Power Series and its Convergence

* Definition of Power Series:

A Series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

Where $a_0, a_1, a_2 \dots$ are positive or negative numbers and in which only the integral powers of x appear in each term is called a power Series of x,

or, more generally

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots + a_n(x-x_0)^n + \dots$$

where $a_0, a_1, a_2 \dots$ etc are positive or negative numbers and in which only the integral powers of x appear in each term is called a power Series of $(x-x_0)$.

* Interval of Convergence and Radius of Convergence of a Power Series:

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{R}, \frac{1}{R} \neq 0$

then the power Series $\sum a_n x^n$ converges absolutely and therefore converges, for all values of x within the interval $-R < x < R$ and diverges outside the interval.

Proof. By Ratio Test, we have

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n} \cdot x\right| = \frac{1}{R}|x|$$

Hence the Series converges absolutely

if $\frac{1}{R}|x| < 1$

i.e. if $|x| < R$

and diverges if $\frac{1}{R}|x| > 1$

i.e. if $|x| > R$.

Hence the power Series $\sum a_n x^n$ converges absolutely for all values of x within $-R < x < R$ and diverges outside it.

The interval $(-R, R)$ is called the interval of Convergence and R is called the radius of Convergence.

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Interval of Convergence:

If a power series $\sum a_n x^n$ converges for every $|x| < R$ (indeed absolutely), but diverges for $|x| > R$ then the interval $(-R, R)$ is called the interval of convergence of the power series and R is very often called the radius of convergence.

Convergence of Power Series:

A power series $\sum a_n x^n$ whose terms are functions of x is said to be convergent for $x = x_0$ if given any pre-assigned positive number ϵ however small, there exists a positive ~~number~~ integer m such that

$$|S(x_0) - S_n(x_0)| < \epsilon \text{ whenever } n \geq m$$

Where m depends on ϵ and $S(x)$ denoting the sum of the series.

This implies that the power series $\sum a_n x^n$ is convergent at $x = x_0$ if the sequence

$$\{a_n x_0^n\} \to 0 \text{ as } n \to \infty$$

$$\Rightarrow |a_n x_0^n| < \epsilon \text{ for } n \geq m$$

$\Rightarrow$ There exists a finite number M such that $|a_n x_0^n| < M$ for all values of n.

Uniform Convergence:

Let the power series $\sum a_n x^n$ converges for all values of x in $\alpha \leq x \leq \beta$.

Then the power series is said to be uniformly convergent in $\alpha \leq x \leq \beta$, if given any pre-assigned positive number ϵ, however small, there exists a positive number m such that for all $x \in (\alpha, \beta)$

$$|S(x) - S_n(x)| < \epsilon \text{ whenever } n \geq m$$

$$\text{i.e. } |R_n(x)| < \epsilon \text{ whenever } n \geq m$$

Where m depending on ϵ only and not on x.

So if x_1 and x_2 be any two values of x in the interval (α, β), then each of

$|R_n(x_1)|$ and $R_n(x_2) < \epsilon$ for $n \geq$ the same value of m.

* The Fundamental Theorem of Power Series:

If $\sum a_n x^n$ does not merely converge every... where, then a unique positive number ρ exists ... $\sum a_n x^n$ converges for $|x| < \rho$, and diverges for $|x| ...$

Proof: The nature of statement implies that there exists at least one value $x_1(\neq 0)$ of x such that $\sum a_n x_1^n$ converges and also there exists at least one value $y_1(\neq 0)$ of x such that $\sum a_n y_1^n$ diverges, so that $0 < x_1 < y_1$.

Let D_1 be the mode of the interval formed by the points x_1 and y_1 and d_1 be the length of this interval. Then $d_1 = y_1 - x_1$.

Now the power series $\sum a_n x^n$ may diverge or converge at the mid point of D_1.

If $\sum a_n x^n$ diverges at the mid point of D_1 at $x = y_2$ then we must have $x_1 \leq x_2 < y_2$ and converges at $x = x_2$.

If $\sum a_n x^n$ converges at the mid point of D_1 at $x = x_2$ and diverges at $x = y_2$, then we must have $x_2 < y_2 \leq y_1$.

Let D_2 be the mode of the interval formed by the points x_2 and y_2 in the above two cases and d_2 be the length of this new interval.

Then for D_2 we must have

$$x_1 \leq x_2 < y_2 \leq y_1$$

$$\text{and } d_2 = (y_2 - x_2)$$

Proceeding in this way, in the n^{th} stage we shall obtain the mode D_n formed by the points x_n and y_n at the end points of D_n of length d_n such that $d_n = y_n - x_n$ and

$$x_1 \leq x_2 \leq x_3 \cdots \leq x_n < y_n \leq \cdots \leq y_2 \leq y_1$$

Then clearly $Lt\, x_n$ and $Lt\, y_n$ exists and $Lt\, x_n = Lt\, y_n = \rho$

$$\Rightarrow x_n \leq \rho \leq y_n$$

Let x_0 be the value of x such that $|x_0| < \rho$, then by making n sufficiently large we can make $|x_0| < x_n$ so that

$$d_n = y_n - x_n \quad \text{i.e.,} < \rho - |x_0|$$

$$< y_n - |x_0|$$

$\Rightarrow x = x_0$ is the point of convergence.

Let y_0 be the value of x such that $|y_0| > \rho$, then by making n sufficiently large we can make $|y_0| > y_n$ so that

$$d_n = y_n - x_n \quad \text{i.e.} < |y_0| - \rho$$

$$< |y_0| - x_n$$

$\Rightarrow x = y_0$ is a point of divergence satisfying the inequality $d_n < \rho - |x_0|$ and $d_n < |y_0| - \rho$

Note: The number ρ is called the radius of convergence, and the interval within which the power series is convergent is called interval of convergence.

Theorem for the determination of radius of Convergence:

In a power Series $\sum a_n x^n$ if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \mu$ then $\sum a_n x^n$ converges for all x if $|x| < \rho$ and diverges for all x if $|x| > \rho$ where $\rho = \frac{1}{\mu}$ and $0 < \mu < \infty$.

Proof: Now
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right||x|$$
$$= \mu|x|$$
$$= \frac{|x|}{\rho}$$

So by D'Alembert's Test,

$\sum a_n x^n$ converges if $\frac{|x|}{\rho} < 1$ i.e. $|x| < \rho$ and diverges if $\frac{|x|}{\rho} > 1$ i.e. $|x| > \rho$.

No conclusion can however be drawn if $x = \pm\rho$.

