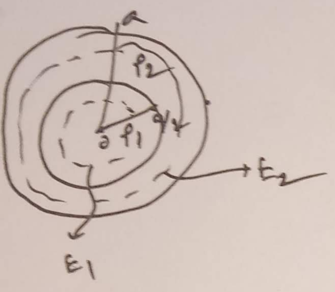
Prob: A spherical charge distribution consists of a uniform charge density ρ_1 from $r=0$ to $a/2$ and ρ_2 from $r=a/2$ to a. Find electric potentials at $r=0, a/2, a$.

Solutions.

Total charge $Q = \int_0^{a/2} 4\pi r^2 \rho_1 \, dr + \int_{a/2}^{a} 4\pi r^2 \rho_2 \, dr.$

$$= 4\pi \rho_1 \frac{r^3}{3}\Big|_0^{a/2} + 4\pi \rho_2 \frac{r^3}{3}\Big|_{a/2}^{a}.$$



$$= 4\pi \rho_1 \frac{a^3}{24} + 4\pi \rho_2 \frac{a^3}{3} - 4\pi \rho_2 \frac{a^3}{24}.$$

$$= 4\pi \rho_1 \frac{a^3}{24} + 4\pi \rho_2 \frac{a^3}{3}\left(1 - \frac{1}{8}\right)$$

$$= 4\pi \rho_1 \frac{a^3}{24} + 4\pi \rho_2 \frac{7a^3}{24}.$$

$$= \frac{4\pi a^3}{24}\left(\rho_1 + 7\rho_2\right).$$

$$\therefore V_s = V(a) = \frac{Q}{4\pi\epsilon_0 . a.} = \frac{4\pi a^2}{4\pi\epsilon_0 . 24}\left(\rho_1 + 7\rho_2\right)$$

$$= \frac{a^2}{24\epsilon_0}\left(\rho_1 + 7\rho_2\right). \text{ Proved.}$$

Between, $r=0$ to $a/2$. Let, $E=E_1$. Case-1. According Gauss's law.

$$E_1 . \cancel{4\pi r^2} = \frac{\int_0^r 4\pi r^2 \rho_1 \, dr}{\epsilon_0.} = \frac{\cancel{4\pi} \rho_1 \cancel{r^3} r.}{3\epsilon_0.}$$

$$\therefore E_1 = \frac{\rho_1 r}{3\epsilon_0.}$$

Case: 2. (between $r = a/2$ to r). $r \le a$. According to Gauss's law.

$$E_2 \cdot 4\pi r^2 = \frac{\int_0^{a/2} 4\pi r^2 \rho_1 \, dr}{\epsilon_0.} + \frac{\int_{a/2}^{r} 4\pi r^2 \rho_2 \, dr}{\epsilon_0.}$$

or. $E_2 \cdot 4\pi r^2 = \frac{1.4\pi}{\epsilon_0}\left[\rho_1 \frac{r^3}{3}\Big|_0^{a/2}\right] + \frac{4\pi \rho_2}{\epsilon_0} \frac{r^3}{3}\Big|_{a/2}^{r}$

$$= \frac{\cancel{4\pi}\rho_1}{\epsilon_0.}\frac{a^3}{24} + \frac{4\pi}{\epsilon_0}\rho_2\left(\frac{r^3}{3} - \frac{a^3}{24}\right).$$

$$= \frac{\rho_1 a^3}{24\epsilon_0} + \cancel{\frac{\rho_2 a^3 7}{\epsilon_0 . 24}} + \frac{\rho_2 r^3}{3\epsilon_0} - \frac{\rho_2 a^3}{24\epsilon_0}.$$

or $E_2 = \frac{\rho_1 a^3}{24\epsilon_0 r^2} + \frac{\rho_2}{3\epsilon_0}\left(r - \frac{a^3}{8r^2}\right).$

$$V(a/2) - V(a) = -\int_a^{a/2} E_2 \cdot dr.$$

$$= -\int_a^{a/2} \frac{\rho_1 a^3\, dr}{24\epsilon_0 r^2} - \int_a^{a/2} \frac{\rho_2 r\, dr}{3\epsilon_0} + \int_a^{a/2} \frac{\rho_2 a^3\, dr}{24\epsilon_0 r^2}$$

$$= \frac{\rho_1 a^3}{24\epsilon_0 . r}\Big|_a^{a/2} - \frac{\rho_2}{3\epsilon_0}\frac{r^2}{2}\Big|_a^{a/2} - \frac{\rho_2 a^3}{24\epsilon_0 . r}\Big|_a^{a/2}$$

$$= \frac{\rho_1 a^3}{24\epsilon_0}\left(\frac{2}{a} - \frac{1}{a}\right) - \frac{\rho_2}{6\epsilon_0}\left(\frac{a^2}{4} - a^2\right) - \frac{\rho_2 a^3}{24\epsilon_0}\left(\frac{2}{a} - \frac{1}{a}\right).$$

$$= \frac{\rho_1 a^2}{24\epsilon_0} + \frac{\rho_2}{{}_2\cancel{6}\epsilon_0}\frac{\cancel{3}a^2}{4} - \frac{\rho_2 a^3}{24\epsilon_0}$$

