 1$.

Problem: Test for Convergence of the Series

$$1 + \frac{x}{1!} + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \cdots, \quad \text{for } x>0$$

Solution: Ignoring the first term,

$$u_n = \frac{n^n x^n}{n!} \quad \& \quad u_{n+1} = \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!}$$

Now,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n x = ex$$

By Ratio test, the Series converges for $ex<1$ or $x<\frac{1}{e}$ and diverge for $x>\frac{1}{e}$

For $x=\frac{1}{e}$, we have

$$\frac{u_n}{u_{n+1}} = \left(\frac{n}{n+1}\right)^n \cdot e$$

Then $\log\frac{u_n}{u_{n+1}} = \log\left(\frac{n}{n+1}\right)^n + \log_e e$

$= 1 + n\log\left\{\frac{1}{1+\frac{1}{n}}\right\}$

$= 1 + n\log\left(1+\frac{1}{n}\right)^{-1}$

$= 1 - n\log\left(1+\frac{1}{n}\right)$

$\therefore n\log\frac{u_n}{u_{n+1}} = n\left[1 - n\log\left(1+\frac{1}{n}\right)\right]$

Now,

$$\lim_{n\to\infty}\left\{n\log\frac{u_n}{u_{n+1}}\right\} = \lim_{n\to\infty} n\left[1 - n\log\left(1+\frac{1}{n}\right)\right]$$

$$= \lim_{n\to\infty} n\left[1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots\right)\right]$$

$$= \frac{1}{2} < 1$$

Therefore by logarithmic ratio test, the series diverges.

Hence the series converges for $x < \frac{1}{e}$ and diverges for $x \geq \frac{1}{e}$.

Problem: Test the convergence of the series whose general term is $\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$.

Solution: Let $u_n = \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{n^{3/2}}}$, then

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left\{\frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}\right\} = \frac{1}{e} < 1$$

Hence, the series converges.

Problem: Apply Cauchy's integral test or otherwise to show that the p-series $\sum \frac{1}{n^p}$ is convergent if $p>1$ and divergent if $p \leq 1$.

Solution: Here $u_n = \frac{1}{n^p}$

For $x \geq 1$ the function is non-negative monotonic non-increasing and integrable.

$$\therefore \int_1^X u(x)\,dx = \int_1^X x^{-p}\,dx = \left[\frac{x^{-p+1}}{-p+1}\right]_1^X$$

$$= \frac{1}{-p+1}\left[X^{-p+1} - 1\right]$$

$$= \frac{1}{1-p}\left[X^{1-p} - 1\right]$$

Case-I: $p > 1$

$$\therefore \int_1^\infty u(x)\,dx = \lim_{X\to\infty} \int_1^X u(x)\,dx = \frac{1}{1-p} \lim_{X\to\infty} \left[\frac{1}{X^{p-1}} - 1\right]$$

$$= \frac{1}{p-1}$$

$\therefore \int_1^\infty u(x)\,dx$ is convergent and $\sum \frac{1}{n^p}$ is convergent when $p>1$

<u>Case–II</u> : $p = 1$ the series becomes harmonic series $\sum \frac{1}{n}$ and hence divergent.

<u>Case–III</u> : $p<1$, $\int_1^\infty u(x)\,dx = \infty$ i.e. $\int_1^\infty u(x)\,dx$ is divergent

$\therefore \sum \frac{1}{n^p}$ is divergent.

Hence, $\sum \frac{1}{n^p}$ is convergent if $p>1$ and is divergent if $p \le 1$.

<u>Problem</u>: Applying Cauchy's integral test, Show that $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ is convergent if $p>1$ and is divergent if $p \le 1$.

<u>Solution</u>: Let the series be $\sum_{n=2}^{\infty} u_n$

$$\therefore u_n = \frac{1}{n(\log n)^p} = u(n) \ \forall\ n \geq 2$$

$$\therefore u(x) = \frac{1}{x(\log x)^p}, \ x \geq 2 \text{ and } x \in \mathbb{R}$$

The function is non-negative, non-increasing and integrable.