Problem: Find the radius R of Convergence of the power Series $\frac{1}{2}x + \frac{1}{2}\cdot\frac{3}{5}x^2 + \frac{1}{2}\cdot\frac{3}{5}\cdot\frac{5}{8}x^3 + \frac{1}{2}\cdot\frac{3}{5}\cdot\frac{5}{8}\cdot\frac{7}{11}x^4 + \cdots$ and examine whether the Series converges when $x = R$.

Solution: Here
$$a_n = \frac{1\cdot3\cdot5\cdot8\cdots(2n-1)}{2\cdot5\cdot8\cdot11\cdots(3n-1)}$$
and
$$a_{n+1} = \frac{1\cdot3\cdot5\cdot8\cdots(2n-1)(2n+1)}{2\cdot5\cdot8\cdot11\cdots(3n-1)(3n+2)}$$

$$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2n+1}{3n+2}\right| = \lim_{n\to\infty}\left|\frac{2+\frac{1}{n}}{3+\frac{2}{n}}\right| = \frac{2}{3} = \frac{1}{R}$$

$$\therefore R = \frac{3}{2}$$

For $x = R = \frac{3}{2}$, the given Series becomes

$$\frac{1}{2}\left(\frac{3}{2}\right) + \frac{1}{2}\cdot\frac{3}{5}\cdot\left(\frac{3}{2}\right)^2 + \frac{1}{2}\cdot\frac{3}{5}\cdot\frac{5}{8}\left(\frac{3}{2}\right)^3 + \cdots\cdots$$

So that
$$u_n = \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot5\cdot8\cdots(3n-1)}\left(\frac{3}{2}\right)^n$$
and
$$u_{n+1} = \frac{1\cdot3\cdot5\cdots(2n-1)(2n+1)}{2\cdot5\cdot8\cdots(3n-1)(3n+2)}\left(\frac{3}{2}\right)^{n+1}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{3n+2}{2n+1}\cdot\frac{2}{3} \quad \text{or} \quad \frac{u_n}{u_{n+1}} - 1 = \frac{6n+4}{6n+3} - 1 = \frac{1}{6n+3}$$
$$= \frac{1}{n\left(6+\frac{3}{n}\right)}$$

$$\therefore \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty}\frac{n}{n\left(6+\frac{3}{n}\right)} = \lim_{n\to\infty}\frac{1}{6+\frac{3}{n}} = \frac{1}{6} < 1$$

$\therefore$ By Raabe's Test the Series is divergent when $x = \frac{3}{2}$

Other Theorems on Power Series

Theorem-1: If the power series
$$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \dots$$
Converges for a particular value x_0 of x it converges absolutely for all values of x such that $|x| < |x_0|$

Proof: Since the series converges for $x = x_0$, the series $a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$ converges. Then there exists a positive number ϵ such that
$$|a_n x^n| < \epsilon \text{ whenever } n \geq m.$$
So we can write
$$|a_n x^n| < M \text{ for all } n$$
where M is a definite finite number.

Now since $|x| < |x_0|$
$$\Rightarrow \left|\frac{x}{x_0}\right| < 1$$

$\therefore$ $|a_0| + |a_1 x| + |a_2 x^2| + \dots + |a_n x^n| + \dots$
$$= |a_0| + |a_1 x_0| \left|\frac{x}{x_0}\right| + |a_1 x_0^2| \left|\frac{x}{x_0}\right|^2 + \dots + |a_n x_0^n| \left|\frac{x}{x_0}\right|^n + \dots$$
$$< M\left\{1 + \left|\frac{x}{x_0}\right| + \left|\frac{x}{x_0}\right|^2 + \dots + \left|\frac{x}{x_0}\right|^n + \dots\right\}$$
i.e. < a convergent series.

So $|a_0| + |a_1 x| + |a_2 x^2| + \dots + |a_n x^n| + \dots$ is convergent.

Hence the power series converges absolutely for all values of x such that $|x| < |x_0|$.

*__Theorem-2__: If $\sum_{n=0}^{\infty} a_n$ is convergent, then show that the power series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent when $|x| < 1$

Proof: Since $\sum_{n=0}^{\infty} a_n$ is convergent, the series
$$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$
i.e. $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = 1$.

Hence by theorem-1, the series $\sum a_n x^n$ is absolutely convergent when $|x| < 1$.

Theorem-3. A power series is absolutely and uniformly convergent in a range strictly within the interval of convergence $(-R, R)$

or

If R be the radius of convergence of a power series $\sum a_n x^n$, then the series is absolutely and uniformly convergent in $(-\eta, \eta)$ where η is any number $< R$.

Proof. Let x_0 be a number such that $\eta < x_0 < R$, then by theorem-1, $\sum a_n x^n$ is absolutely convergent.

$\therefore |a_n x_0^n| + |a_{n+1} x_0^{n+1}| + \cdots\cdots < \epsilon$ when $n \geq m$

So if x be any point in $(-\eta, \eta)$ i.e. $-\eta \leq x \leq \eta$

$$|a_n x^n| + |a_{n+1} x^{n+1}| + \cdots\cdots$$

$$< |a_n x_0^n| + |a_{n+1} x_0^{n+1}| + \cdots\cdots$$

$< \epsilon$ when $n \geq m$ and for every x such that $-\eta \leq x \leq \eta$.

i.e. $|x| < \eta$

$\therefore$ It follows that the series is uniformly convergent in the closed interval $(-\eta, \eta)$ i.e. in every interval $|x| \leq \eta$ where η is any positive number $< |x_0|$.

Theorem-4: A power series represents a function which is continuous within the interval of convergence.

or

For a power series, the sum function is continuous within the interval of convergence.

Proof. We know that a power series is a series of continuous functions which converges uniformly within the interval of convergence.

Let $S(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$

$$= \sum_{n=0}^{\infty} a_n x^n \quad \cdots \quad \cdots \quad \cdots \quad (1)$$

$$S_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

$$= \sum_{n=0}^{n-1} a_n x^n \quad \cdots \quad \cdots \quad \cdots \quad (2)$$

$$R_n(x) = a_n x^n + a_{n+1} x^{n+1} + \cdots\cdots$$

$$= \sum_{n=n}^{\infty} a_n x^n \quad \cdots \quad \cdots \quad \cdots \quad (3)$$

Since (1) Converges uniformly, for a given positive number ϵ however small it may be there exists a positive integer m such that

$$|S(x) - S_n(x)| < \epsilon/3, \text{ whenever } n \geq m,$$

the same m serving all values of x within the interval of Convergence.

Choosing such value of n

$$S(x) = S_n(x) + R_n(x)$$

where $|R_n(x)| < \epsilon/3$ for all x within the interval of Convergence.