$$= \frac{\rho_1 a^2}{24\epsilon_0} + \frac{\rho_2 a^2}{8\epsilon_0}\left(1 - \frac{1}{3}\right).$$

$$= \frac{\rho_1 a^2}{24\cancel{a}\epsilon_0} + \frac{\rho_2 a^2 . 2}{8\epsilon_0 . 3}.$$

$$= \frac{\rho_1 a^2}{24\epsilon_0} + \frac{2\rho_2 a^2}{24\epsilon_0}.$$

$$\therefore V(a/2) = V(a) + \frac{\rho_1 a^2}{24\epsilon_0} + \frac{2\rho_2 a^2}{24\epsilon_0}.$$

$$= \frac{a^2}{24\epsilon_0}(\rho_1 + 7\rho_2) + \frac{\rho_1 a^2}{24\epsilon_0} + \frac{2\rho_2 a^2}{24\epsilon_0}.$$

$$= \frac{a^2}{24\epsilon_0}(2\rho_1 + 9\rho_2).$$ Proved.

Similarly, $V(0) - V(a/2) = -\int_{a/2}^{0} E_1 \cdot dr.$

$$= -\int_{a/2}^{0} \frac{\rho_1}{3\epsilon_0} r\, dr.$$

$$= -\frac{\rho_1}{3\epsilon_0}.\frac{r^2}{2}\Big|_{a/2}^{0}$$

$$= -\frac{\rho_1}{3\epsilon_0}\left(0 - \frac{a^2}{8}\right) = \frac{\rho_1 a^2}{24\epsilon_0}.$$

$$\therefore V(0) = V(a/2) + \frac{\rho_1 a^2}{24\epsilon_0}.$$

$$= \frac{a^2}{24\epsilon_0}(3\rho_1 + 9\rho_2).$$ Proved.

• A sphere of radius a carries a charge density $\rho(r) = Kr$. where K is a Constant. Find the total electrostatic energy of the system.

Total charge of the system is,

$$Q = \int_0^a \rho(r)\, 4\pi r^2 dr = 4\pi K \int_0^a r^3 dr = \cancel{4}\pi K \frac{a^4}{\cancel{4}}.$$

$$Q = \pi K a^4.$$

$$E_o = \frac{Q}{4\pi\epsilon_0 r^2}.$$

For an internal point, $\oint E_i\, ds = \frac{Q_r}{\epsilon_0} = \frac{1}{\epsilon_0} \int_0^r \rho(r) \cdot 4\pi r^2 dr.$

$$\therefore E_i \cdot 4\pi r^2 = \frac{1}{\epsilon_0} \int_0^r Kr \cdot 4\pi r^2 dr = \frac{4\pi K}{\epsilon_0} \frac{r^4}{4} = \frac{\pi K r^4}{\epsilon_0}.$$

$$\therefore E_i = \frac{\pi K r^4}{\epsilon_0 \cdot 4\pi r^2} = \frac{\pi K r^2}{4\pi\epsilon_0}.$$

Now, the total energy of the system,

$$U = \int_0^a \frac{1}{2}\epsilon_0 E_i^2 \cdot 4\pi r^2 dr + \int_a^\infty \frac{1}{2}\epsilon_0 E_o^2 \cdot 4\pi r^2 dr.$$

$$= \frac{1}{2}\epsilon_0 \cdot 4\pi \int_0^a \left(\frac{\pi K r^2}{4\pi\epsilon_0}\right)^2 4\pi r^2 dr + \frac{1}{2}\epsilon_0 4\pi \int_a^\infty \frac{Q^2 \cdot 4\pi r^2 dr}{16\pi^2\epsilon_0^2 \cdot r^4}$$

$$= \frac{4\pi\epsilon_0 \cdot \pi^2 K^2 \cdot 4\pi}{2 \cdot 16\pi^2 \epsilon_0^4} \int_0^a r^6 dr + \frac{\epsilon_0 \cdot 4\pi Q^2}{2 \cdot 16\pi^2\epsilon_0^4} \int_a^\infty \frac{dr}{r^2}$$

$$= \frac{\pi^2 K^2}{2\epsilon_0} \frac{a^7}{7} + \frac{Q^2}{2\epsilon_0}\left(-\frac{1}{r}\right)\Big|_a^\infty$$

$$= \frac{\pi^2 K^2}{7\epsilon_0}\left(\frac{a^7}{2}\right) - \frac{Q^2}{2\epsilon_0}\left(\frac{1}{\infty} - \frac{1}{a}\right) = \frac{\pi^2 K^2}{7\epsilon_0}\frac{a^7}{2} + \frac{Q^2}{2\epsilon_0}\frac{1}{a}$$

$$\therefore U = \frac{\pi^2 K^2}{7\epsilon_0}\frac{a^7}{2} + \frac{\pi^2 K^2 \cancel{a^8}\, a^7}{2\epsilon_0 \cdot \cancel{a}}$$

$$= \frac{\pi^2 K^2 a^7}{2\epsilon_0}\left(\frac{1}{7} + 1\right).$$

$$= \frac{8\pi^2 K^2 a^7}{2\epsilon_0 \cdot 7}$$

$$= \frac{4\pi^2 K^2 a^7}{7\epsilon_0}.$$