<u>Case–I</u> : Let $p>1$

$$\therefore \int_2^\infty u(x)\,dx = \lim_{X\to\infty} \int_2^X \frac{1}{x(\log x)^p}\,dx$$

$$= \lim_{X\to\infty} \left[\frac{(\log x)^{-p+1}}{-p+1}\right]_2^X$$

$$= \lim_{X\to\infty} \left[\frac{(\log X)^{-p+1}}{-p+1} - \frac{(\log 2)^{-p+1}}{-p+1}\right]$$

$$= \frac{1}{-p+1} \lim_{X\to\infty} \left[\frac{1}{(\log X)^{p-1}} - \frac{1}{(\log 2)^{p-1}}\right]$$

$$= \frac{1}{(p-1)(\log 2)^{p-1}}, \text{ which is finite}$$

$\therefore \int_2^\infty u(x)\,dx$ is convergent and so $\sum \frac{1}{n(\log n)^p}$ is convergent for $p>1$

Case–II ; Let $p<1$.

Then $\int_2^\infty u(x)\,dx = \infty$ as $\lim\limits_{x\to\infty} \dfrac{1}{(\log x)^{p-1}} = \infty$ and $-p+1>0$

$\therefore$ The integral diverges and so $\sum \frac{1}{n(\log n)^p}$ is divergent for $p<1$.

Case–III ; Let $p=1$

$$\therefore\ u(x) = \frac{1}{x\log x}$$

$$\therefore\ \int_2^\infty u(x)\,dx = \lim_{x\to\infty} \int_2^x \frac{dx}{x\log x}$$

$$= \lim_{x\to\infty} \left[\log(\log x)\right]_2^x$$

$$= \lim_{x\to\infty} \left[\log(\log x) - \log(\log 2)\right]$$

$$= \infty$$

$\therefore$ $\int_2^\infty u(x)\,dx$ is divergent and so $\sum \frac{1}{n(\log n)^p}$ is divergent for $p=1$

$\therefore$ the given series is convergent for $p>1$ and divergent for $p\le 1$.

Problem : Find whether the series

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3} + \cdots$$

is convergent or divergent.

Solution :

Here $u_n = \left\{\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right\}^{-n}$

So $u_n^{\frac{1}{n}} = \left\{\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right\}^{-1}$

$$= \left\{\left(\frac{n+1}{n}\right)^{n} - 1\right\}^{-1} \cdot \left(\frac{n+1}{n}\right)^{-1}$$

Therefore $\lim\limits_{n\to\infty} u_n^{1/n} = (e-1)^{-1}\cdot 1$

$$= \frac{1}{e-1} < 1, \text{ Since } e>2$$

Hence by Cauchy's root test, the series is convergent.

Problem: Show that the series, whose general term is given by $u_n = \left(1-\frac{1}{n}\right)^{n^2}$, is convergent.

Solution:

Here $u_n^{1/n} = \left\{\left(1-\frac{1}{n}\right)^{n^2}\right\}^{1/n}$

$$= \left(1-\frac{1}{n}\right)^n$$

So $\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = e^{-1} = \frac{1}{e} < 1$

Hence the series is convergent by Cauchy's root test.

Problem: Examine the convergency or divergency of the series

(i) $\frac{1^2}{2} + \frac{2^2}{2^2} + \frac{3^2}{2^3} + \frac{4^2}{2^4} + \cdots$

(ii) $1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \cdots + \frac{x^n}{n^2+1} + \cdots$

Solution: (i) Here $u_n = \frac{n^2}{2^n}$ and $u_{n+1} = \frac{(n+1)^2}{2^{n+1}}$

Now $\frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \frac{1}{2}\left(\frac{n+1}{n}\right)^2 = \frac{1}{2}\left(1+\frac{1}{n}\right)^2$

$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1}{2} < 1$

Hence by D' Alembert's ratio test, the given series is convergent.