Now $S_n(x)$ being the sum of a finite number n of continuous functions, is continuous within the interval.

$\therefore$ There is a positive number δ such that x' and x being any two values of x within the interval,

$$|S_n(x') - S_n(x)| < \epsilon/3 \quad \text{whenever } |x'-x| \leq \delta$$

Now $S(x) = S_n(x) + R_n(x)$ where $|R_n(x)| < \epsilon/3$

and $S(x') = S_n(x') + R_n(x')$ where $|R_n(x')| < \epsilon/3$

$$\therefore\ S(x') - S(x) = S_n(x') - S_n(x) + R_n(x') - R_n(x)$$

or $$|S(x') - S(x)| = |S_n(x') - S_n(x) + R_n(x') - R_n(x)|$$

$$\leq |S_n(x') - S_n(x)| + |R_n(x')| + |R_n(x)|$$

$$< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$$

i.e. $< \epsilon$

Hence $S(x)$ is continuous within the interval of Convergence.

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<u>Theorem-5:</u> A power series can be integrated term by term within the interval of Convergence.

<u>Proof:</u> Let $S(x) = a_0 + a_1x + a_2x^2 + \dots \dots + a_nx^n + \dots \dots$

$$= \sum_{n=0}^{\infty} a_nx^n$$

and $S_n(x) = a_0 + a_1x + a_2x^2 + \dots \dots + a_{n-1}x^{n-1}$

$$= \sum_{n=0}^{n-1} a_nx^n$$

$$R_n(x) = a_n x^n + a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \cdots$$
$$= \sum_{n}^{\infty} a_n x^n$$

Then $S(x) = S_n(x) + R_n(x)$

Since a power series is uniformly convergent within the interval of convergence, for any x within the interval

$$|R_n(x)| < \epsilon \text{ for } n \geq m$$

Now if c_1 and c_2 $(c_1 < c_2)$ be two points within the interval

$$\left|\int_{c_1}^{c_2} R_n(x)\,dx\right| \leq \int_{c_1}^{c_2} |R_n(x)|\,dx < \int_{c_1}^{c_2} \epsilon\,dx = \epsilon(c_2 - c_1) \quad \text{--- (1)}$$

Now $$\int_{c_1}^{c_2} S(x)\,dx = \int_{c_1}^{c_2} S_n(x)\,dx + \int_{c_1}^{c_2} R_n(x)\,dx$$

$$\therefore \left|\int_{c_1}^{c_2} S(x)\,dx - \int_{c_1}^{c_2} S_n(x)\,dx\right| = \left|\int_{c_1}^{c_2} R_n(x)\,dx\right|$$

$$\leq \int_{c_1}^{c_2} |R_n(x)|\,dx$$

Hence $\int_{c_1}^{c_2} S(x)\,dx$ $\qquad < \epsilon(c_2 - c_1)$ by (1)

$$= \lim_{n\to\infty} \int_{c_1}^{c_2} S_n(x)\,dx$$

$$= \int_{c_1}^{c_2} a_0\,dx + \int_{c_1}^{c_2} a_1 x\,dx + \int_{c_1}^{c_2} a_2 x^2\,dx + \cdots$$

Hence the theorem is proved.

Theorem-6: A power series can be differentiated term by term within the interval of convergence.

Proof: Let $S(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$

$$= \sum_{n=0}^{\infty} a_n x^n \cdots \qquad \cdots \qquad \cdots \text{ (1)}$$

Let us now form a new series $\varphi(x)$ by differentiating $S(x)$ term by term,

i.e. $$\varphi(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + (n+1)a_{n+1}x^n + \cdots$$

$$= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \cdots \qquad \cdots \text{ (2)} \qquad \cdots$$

As (1) and (2) are both power series, they are both uniformly convergent within their respective intervals of convergence.

We shall now prove that (2) has the same interval of convergence as (1).

If R be the radius of Convergence of (1), then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{R}$$

for the Series (2)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\,a_{n+1}}{n\,a_n}\right|$$

$$= \lim_{n\to\infty}\left|1+\frac{1}{n}\right| \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

$$= 1\cdot\frac{1}{R} = \frac{1}{R}$$

Thus the derived Series $\varphi(x)$ has the Same interval of Convergence as the original Series $S(x)$.

Now $\varphi(x)$ being a power Series, it can be integrated term by term within the interval of Convergence $(-R,R)$.

If x be a point of this interval,

$$\int_0^x \varphi(x)\,dx = \sum \int_0^x (n+1)\,a_{n+1}\,x^n\,dx$$

$$= \sum a_{n+1}\,x^{n+1}$$

Again $\varphi(x)$ being the Sum of the power Series (2) is a Continuous function of x in $(-R,R)$ and by the property of definite integrals

$$S'(x) = \varphi(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots\cdots + (n+1)a_{n+1}x^n + \cdots$$

Hence the theorem is proved.

* <u>Theorem-7</u>: If $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ for $|x|<1$, then

$$f'(x) = \sum_{n=1}^{\infty} n\,a_n x^{n-1} \text{ for } |x|<1$$

<u>Proof</u>: The Series $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$

$$= a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \quad (1)$$

is given to be Convergent within $(-1,1)$.

So the derived Series

$$a_1 + 2a_2x + 3a_3x^2 + \cdots\cdots + n\,a_n x^{n-1} + \cdots$$

i.e. $$\sum_{n=1}^{\infty} n\,a_n x^{n-1} \cdots\cdots\cdots (2)$$

is also Convergent within $(-1,1)$.

So we get

$$\left|\sum n(n-1)\,a_n x^{n-2}\right| \le \sum\left|n(n-1)\,a_n x^{n-2}\right|$$

whenever $|x|<R$

If now $f(x)$ and $g(x)$ be the sum of the ①, ② respectively within $(-1,1)$ then

$$f(x) = \sum a_n x^n$$

$$\text{and } g(x) = \sum n a_n x^{n-1}$$

So that by Mean value Theorem

$$\frac{f(x+h)-f(x)}{h} = \sum a_n \frac{(x+h)^n - x^n}{h}$$

$$= \sum a_n n \xi_1^{n-1} \quad \text{for } x < \xi_1 < x+h$$

$$\therefore \left| \frac{f(x+h)-f(x)}{h} - g(x) \right| = \left| \sum a_n n \xi_1^{n-1} - g(x) \right|$$

$$= \left| \sum a_n n \xi_1^{n-1} - \sum n a_n x^{n-1} \right|$$

$$= \sum n a_n \left| \xi_1^{n-1} - x^{n-1} \right|$$

$$= \sum n(n-1) a_n (\xi_1 - x) \xi_2^{n-2} \text{ (by M.V.T.)}$$

So if $|\xi_2| < x$ and $|\xi_1 - x| < h$, for $x < \xi_2 < \xi_1$

$$\left| \frac{f(x+h)-f(x)}{h} - g(x) \right| \leq |h| \left| \sum n(n-1) a_n x^{n-2} \right|$$

ie, $\leq |h|$, a bounded quantity.

Hence proceeding to the limit as $h \to 0$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = g(x)$$

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$$= \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{for } |x| < 1.$$

Theorem-9: Prove that the representation of a function $f(x)$ as a power series, whenever possible, is unique.

Proof: We shall show that there can not be two different power series representing the same function in an interval $(-R, R)$.

If possible let the two series

$$\sum_{n=0}^{\infty} a_n x^n \text{ and } \sum_{n=0}^{\infty} b_n x^n$$

represent the same function $f(x)$ in $|x| < R$.

Then both the series are convergent and have the same sum $f(x)$.

$$\therefore a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n + \dots$$

$$= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \dots + b_n x^n + \dots \quad ①$$

Since term by term differentiation is possible in a power series, we get after differentiating both sides successively

$a_1 + 2a_2x + 3a_3x^2 + \dots + na_nx^{n-1} + \dots$

$= b_1 + 2b_2x + 3b_3x^2 + \dots + nb_nx^{n-1} + \dots$ (2)

$2a_2 + 3\cdot 2a_3x + \dots + n(n-1)x^{n-2} + \dots$

$= 2b_2 + 3\cdot 2b_3x + \dots + n(n-1)x^{n-2} + \dots$ (3)

Putting $x = 0$ in (1), (2) and (3), we get

$a_0 = b_0;\ a_1 = b_1;\ a_2 = b_2;\ \dots\ a_n = b_n.$

** $\therefore$ The two series $\sum a_nx^n$ and $\sum b_nx^n$ are unique.

Theorem-9: If one and the same function $f(x)$ can be expanded in a power series in two different way so that

$$f(x) = \sum_{n=0}^{\infty} a_nx^n = \sum_{n=0} b_nx^n$$

both series being convergent when $|x| < R$, then show that $a_n = b_n$ for all values of n.

Proof: Given $f(x) = \sum_{n=0}^{\infty} a_nx^n = \sum_{n=0}^{\infty} b_nx^n$ for $|x| < R$.

We know that a power series can be differentiate term by term within the interval of convergence.

Hence differentiating $f(x)$ term by term successively for x within the radius of convergence i.e. in $|x| < R$, we get

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + \dots + na_nx^{n-1} + \dots$$
$$= b_1 + 2b_2x + 3b_3x^2 + \dots + nb_nx^{n-1} + \dots$$

$$f''(x) = 2a_2 + 3\cdot 2a_3x + \dots + n(n-1)a_nx^{n-2} + \dots$$

$$= 2b_2 + 3\cdot 2b_3x + \dots + n(n-1)b_nx^{n-2} + \dots$$

and so on for $f^n(x)$

Now putting $x = 0$ in the above relations we get

$a_1 = b_1;\ a_2 = b_2;\ \dots\dots\ a_n = b_n$

Hence $a_n = b_n$ for all values of n.

Problem: Determine the interval of convergence of the following power series $\sum a_n x^n$ where a_n is given by

(i) $a_n = \frac{1}{\log(n+1)}$ (ii) $a_n = (-1)^{n-1}\frac{1}{2n-1}$ (iii) $a_n = n^n$

** (iv) $a_n = \{\sqrt[n]{n} - 1\}^n$

Solution: (i) Here $a_n = \frac{1}{\log(n+1)}$

Now $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{\log(n+1)}}{\frac{1}{\log(n+2)}}\right| = \lim_{n\to\infty}\left|\frac{\log(n+2)}{\log(n+1)}\right|$

$= \lim_{n\to\infty}\left|\frac{\log n\left(1+\frac{2}{n}\right)}{\log n\left(1+\frac{1}{n}\right)}\right|$

$= \lim_{n\to\infty}\left|\frac{\log n}{\log n}\right| = 1$

Hence the interval of convergence is $-1<x<1$.

(ii) $a_n = (-1)^{n-1}\cdot\frac{1}{2n-1}$

Here $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n-1}\cdot\frac{1}{2n-1}}{(-1)^n\cdot\frac{1}{2n+1}}\right|$

$= \lim_{n\to\infty}\left|\frac{2n+1}{2n-1}\right| = \lim_{n\to\infty}\left|\frac{2+\frac{1}{n}}{2-\frac{1}{n}}\right| = 1$

Hence the interval of convergence is $-1<x<1$.

(iii) $a_n = n^n$

Here $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n^n}{(n+1)^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n^n}{n^{n+1}\left(1+\frac{1}{n}\right)^{n+1}}\right|$

$= \lim_{n\to\infty}\left|\frac{1}{n\left(1+\frac{1}{n}\right)}\right| = \lim_{n\to\infty}\left|\frac{1/n}{1+\frac{1}{n}}\right| = 0$

$\therefore$ The test fails.

Again $\lim_{n\to\infty}|a_n|^{1/n} = \lim_{n\to\infty}|n| = \infty$

$\therefore$ The given power series never converges.

(iv) $a_n = (\sqrt[n]{n} - 1)^n$

Here $\lim_{n\to\infty}|a_n|^{1/n} = \lim_{n\to\infty}|\sqrt[n]{n}-1| = \lim_{n\to\infty}|n^{1/n}-1|$

$= \lim_{n\to\infty} n^{\frac{1}{n}} - 1 = 1-1=0$

$\therefore$ The power series converges for all values of x, $-\infty < x < \infty$.

Problem: Show that the power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$ have the same radius of convergence.

Solution: Let R and R' be the radii of convergence of the first and second series respectively.

Then for the first series

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{R}$$

For the second series

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = a_1 + \sum_{n=1}^{\infty} (n+1) a_{n+1} x^n$$

$$= a_1 + \sum_{n=1}^{\infty} b_n x^n$$

where $b_n = (n+1) a_{n+1}$

$\therefore$ For radius of convergence of $\sum_{n=1}^{\infty} b_n x^n$

$$\lim_{n\to\infty} \left|\frac{b_n}{b_{n-1}}\right| = \frac{1}{R'}$$

$$\therefore \frac{1}{R'} = \lim_{n\to\infty} \left|\frac{(n+1) a_{n+1}}{n\, a_n}\right|$$

$$= \lim_{n\to\infty} \left|\left(1+\frac{1}{n}\right) \frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{R}$$

$$\therefore R' = R$$

$\therefore$ Radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=1}^{\infty} b_n x^n$ are the same.