(ii) From after the first term,

$$u_n = \frac{x^n}{n^2+1}, \quad u_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}$$

Then $\cancel{\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(n+1)^2+1}}$.

$$\frac{u_n}{u_{n+1}} = \frac{x^n}{n^2+1} \cdot \frac{(n+1)^2+1}{x^{n+1}}$$

$$= \frac{n^2+2n+1+1}{n^2+1} \cdot \frac{1}{x}$$

Therefore $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n^2+2n+2}{n^2+1} \cdot \frac{1}{x} = \frac{1}{x}$

Hence if $\frac{1}{x} > 1$ that is, if $x < 1$, then the series is convergent. If $\frac{1}{x} < 1$ that is, if $x > 1$ then the series is divergent, by D' Alembert's ratio test.

If $x = 1$, then $u_n = \frac{1}{n^2+1}$

Comparing the series with the convergent series for which $v_n = \frac{1}{n^2}$

We get $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$, finite quantity

Therefore, if $x=1$, then the series is convergent.

Thus the given series is convergent if $x \le 1$ and divergent if $x > 1$.

<u>Problem</u>: Test the convergent of the series

$$\frac{1}{(\log 2)^p} + \frac{1}{(\log 3)^p} + \cdots + \frac{1}{(\log n)^p} + \cdots$$

<u>Solution</u>: Here $u_n = \frac{1}{\{\log(n+1)\}^p}$, $u_{n+1} = \frac{1}{\{\log(n+2)\}^p}$

Now $\frac{u_n}{u_{n+1}} = \left\{\frac{\log(n+2)}{\log(n+1)}\right\}^p = \left\{\frac{\log(n+1) + \log\left(1+\frac{1}{n+1}\right)}{\log(n+1)}\right\}^p$

$$= \left[1 + \frac{1}{\log(n+1)}\left\{\frac{1}{n+1} - \frac{1}{2(n+1)^2} + \cdots\right\}\right]^p$$

$$= \left\{1 + \frac{1}{(n+1)\log(n+1)} - \frac{1}{2(n+1)^2\log(n+1)} + \cdots\right\}^p$$

Therefore $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1$ and test fails to give any definite information.

Then $\log \frac{u_n}{u_{n+1}} = p \log\left\{1 + \frac{1}{(n+1)\log(n+1)} - \frac{1}{2(n+1)^2\log(n+1)} + \cdots\right\}$

$$= p\left\{\frac{1}{(n+1)\log(n+1)} + \cdots\right\}$$

Therefore $\lim_{n\to\infty}\left\{n \log \frac{u_n}{u_{n+1}}\right\}$

$$= \lim_{n\to\infty} p\left\{\frac{n}{(n+1)\log(n+1)}\right\}$$

$= 0$ for all values of p

Hence the series is divergent for all values of p, by logarithmic ratio test.

Problem: Find whether the series are convergent or divergent

(i) $\frac{1}{2} + \frac{1}{2+1} + \frac{1}{2^2+1} + \frac{1}{2^3+1} + \cdots$

(ii) $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$

Theorem: If $\{u_n\}$ be a decreasing sequence of positive terms tending to zero, then the series

$$u_1 - \frac{1}{3}(u_1+u_3) + \frac{1}{5}(u_1+u_3+u_5) \cdots \cdots$$ is convergent.

Proof: Let the given series be

$$v_1 - v_2 + v_3 - v_4 + \cdots \cdots$$

where $v_n = \frac{1}{2n-1}(u_1 + u_3 + u_5 + \cdots \cdots + u_{2n-1})$

But $\lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{u_1+u_3+u_5+\cdots+u_{2n-1}}{2n-1}$

$= 0$ by Cauchy's first theorem on limit

$\therefore \lim_{n\to\infty} v_n = 0$

Since $u_1 > u_2 > u_3 > \cdots \cdots$

$$3u_1 = u_1 + u_1 + u_1 > u_1 + u_3$$

$$5(u_1+u_3) > 3(u_1+u_3+u_5)$$

$7(u_1+u_3+u_5) > 5(u_1+u_3+u_5+u_7)$ and so on

$\therefore u_1 > \frac{1}{3}(u_1+u_3) > \frac{1}{5}(u_1+u_3+u_5) > \cdots$ and so on.

i.e. $v_1 > v_2 > v_3 > \cdots \cdots \cdots$

So the given series is an alternating series with terms decreasing and ultimately tending to zero as a limit. Hence the given series is convergent.

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