But the radius of convergence of $\sum_{n=1}^{\infty} b_n x^n$ is the same as the radius of convergence of $\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$.

Hence the radius of convergence of the two given series is the same.

* **Problem:** Show by an example that a differentiation is not always permissible for power series in general.

Solution: Let us consider the power series

$$f(x) = 1 + \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{3}}x^2 + \frac{1}{\sqrt{4}}x^3 + \cdots$$

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Here $\underset{n\to\infty}{Lt}\left|\frac{u_n}{u_{n+1}}\right| = \underset{n\to\infty}{Lt}\left|\frac{\frac{x^{n-1}}{\sqrt{n}}}{\frac{x^n}{\sqrt{n+1}}}\right|$

$$= \underset{n\to\infty}{Lt}\left|\frac{1}{x}\sqrt{1+\frac{1}{n}}\right| = \frac{1}{x}$$

$\therefore$ The Series is convergent if $|x|<1$

$\therefore$ The derived Series $f'(x)$ will also represent a power Series within the same radius of convergence i.e. for $|x|<1$.

Now $f'(x) = \frac{1}{\sqrt{2}} + \frac{2x}{\sqrt{3}} + \frac{3x^2}{\sqrt{4}} + \cdots$

$$\therefore f'(2) = \frac{1}{\sqrt{2}} + \frac{2.2}{\sqrt{3}} + \frac{3.4}{\sqrt{4}} + \cdots$$

which is divergent series.

This shows that a divergent Series has a sum $f'(2)$ which is impossible.

$\therefore$ $f'(2)$ does not exist.

Absurdity is due to the fact that 2 lies outside the radius of Convergence and term by term differentiation can not be possible for $x=2$ which is a point outside the radius of Convergence.

Problem: Prove that the Series $1-x+x^2-x^3+\cdots$ can be integrated term by term over the interval $(0,1)$ and hence deduce that

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots = \log 2$$

Solution: The Series $1-x+x^2-x^3+\cdots$ is a geometric Series of common ratio $-x$.

$\because$ The G.P. Series converges for every x between -1 and $+1$, the Series

$1-x+x^2-x^3+\cdots$ can be integrated term by term within $(-1,1)$ i.e., within $(0,1)$

But $1-x+x^2-x^3+\cdots = (1+x)^{-1} = \frac{1}{1+x}$

Integrating, term by term within $(0,1)$ we get

$$\left[x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right]_0^1 = \left[\log(1+x)\right]_0^1$$

i.e., $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots = \log 2$

Hence the results.

Problem**: From the equation $\sin^{-1}x = \int_0^x \frac{dx}{\sqrt{1-x^2}}$, obtain power Series for $\sin^{-1}x$

Solution: Now

$$\sin^{-1}x = \int_0^x \frac{dx}{\sqrt{1-x^2}} = \int_0^x (1-x^2)^{-\frac{1}{2}}dx$$

$$= \int_0^x \left(1+\frac{1}{2}x^2+\frac{1}{2}\cdot\frac{3}{4}x^4+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}x^6+\cdots\right)dx$$

The integrand being a Binomial expansion for negative index is valid for $-1<x<1$.

$\therefore$ The radius of Convergence of the power Series is $|x|<1$.

$\therefore$ The Series can be integrated term by term with this radius of Convergence.

$\therefore$ Integrating Term by Term between 0 and x

$$\sin^{-1}x = x+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{x^7}{7}+\cdots \quad \text{when } |x|<1$$

But the Series is also convergent at $x=\pm1$.

**Hence the Series is valid for $|x|\le1$.

Problem: Show that the Series

$$1+\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{5}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots$$

is convergent and converges to $\pi/2$ by using the theorem of term by term integration of the power Series expansion of the expression $(1-x^2)^{-\frac{1}{2}}$

Solution: Now $(1-x^2)^{-\frac{1}{2}} = 1+\frac{1}{2}x^2+\frac{1}{2}\cdot\frac{3}{4}x^4+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}x^6+\cdots$

This being a Binomial expansion for negative index is valid for $-1\le x\le1$.

$\therefore$ The radius of Convergence of the power Series is $|x|\le1$.

So term by term integration between $(0,1)$ is possible

$\therefore$ Integrating between $(0,1)$

$$\int_0^1\left(1+\frac{1}{2}x^2+\frac{1}{2}\cdot\frac{3}{4}x^4+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot x^6+\cdots\right)dx = \int_0^1\frac{dx}{\sqrt{1-x^2}}$$

or $1+\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{5}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{7}+\cdots = \left[\sin^{-1}x\right]_0^1$

Hence the Series Converges to $\pi/2$. $= \pi/2$

** Cauchy - Hadamard Theorem: Every P.S. $\sum_{n=0}^{\infty} a_n x^n$ being a radius of Convergence R, $0 \le R \le \infty$ has the following properties.

① The Series converges absolutely for $|x| < R$.

② The Series Converges uniformly for $|x| < P$, where $0 \le P \le R$.

③ The Series diverges for $|x| > R$ i.e, the terms of the Series are unbounded.

Proof: ① We define by

$$\frac{1}{R} = \overline{\lim_{n\to\infty}} |a_n|^{1/n} \quad \dots \dots \quad (1)$$

We choose a real number P such that $|x| < P < R$.
Then $\frac{1}{P} > \frac{1}{R}$.
Then from (1), there exists a positive integer m such that

$$|a_n|^{1/n} < \frac{1}{P}, \forall n \ge m$$

$$\Rightarrow |a_n| < \frac{1}{P^n}, \forall n \ge m$$

$$\Rightarrow |a_n x^n| < \left|\frac{x}{P}\right|^n = \left\{\frac{|x|}{P}\right\}^n \quad [P > 0]$$

Now $\sum_{n=0}^{\infty} \left(\frac{|x|}{P}\right)^n$ is a G.P. Series with common ratio<1 $\left[\text{as } |x| < P \Rightarrow \frac{|x|}{P} < 1\right]$ and so $\sum_{n=0}^{\infty}\left(\frac{|x|}{P}\right)^n$ Converges.
Therefore, the given Series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely.

② Let $0 \le P < R$. We have to show that the Series Converges uniformly for $|x| \le P$. We choose real number P' such that $P < P' < R$. It follows from case (1), that

$$|a_n| < \frac{1}{(P')^n}, \forall n \ge m$$

$$\Rightarrow |a_n x^n| < \left(\frac{|x|}{P'}\right)^n \le \left(\frac{P}{P'}\right)^n \text{ as } |x| \le P.$$

Now the Series $\sum\left(\frac{P}{P_1}\right)^n$ of positive constants is Convergent as being a G.P. Series with common ratio < 1.
Therefore, by Weierstrass's test, the P.S. $\sum_{n=0}^{\infty} a_n x^n$ Converges absolutely for $|x| < R$ and hence it converges uniformly for $|x| \le P < R$.

③ Let $|x| > R$. We choose P such that $R < P < |x|$.
Then $\frac{1}{R} > \frac{1}{P}$.
Now from $\frac{1}{R} = \overline{\lim_{n\to\infty}} |a_n|^{1/n}$,
We have $|a_n|^{1/n} > \frac{1}{P}, \forall n \ge m$

$$= \frac{1}{R}\cdot R = 1$$
$$\Rightarrow R' = 1.$$

So let $\sum_{n=0}^{\infty} a_n x^n$ be a power Series with radius of Convergence 1.

Thus we have $f(x) = \sum a_n x^n$ for $-1 < x < 1$.

Also $f(1) = \sum_{n=0}^{\infty} a_n$

Now $\frac{1}{1-x} f(x) = \frac{1}{1-x}(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots)$

$$= (1 + x + x^2 + \dots + x^n + \dots)(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots)$$

$$= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \dots \dots$$

$$= \sum_{n=0}^{\infty} c_n x^n \text{ where } c_n = a_0 + a_1 + a_2 + \dots + a_n$$
$$= \sum_{0}^{n} a_k$$

$$\Rightarrow f(x) = (1-x)\sum_{n=0}^{\infty} c_n x^n$$

$$\Rightarrow f(x) - f(1) = (1-x)\sum_{n=0}^{\infty} c_n x^n - f(1) \text{ --- } (1)$$

Also $(1-x)\sum_{n=0}^{\infty} x^n = 1$ for $-1 < x < 1$ --- (2)

Since $(1-x)\lim_{n\to\infty} \frac{1-x^n}{1-x} = (1-x)\cdot\frac{1}{(1-x)} = 1$ as $\lim_{n\to\infty} x^n = 0$

Then using this result, (1) can be written as

$$f(x) - f(1) = (1-x)\sum_{n=0}^{\infty} c_n x^n - f(1)(1-x)\sum_{0}^{\infty} x^n$$
$$= (1-x)\sum^{\infty}\{c_n - f(1)\}x^n \text{ for } -1 < x < 1 \text{ --- } (3)$$

Now $c_n = \sum_{k=0}^{n} a_k \to f(1)$ as $n \to \infty$

$$\Rightarrow c_n - f(1) \longrightarrow 0 \text{ as } n \to \infty$$
$$\Rightarrow |c_n - f(1)| < \epsilon/2, \ \forall n \geq m \text{ --- } (4)$$

From eqn. (3),

$$|f(x) - f(1)| = \left|(1-x)\sum_{n=0}^{m}\{c_n - f(1)\}x^n + (1-x)\sum_{n=m+1}^{\infty}\{c_n - f(1)\}x^n\right|$$
$$\leq (1-x)\sum_{n=0}^{m-1}|c_n - f(1)||x^n| + (1-x)\sum_{m+1}^{\infty}|c_n - f(1)|x^n$$
$$\leq (1-x)\sum_{0}^{m-1}|c_n - f(1)|x^n + \epsilon/2\cdot 1 \quad [\text{by } (2) \ \& \ (4)] \text{ --- } (5)$$

So for a fixed m,

$$(1-x)\sum_{n=0}^{m-1}|c_n - f(1)|x^n$$
$$= (1-x)[|c_0 - f(1)| + |c_1 - f(1)|x + |c_2 - f(1)|x^2 + \dots + |c_{m-1} - f(1)|x^{m-1}]$$

$= \frac{1}{R} \cdot R = 1$

$\Rightarrow R' = 1.$

So let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence 1.

Thus we have $f(x) = \sum a_n x^n$ for $-1 < x < 1$.

Also $f(1) = \sum_{n=0}^{\infty} a_n$

Now $\frac{1}{1-x} f(x) = \frac{1}{1-x}(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots)$

$= (1 + x + x^2 + \dots + x^n + \dots)(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots)$

$= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \dots \dots$

$= \sum_{n=0}^{\infty} C_n x^n$ where $C_n = a_0 + a_1 + a_2 + \dots + a_n$

$= \sum_{0}^{n} a_k$

$\Rightarrow f(x) = (1-x) \sum_{n=0}^{\infty} C_n x^n$

$\Rightarrow f(x) - f(1) = (1-x) \sum_{n=0}^{\infty} C_n x^n - f(1)$ —— (1)

Also $(1-x) \sum_{n=0}^{\infty} x^n = 1$ for $-1 < x < 1$ —— (2)

Since $(1-x) \lim_{n\to\infty} \frac{1-x^n}{1-x} = (1-x) \cdot \frac{1}{(1-x)} = 1$ as $\lim_{n\to\infty} x^n = 0$

Then using this result, (1) can be written as

$f(x) - f(1) = (1-x) \sum_{n=0}^{\infty} C_n x^n - f(1)(1-x) \sum_{0}^{\infty} x^n$

$= (1-x) \sum_{0}^{\infty} \{C_n - f(1)\} x^n$ for $-1 < x < 1$ —— (3)

Now $C_n = \sum_{k=0}^{n} a_k \to f(1)$ as $n \to \infty$

$\Rightarrow C_n - f(1) \longrightarrow 0$ as $n \to \infty$

$\Rightarrow |C_n - f(1)| < \epsilon/2, \ \forall n \geq m$ —— (4)

From eqn. (3),

$|f(x) - f(1)| = \left|(1-x) \sum_{n=0}^{m} \{C_n - f(1)\} x^n + (1-x) \sum_{n=m+1}^{\infty} \{C_n - f(1)\} x^n\right|$

$\leq (1-x) \sum_{n=0}^{m-1} |C_n - f(1)| x^n + (1-x) \sum_{m+1}^{\infty} |C_n - f(1)| x^n$

$\leq (1-x) \sum_{0}^{m-1} |C_n - f(1)| x^n + \epsilon/2 \cdot 1$ [by (2) & (4)] —— (5)

So for a fixed m,

$(1-x) \sum_{n=0}^{m-1} |C_n - f(1)| x^n$

$= (1-x)[|C_0 - f(1)| + |C_1 - f(1)| x + |C_2 - f(1)| x^2 + \dots + |C_{m-1} - f(1)| x^{m-1}]$

is a positive continuous function of x and takes its value at $x=1$.

$$\Rightarrow |(1-x)| \sum_{n=0}^{n-1} |c_n - f(1)| x^n < \epsilon/2, \forall x \in (c-\delta, 1)$$

$$\Rightarrow (1-x) \sum_{0}^{n-1} |c_n - f(1)| x^n < \epsilon/2, \forall x \in (c-\delta, 1)$$

Thus from (5),

$$|f(x) - f(1)| < \epsilon/2 + \epsilon/2 = \epsilon, \forall x \in (c-\delta, 1)$$

$$\Rightarrow \lim_{x \to 1-0} f(x) = f(1) = \sum_{0}^{\infty} a_n$$

Abel's Theorem (First Form):

If a power series $\sum a_n x^n$ has a finite non zero radius of convergence R and if it converges at $x=R$, then it converges uniformly in $0 \le x \le R$. If again it converges at $x=-R$, then it converges uniformly in $-R \le x \le 0$.

Proof: There is no loss of generality in considering the radius of convergence of the given series to be unity. Since a substitution of $x = Ry$ carries $\sum a_n x^n$ into $\sum a_n R^n y^n = \sum b_n y^n$ where $b_n = a_n R^n$ and the series $\sum b_n y^n$ has a radius of convergence equal to 1. Hence we assume that $R=1$. Thus our assumption that $\sum a_n x^n$ converges at $x=R$ becomes an assumption that $\sum a_n$ converges.

Next we apply Dirichlet's test, using theorem with
$u_n(x) = a_n$, $v_n(x) = x^n$

See that we can apply the test, since
(i) $\sum u_n(x)$ converges uniformly mean $\sum a_n$ converges, which is an explicit assumptions;
(ii) if $x \neq 1$,

$$\sum_{n=1}^{\infty} |v_n(x) - v_{n-1}(x)| = \sum_{n=1}^{\infty} |x^n - x^{n-1}|$$

$$= \sum_{n=1}^{\infty} x^{n-1} |x-1|$$

$$= (1-x) \sum_{n=1}^{\infty} x^{n-1}$$

$$= (1-x)/(1-x) = 1$$

and if $x=1$, $\sum_{n=1}^{\infty} |v_n(x) - v_{n-1}(x)| = 0$

(iii) $|v_0(x)| = 1$

Thus (ii) and (iii) are satisfied when $K=1$. Hence by Dirichlet's test, the result follows.

The Weierstrass Approximation Theorem:

If $f(x)$ is a real continuous function defined on a closed interval $[a,b]$ then there exists a sequence of real polynomials $\{P_n\}$ which converges uniformly to $f(x)$ on $[a,b]$ i.e. $\lim_{n\to\infty} P_n(x) = f(x)$, converges uniformly on $[a,b]$

Proof: If $a=b$, the conclusion follows by taking $P_n(x)$ to be a constant polynomial, defined by $P_n(x) = f(a)$ for all n.

We may thus assume $a<b$.

We next observe that a linear transformation $x' = (x-a)/(b-a)$ is a continuous mapping of $[a,b]$ onto $[0,1]$. Accordingly, we assume without loss of generality that $a=0$, $b=1$.

Consider

$$F(x) = f(x) - f(0) - x[f(1) - f(0)],$$

for $0 \le x \le 1$.

Here $F(0) = 0 = F(1)$, and if F can be expressed as a limit of a uniformly convergent sequence of polynomials then the same is true for f, since $f - F$ is a polynomial. So we ~~can~~ may assume that $f(1) = f(0) = 0$.

Let us further define $f(x)$ to be zero for x outside $[0,1]$. Thus, f is now uniformly continuous on the whole real line.

Let us consider the polynomial (non-negative for $|x| \le 1$)

$$B_n(x) = C_n(1-x^2)^n, \quad n = 1, 2, 3, \dots \qquad (1)$$

where C_n, independent of x, is so chosen that

$$\int_{-1}^{1} B_n(x)\,dx = 1 \text{ for } n = 1, 2, 3, \dots \qquad (2)$$

$$\therefore \quad 1 = \int_{-1}^{1} C_n(1-x^2)^n dx = 2C_n \int_0^1 (1-x^2)^n dx$$

$$\text{i.e. } \ge 2C_n \int_0^{1/\sqrt{n}} (1-x^2)^n dx$$

$$\text{i.e. } \ge 2C_n \int_0^{1/\sqrt{n}} (1-nx^2) dx$$

$$= 2C_n \cdot \left[x - \frac{n}{3}x^3\right]_0^{1/\sqrt{n}}$$

$$= 2C_n \cdot \frac{2}{3\sqrt{n}}$$

$$= \frac{4C_n}{3\sqrt{n}} > \frac{C_n}{\sqrt{n}}$$

$$\Rightarrow \sqrt{n} > C_n$$

$$\Rightarrow C_n < \sqrt{n} \qquad (3)$$

which gives some information about the order of magnitude of C_n.

Therefore, for any $\delta > 0$, ③ gives
$$B_n(x) \le \sqrt{n}\,(1-\delta^2)^n, \text{ when } \delta \le |x| \le 1.$$
So that $B_n \to 0$ uniformly, for $\delta \le |x| \le 1$.

Again, let
$$P_n(x) = \int_{-1}^{1} f(x+t)\, B_n(t)\, dt, \quad 0 \le x \le 1 \quad \cdots \cdots ⑤$$
$$= \int_{-1}^{-x} f(x+t) B_n(t) dt + \int_{-x}^{1-x} f(x+t) B_n(t) dt + \int_{1-x}^{1} f(x+t) B_n(t)$$

For $|x| \le 1$, $-1+x \le x+t \le 0$, for $-1 \le t \le -x$, so that $x+t$ lie outside $[0,1]$ and therefore $f(x+t) = 0$ and hence the first integral on the right vanishes. Similiarly, the third inte is also equal to zero.

Hence
$$P_n(x) = \int_{-x}^{1-x} f(x+t)\, B_n(t)\, dt$$
$$= \int_0^1 f(t)\, B_n(t-x)\, dt$$
which clearly is a real polynomial.

We now proceed to show that the sequence $\{P_n(x)\}$ converges uniformly to f on $[0,1]$.

Continuity of f on the closed interval $[0,1]$ implies that f is bounded and uniformly continuous on $[0,1]$.

Therefore, there exists M such that
$$M = \sup |f(x)| \quad \cdots \cdots ⑥$$
and for any $\epsilon > 0$, we can choose $\delta > 0$ such that for any two points x_1, x_2 in $[0,1]$,
$$|f(x_1) - f(x_2)| < \epsilon/2, \text{ when } |x_1 - x_2| < \delta \le 1 \cdots$$

For $0 \le x \le 1$, we have
$$|P_n(x) - f(x)| = \left| \int_{-1}^{1} f(x+t) B_n(t) dt - f(x) \right|$$
$$= \left| \int_{-1}^{1} \{f(x+t) - f(x)\} B_n(t) dt \right| \quad [\text{using ②}$$
$$\le \int_{-1}^{1} |f(x+t) - f(x)|\, B_n(t) dt \quad \because B_n(t)$$
$$= \int_{-1}^{-\delta} |f(x+t) - f(x)|\, B_n(t) dt + \int_{-\delta}^{\delta} |f(x+t) - f(x)|\, B_n(t) dt + \int_{\delta}^{1} |f(x+t) - f(x)|\, B_n(t)$$
$$\le 2M \int_{-1}^{-\delta} B_n(t)\, dt + \epsilon/2 \int_{-\delta}^{\delta} B_n(t) dt + 2M \int_{\delta}^{1} B_n(t)\, dt \quad [\text{using equation ⑥}$$
$$\le 2M\sqrt{n}\,(1-\delta^2)^n \left\{ \int_{-1}^{-\delta} dt + \int_{\delta}^{1} dt \right\} + \epsilon/2 \quad [\text{using eqn ②}$$
$$\le 4M\sqrt{n}\,(1-\delta^2)^n + \epsilon/2$$
i.e. for large values of n.

Thus for $\epsilon > 0$, there exists N (independent of x) such that

$$|P_n(x) - f(x)| < \epsilon, \quad \forall\, n \geq N$$

$$\Rightarrow \lim_{n \to \infty} P_n(x) = f(x), \text{ uniformly on } [0,1].$$

$$\Rightarrow |a_n x^n| > \left(\frac{|x|}{\rho}\right)^n \quad \ldots\ldots \text{(ii)}$$

Here $\frac{|x|}{\rho} > 1$.

So $\sum_{n=0}^{\infty} |a_n x^n|$ diverges, as the $\sum_{n=0} \left(\frac{|x|}{\rho}\right)^n$ being a G.P. series with common ratio >1, diverges.

Each term of the series is unbounded since $\lim_{n\to\infty} \left(\frac{|x|}{\rho}\right)^n \to \infty$ as $\frac{|x|}{\rho} > 1$.

** Problem: Find the power series for $\tan^{-1} x$ by integration and deduce the sum $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$

Solution: We have

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} = (1+x^2)^{-1} = 1-x^2+x^4-x^6+\ldots$$

It is clear from the ratio test that the series converges $|x|<1$, and oscillates at the end points. Thus, integrating term by term in $-1<x<1$,

$$\int_0^x \frac{dx}{1+x^2} = \int_0^x 1\,dx - \int_0^x x^2\,dx + \int_0^x x^4\,dx - \int_0^x x^6\,dx + \cdots$$

i.e. $\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots \ldots$ for $|x|<1$

Also at $x=1$, the integrated series becomes

$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$ which is convergent

and $x=-1$, $-\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots+\ldots\right)$ is convergent

Thus $\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots + \ldots$ for $|x$

Hence the series represents a continuous function throughout $-1 \le x \le 1$ and as $x \to 1^-$ we obtain from (I)

** $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots+\ldots = \pi/4$

Abel's Theorem on limit form: If the P.S. $\sum_{n=0}^{\infty} a_n x^n$ converges with finite radius of convergence R and the series also converges at $x=R$, then $\lim_{x\to R-0} f(x) = \sum_{n=0}^{\infty} a_n R^n$, where $f(x) = \sum a_n x^n$ for $-R<x<R$.

Proof: Without any loss of generality we can take R
To justify it, we put $x = Ry$.

Then $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n R^n y^n = \sum_{n=0} b_n y^n$ where $b_n = a_n R^n$.

Let the P.S. $\sum_{n=0}^{\infty} b_n y^n$ converges for $|y|<R'$

Then $\frac{1}{R'} = \overline{\lim}|b_n|^{1/n} = \overline{\lim}|a_n R^n|^{1/n} = \overline{\lim}|a_n|^{1/n} \cdot R$

$$\Rightarrow |a_n x^n| > \left(\frac{|x|}{\rho}\right)^n \quad \ldots\ldots\ldots \text{(ii)}$$

Here $\frac{|x|}{\rho} > 1$.

So $\sum_{n=0}^{\infty} |a_n x^n|$ diverges, as the $\sum_{n=0}^{\infty} \left(\frac{|x|}{\rho}\right)^n$ being a G.P. series with common ratio >1, diverges. Since each term of the series is unbounded

$$\lim_{n\to\infty} \left(\frac{|x|}{\rho}\right)^n \longrightarrow \infty \text{ as } \frac{|x|}{\rho} > 1.$$

**

Problem: Find the power series for $\tan^{-1} x$ by ... and deduce the sum $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

Solution: We have

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} = (1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \ldots$$

It is clear from the ratio test that the series converges $|x| < 1$, and oscillates at the end points.

Thus, integrating term by term in $-1 < x < 1$,

$$\int_0^x \frac{dx}{1+x^2} = \int_0^x 1\,dx - \int_0^x x^2\,dx + \int_0^x x^4\,dx - \int_0^x x^6\,dx + \ldots \quad \text{for } |x| < 1$$

$$\text{i.e. } \tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots \quad \text{(1)}$$

Also at $x = 1$, the integrated series becomes

$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ which is convergent

and $x = -1$, $-(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots + \ldots)$ is convergent

Thus $\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots + \ldots$ for $|x| \le 1$.

Hence the series represents a continuous function throughout $-1 \le x \le 1$ and as $x \to 1^-$ we obtain from (1)

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots + \ldots = \pi/4$$

**

Abel's Theorem on limit form: If the P.S. $\sum_{n=0}^{\infty} a_n x^n$ converges with finite radius of convergence R and the series also converge at $x = R$, then $\lim_{x\to R-0} f(x) = \sum_{n=0}^{\infty} a_n R^n$, where $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $-R < x < R$.

Proof: Without any loss of generality we can take $R = 1$.

To justify it, we put $x = Ry$.

Then $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n R^n y^n = \sum_{n=0}^{\infty} b_n y^n$ where $b_n = a_n R^n$.

Let the P.S. $\sum_{n=0}^{\infty} b_n y^n$ converges for $|y| < R'$

Then $\frac{1}{R'} = \overline{\lim} |b_n|^{1/n} = \overline{\lim} |a_n R^n|^{1/n} = \overline{\lim} |a_n|^{1/n} \cdot R$